Supplementary material

Appendix D: Energetics of the NP component

Following the reasoning which led to (2.27), but without the wave-mode contributions, (3.4), without the forcing term, gives

$$\frac{\partial a_0(\mathbf{k})}{\partial t} = \int N^*_{000}(\mathbf{k},\mathbf{p})\, a_0^*(\mathbf{p})\, a_0^*(-\mathbf{k}-\mathbf{p})\, d^3\mathbf{p} - D_{00}(\mathbf{k})\, a_0(\mathbf{k}). \tag{D.1}$$

Taking the ensemble average of the complex conjugate of (D.1), multiplied by $a_0(\mathbf{k}')$,

$$\overline{\frac{\partial a_0^*(\mathbf{k})}{\partial t} a_0(\mathbf{k}')} = \left(\tau(\mathbf{k}) - D_{00}(\mathbf{k})\, A_{00}(\mathbf{k})\right)\delta(\mathbf{k}-\mathbf{k}'), \tag{D.2}$$

where

$$\tau(\mathbf{k}) = \int N_{000}(\mathbf{k},\mathbf{p})\,\Theta(\mathbf{k},\mathbf{p})\, d^3\mathbf{p} \tag{D.3}$$

and we have used (2.35), the fact that D_{00} is real, as well as

$$\overline{a_0(\mathbf{k})\, a_0(\mathbf{p})\, a_0(\mathbf{k}'')} = \Theta(\mathbf{k},\mathbf{p})\,\delta(\mathbf{k}+\mathbf{p}+\mathbf{k}''), \tag{D.4}$$

which follows from homogeneity. Note that, given the Dirac function in (D.2), $\Theta(\mathbf{k}',\mathbf{p})$ has been replaced by $\Theta(\mathbf{k},\mathbf{p})$ in (D.3). Taking the complex conjugate of (D.2), permuting $\mathbf{k} \leftrightarrow \mathbf{k}'$ and using the fact that $D_{00}A_{00}$ is real,

$$\begin{aligned}\overline{a_0^*(\mathbf{k})\frac{\partial a_0(\mathbf{k}')}{\partial t}} &= \left(\tau^*(\mathbf{k}') - D_{00}(\mathbf{k}')\, A_{00}(\mathbf{k}')\right)\delta(\mathbf{k}-\mathbf{k}') \\ &= \left(\tau^*(\mathbf{k}) - D_{00}(\mathbf{k})\, A_{00}(\mathbf{k})\right)\delta(\mathbf{k}-\mathbf{k}')\end{aligned}. \tag{D.5}$$

Taking the time derivative of (2.35) and using (D.2) and (D.5),

$$\frac{\partial A_{00}}{\partial t} = 2\Re\{\tau\} - 2D_{00}A_{00}. \tag{D.6}$$

Using (3.7), (D.6) gives the NP-mode energy equation

$$\frac{\partial e_{NP}}{\partial t} = \Re\{\tau\} - 2D_{00}e_{NP}, \tag{D.7}$$

of which the first term on the right-hand side represents nonlinear energy transfer between NP modes, while the second expresses visco-diffusive dissipation.

Taking the integral of (D.7) over $\mathbf{k}$ and using (D.3),

$$\frac{d}{dt}\int e_{NP}(\mathbf{k})\,d^3\mathbf{k} = \Re\left\{\iint N_{000}(\mathbf{k},\mathbf{p})\Theta(\mathbf{k},\mathbf{p})\,d^3\mathbf{p}d^3\mathbf{k}\right\} - 2\int D_{00}(\mathbf{k})e_{NP}(\mathbf{k})\,d^3\mathbf{k} \quad . \tag{D.8}$$

Since, according to (2.13), $\left(k_j + p_j\right)e_j^{(l)}(-\mathbf{k}-\mathbf{p}) = 0$, hence (2.12) implies

$$k_j v_j^{(s_q)}(-\mathbf{k}-\mathbf{p}) = -p_j v_j^{(s_q)}(-\mathbf{k}-\mathbf{p}), \tag{D.9}$$

thus $N_{ss_ps_q}(\mathbf{k},\mathbf{p}) = -N_{s_pss_q}(\mathbf{p},\mathbf{k})$ from (2.28). Given $\Theta(\mathbf{k},\mathbf{p}) = \Theta(\mathbf{p},\mathbf{k})$, which follows from (D.4), the integrand of the first term on the right-hand side of (D.8) changes sign under the permutation $\mathbf{p} \leftrightarrow \mathbf{k}$, hence the integral is zero, yielding (3.8).

Appendix E: Energetics of the wave component

Prior to introducing scaled variables, (3.13) can be rewritten as

$$\frac{\partial A_{ss}(\mathbf{k})}{\partial t} + 2D(\mathbf{k})A_{ss}(\mathbf{k}) = 2\pi\Re\left(E_s(\mathbf{k})\right), \tag{E.1}$$

where

$$E_s(\mathbf{k}) = \sum_{s_p,s_q=\pm1}\int G_{ss_ps_q}(\mathbf{k},\mathbf{p})A_{s_ps_p}(\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}, \tag{E.2}$$

$$G_{ss_ps_q}(\mathbf{k},\mathbf{p}) = \lambda_{ss_ps_q}(\mathbf{k},\mathbf{p})A_{s_qs_q}(-\mathbf{k}-\mathbf{p}) + \zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})A_{ss}(\mathbf{k}), \tag{E.3}$$

we have used reality of $\lambda_{ss_ps_q}$ and the diagonal elements of $A_{ss'}$ and converted the surface integral into a volume integral using a Dirac function. Taking the sum over $s = \pm1$, using the first of equations (2.40) and integrating over $\mathbf{k}$, (E.1) yields

$$\frac{d}{dt}\int e_w(\mathbf{k})\,d^3\mathbf{k} = \pi\Re\left(\sum_{s=\pm1}\int E_s(\mathbf{k})\,d^3\mathbf{k}\right) - 2\int D(\mathbf{k})e_w(\mathbf{k})\,d^3\mathbf{k} \tag{E.4}$$

as the evolution equation of the total energy of the wave component.

Using (2.31), (3.16), (3.17) and (E.3),

$$G_{ss_ps_q}(\mathbf{k},\mathbf{p}) = \left(N_{ss_ps_q}(\mathbf{k},\mathbf{p}) + N_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{p})\right)H_{ss_ps_q}(\mathbf{k},\mathbf{p}), \tag{E.5}$$

where

$$H_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)=M^{*}_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)A_{s_qs_q}\left(-\mathbf{k}-\mathbf{p}\right)+2M^{*}_{s_qss_p}\left(-\mathbf{k}-\mathbf{p},\mathbf{k}\right)A_{ss}\left(\mathbf{k}\right). \qquad \text{(E.6)}$$

Thus, (E.2) gives

$$\begin{gathered}E_s\left(\mathbf{k}\right)= \\ \sum_{s_p,s_q=\pm1}\int N_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)H_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)A_{s_ps_p}\left(\mathbf{p}\right)\delta\left(F_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)\right)d^3\mathbf{p}+ \\ \sum_{s_p,s_q=\pm1}\int N_{ss_qs_p}\left(\mathbf{k},-\mathbf{k}-\mathbf{p}\right)H_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)A_{s_ps_p}\left(\mathbf{p}\right)\delta\left(F_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)\right)d^3\mathbf{p}\end{gathered}. \qquad \text{(E.7)}$$

The final term in (E.7) is treated by changing the integration variable to $\mathbf{q}=-\mathbf{k}-\mathbf{p}$ and permuting the summation indices according to $s_p\leftrightarrow s_q$, hence it becomes

$$\sum_{s_p,s_q=\pm1}\int N_{ss_ps_q}\left(\mathbf{k},\mathbf{q}\right)H_{ss_qs_p}\left(\mathbf{k},-\mathbf{k}-\mathbf{q}\right)A_{s_qs_q}\left(-\mathbf{k}-\mathbf{q}\right)\delta\left(F_{ss_ps_q}\left(\mathbf{k},\mathbf{q}\right)\right)d^3\mathbf{q}, \qquad \text{(E.8)}$$

where $F_{ss_qs_p}\left(\mathbf{k},-\mathbf{k}-\mathbf{q}\right)=F_{ss_ps_q}\left(\mathbf{k},\mathbf{q}\right)$ has been used. Thus, (E.7) yields

$$\begin{gathered}E_s\left(\mathbf{k}\right)= \\ 2\sum_{s_p,s_q=\pm1}\int N_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)I_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)\delta\left(F_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)\right)d^3\mathbf{p}\end{gathered}, \qquad \text{(E.9)}$$

where

$$I_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)=\frac{1}{2}\left(H_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)A_{s_ps_p}\left(\mathbf{p}\right)+H_{ss_qs_p}\left(\mathbf{k},-\mathbf{k}-\mathbf{p}\right)A_{s_qs_q}\left(-\mathbf{k}-\mathbf{p}\right)\right). \qquad \text{(E.10)}$$

Using $M^{*}_{ss_qs_p}\left(\mathbf{k},-\mathbf{k}-\mathbf{p}\right)=M^{*}_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)$ and (E.6),

$$\begin{gathered}I_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)=M^{*}_{s_qss_p}\left(-\mathbf{k}-\mathbf{p},\mathbf{k}\right)A_{ss}\left(\mathbf{k}\right)A_{s_ps_p}\left(\mathbf{p}\right)+ \\ M^{*}_{s_pss_q}\left(\mathbf{p},\mathbf{k}\right)A_{ss}\left(\mathbf{k}\right)A_{s_qs_q}\left(-\mathbf{k}-\mathbf{p}\right)+M^{*}_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)A_{s_ps_p}\left(\mathbf{p}\right)A_{s_qs_q}\left(-\mathbf{k}-\mathbf{p}\right)\end{gathered}. \qquad \text{(E.11)}$$

Observe that the symmetry relation $I_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)=I_{s_pss_q}\left(\mathbf{p},\mathbf{k}\right)$ follows from (E.11) and $M^{*}_{s_qss_p}\left(-\mathbf{k}-\mathbf{p},\mathbf{k}\right)=M^{*}_{s_qs_ps}\left(-\mathbf{k}-\mathbf{p},\mathbf{p}\right)$.

(E.9) implies

$$\begin{gathered}\sum_{s=\pm1}\int E_s\left(\mathbf{k}\right)d^3\mathbf{k}= \\ 2\sum_{s,s_p,s_q=\pm1}\int\int N_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)I_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)\delta\left(F_{ss_ps_q}\left(\mathbf{k},\mathbf{p}\right)\right)d^3\mathbf{p}\,d^3\mathbf{k}\end{gathered}. \qquad \text{(E.12)}$$

As was shown in appendix D using (D.9), $N_{ss_ps_q}(\mathbf{k},\mathbf{p}) = -N_{s_pss_q}(\mathbf{p},\mathbf{k})$. Using this result and $I_{ss_ps_q}(\mathbf{k},\mathbf{p}) = I_{s_pss_q}(\mathbf{p},\mathbf{k})$, $F_{ss_ps_q}(\mathbf{k},\mathbf{p}) = F_{s_pss_q}(\mathbf{p},\mathbf{k})$, it is apparent that the integral in (E.12) changes sign under the permutation $s \leftrightarrow s_p$, thus (E.12) is zero, hence (E.4) gives (3.18).

Appendix F: Numerical scheme for (3.9)-(3.11)

(3.9) and (3.10) are

$$\hat{u}_i^{NP}(\mathbf{x}) = \int \hat{a}_0(\mathbf{k}) v_i^{(0)}(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x}) d^3\mathbf{k}, \tag{F.1}$$

$$\hat{\eta}^{NP}(\mathbf{x}) = \int \hat{a}_0(\mathbf{k}) \eta^{(0)}(\mathbf{k}) \exp(i\mathbf{k}.\mathbf{x}) d^3\mathbf{k}. \tag{F.2}$$

Using (2.15), (3.11) implies

$$\frac{\partial \hat{a}_0}{\partial \hat{t}} + \hat{D}_{00}(\mathbf{k}) \hat{a}_0 = \gamma(\mathbf{k}), \tag{F.3}$$

where

$$\gamma(\mathbf{k}) = -\frac{Nk_\perp e_i^{(1)} k_j \widetilde{\hat{u}_i^{NP} \hat{u}_j^{NP}} + \Omega k_\parallel k_j \widetilde{\hat{\eta}^{NP} \hat{u}_j^{NP}}}{\left(N^2 k_\perp^2 + \Omega^2 k_\parallel^2\right)^{1/2}}. \tag{F.4}$$

(F.1)-(F.4) describe the evolution of the NP component.

In what follows, we use Cartesian coordinates, x_1, x_2 x_3, where $\mathbf{e} = (0,0,1)$ is the axial unit vector. DNS approximates the flow as periodic with respect to x_1, x_2 and x_3. Thus, (F.1) and (F.2) are replaced by the Fourier series

$$\hat{u}_i^{NP}(\mathbf{x}) = \sum_{l_1,l_2,l_3=-\infty}^{\infty} \tilde{u}_i(l_1,l_2,l_3) \exp(i\mathbf{k}.\mathbf{x}), \quad \hat{\eta}^{NP}(\mathbf{x}) = \sum_{l_1,l_2,l_3=-\infty}^{\infty} \tilde{\eta}(l_1,l_2,l_3) \exp(i\mathbf{k}.\mathbf{x}), \tag{F.5}$$

where $\mathbf{k}$ has the components $k_1 = 2\pi l_1 / L_1$, $k_2 = 2\pi l_2 / L_2$, $k_3 = 2\pi l_3 / L_3$, L_i are the spatial periods and, for $\mathbf{k} \neq 0$,

$$\tilde{u}_i(l_1,l_2,l_3) = a(l_1,l_2,l_3) v_i^{(0)}(\mathbf{k}), \quad \tilde{\eta}(l_1,l_2,l_3) = a(l_1,l_2,l_3) \eta^{(0)}(\mathbf{k}), \tag{F.6}$$

in which $a(l_1,l_2,l_3)$ gives the amplitudes of the discrete NP modes which constitute the periodized NP flow considered here. $a(-l_1,-l_2,-l_3) = a^*(l_1,l_2,l_3)$ because $\hat{u}_i^v$ and $\hat{\eta}^v$ are real. Using (2.12) and (2.15),

$$\begin{pmatrix} \tilde{u}_1(l_1,l_2,l_3) \\ \tilde{u}_2(l_1,l_2,l_3) \\ \tilde{u}_3(l_1,l_2,l_3) \end{pmatrix} = i\frac{N}{\left(N^2k_\perp^2+\Omega^2k_3^2\right)^{1/2}}\begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} a(l_1,l_2,l_3), \tag{F.7}$$

$$\tilde{\eta}(l_1,l_2,l_3) = i\frac{\Omega k_3}{\left(N^2k_\perp^2+\Omega^2k_3^2\right)^{1/2}} a(l_1,l_2,l_3). \tag{F.8}$$

There are divisions by zero in (F.7) and (F.8) when $\mathbf{k}=0$, which is the reason why they can only be used for $\mathbf{k}\neq 0$. The case $\mathbf{k}=0$ concerns only a single point in spectral space, so we expect its numerical treatment to be unimportant as regards the overall description of the flow. For simplicity's sake we choose to set $\tilde{u}_i(0,0,0)=\tilde{\eta}(0,0,0)=0$ in (F.5) and to restrict attention to the time evolution of components with $\mathbf{k}\neq 0$. As usual in DNS, the spatial periods, L_i, need to be sufficiently large that periodicity is unimportant. Another way of putting this is that the spacing, $2\pi/L_i$, in spectral space should be sufficiently small. The numerical equivalent of (F.3) is

$$\frac{da}{d\hat{t}} + \hat{D}_{00}(\mathbf{k})\,a = \tilde{\gamma}(\mathbf{k}) \tag{F.9}$$

for $\mathbf{k}\neq 0$, where $\tilde{\gamma}(\mathbf{k})$ is given by (F.4), but with suitable interpretations of $\widetilde{\hat{u}_i^{NP}\hat{u}_j^{NP}}$ and $\widetilde{\hat{\eta}^{NP}\hat{u}_j^{NP}}$ in terms of discrete Fourier transforms.

Although the case $N=0$ has been excluded in the main text, it is perhaps appropriate to consider it here. Analytically, $\hat{u}_i^{NP}=0$ follows from (2.12), (2.15) and (F.1), hence $\gamma=0$ from (F.3). Thus, the nonlinear term in (F.3) vanishes and the NP component should undergo simple visco-diffusive decay. From a numerical point of view, when $N=0$, (F.7) is zero unless $k_3=0$, where there is a zero-by-zero division. In an attempt to approach the case $N=0$, consider an extremely small value of N. This makes (F.7) very small, apart from the $k_3=0$ component, which is $O(1)$. Thus, $\tilde{u}_i(\mathbf{k})$ other than $k_3=0$ are correctly treated by the numerics, but those with $k_3=0$ remain and pollute the numerical results for all other wave vectors over time. These numerical problems reflect the singular nature of the $N\to 0$ limit for the NP component. The case $\Omega=0$, excluded in the main text, also leads to numerical difficulties since both (F.7) and (F.8) have zero-by-zero divisions when $k_\perp=0$.

In addition to supposing periodicity, DNS truncates the sums in (F.5) to $|l_i|\le K_i$, where K_i are positive integers, which must be sufficiently large that the smallest dynamically significant length scales of the flow are resolved. Evaluating (F.5) at the discrete points $x_1=p_1L_1/N_1$, $x_2=p_2L_2/N_2$, $x_3=p_3L_3/N_3$, where $N_i>2K_i$ and $0\le p_i<N_i$ are integers, the result can be expressed as

$$\begin{pmatrix} \hat{u}_1^v(p_1,p_2,p_3) \\ \hat{u}_2^v(p_1,p_2,p_3) \\ \hat{\eta}^v(p_1,p_2,p_3) \end{pmatrix} = \sum_{q_1=0}^{N_1-1}\sum_{q_2=0}^{N_2-1}\sum_{q_3=0}^{N_3-1}\begin{pmatrix} \xi_1(q_1,q_2,q_3) \\ \xi_2(q_1,q_2,q_3) \\ \xi_3(q_1,q_2,q_3) \end{pmatrix}\exp\left(2\pi i\left(\frac{q_1p_1}{N_1}+\frac{q_2p_2}{N_2}+\frac{q_3p_3}{N_3}\right)\right), \tag{F.10}$$

where

$$\begin{pmatrix} \xi_1(q_1,q_2,q_3) \\ \xi_2(q_1,q_2,q_3) \\ \xi_3(q_1,q_2,q_3) \end{pmatrix} = \begin{pmatrix} \tilde{u}_1(l_1,l_2,l_3) \\ \tilde{u}_2(l_1,l_2,l_3) \\ \tilde{\eta}(l_1,l_2,l_3) \end{pmatrix}, \tag{F.11}$$

$q_i = l_i$ for $0 \le l_i \le K_i$ and $q_i = N_i + l_i$ for $-K_i \le l_i < 0$. All $\xi_i(q_1,q_2,q_3)$ which are not covered by (F.11) are zero. (F.10) is a three-dimensional discrete Fourier transform. Given a and $\tilde{u}_i(0,0,0) = \tilde{\eta}(0,0,0) = 0$, (F.7), (F.8), (F.10) and (F.11) allow the determination of $\hat{u}_i^{NP}$ and $\hat{\eta}^{NP}$, hence $\hat{u}_i^{NP}\hat{u}_j^{NP}$ and $\hat{\eta}^{NP}\hat{u}_j^{NP}$, in physical space. Applying the inverse transform,

$$\begin{pmatrix} \chi_{ij}(q_1,q_2,q_3) \\ \lambda_j(q_1,q_2,q_3) \end{pmatrix} = \frac{1}{N_1N_2N_3}\sum_{p_1=0}^{N_1-1}\sum_{p_2=0}^{N_2-1}\sum_{p_3=0}^{N_3-1}\begin{pmatrix} \hat{u}_i^v\hat{u}_j^v(p_1,p_2,p_3) \\ \hat{\eta}^v\hat{u}_j^v(p_1,p_2,p_3) \end{pmatrix}\exp\left(-2\pi i\left(\frac{q_1p_1}{N_1}+\frac{q_2p_2}{N_2}+\frac{q_3p_3}{N_3}\right)\right) \tag{F.12}$$

gives the DNS approximations of $\widetilde{\hat{u}_i^{NP}\hat{u}_j^{NP}}$ and $\widetilde{\hat{\eta}^{NP}\hat{u}_j^{NP}}$ as

$$\begin{pmatrix} \widetilde{\hat{u}_i^{NP}\hat{u}_j^{NP}}(l_1,l_2,l_3) \\ \widetilde{\hat{\eta}^{NP}\hat{u}_j^{NP}}(l_1,l_2,l_3) \end{pmatrix} = \begin{pmatrix} \chi_{ij}(q_1,q_2,q_3) \\ \lambda_j(q_1,q_2,q_3) \end{pmatrix}, \tag{F.13}$$

where, as before, $q_i = l_i$ for $0 \le l_i \le K_i$ and $q_i = N_i + l_i$ for $-K_i \le l_i < 0$. This allows the calculation of $\tilde{\gamma}(\mathbf{k})$ using (F.4). As usual, FFT is used to implement the Fourier transforms in (F.10) and (F.12).

A time-stepping scheme for integration of (F.9) is introduced. Time is discretized to the values $\hat{t}_n = n\Delta$, where Δ is the time step, and (F.9) gives

$$a(\hat{t}_{n+1}) = a(\hat{t}_n)\exp\left(-\hat{D}_{00}\Delta\right) + \int_{\hat{t}_n}^{\hat{t}_{n+1}} \tilde{\gamma}(\hat{t}')\exp\left(\hat{D}_{00}(\hat{t}' - \hat{t}_{n+1})\right)d\hat{t}'. \tag{F.14}$$

Given $a\left(\hat{t}_n\right)$, $\tilde{\gamma}\left(\hat{t}_n\right)$ is determined as described above. As a first-order approximation, $\tilde{\gamma}\left(\hat{t}'\right)$ is taken as constant and equal to $\tilde{\gamma}\left(\hat{t}_n\right)$, hence

$$a^{\dagger} = a\left(\hat{t}_n\right)\exp\left(-\hat{D}_{00}\Delta\right) + \tilde{\gamma}\left(\hat{t}_n\right)\frac{1-\exp\left(-\hat{D}_{00}\Delta\right)}{\hat{D}_{00}} \tag{F.15}$$

as an approximate value of $a\left(\hat{t}_{n+1}\right)$. To increase the precision to second order, let $\tilde{\gamma}^{\dagger}$ be the value obtained using $\left(a\left(\hat{t}_n\right)+a^{\dagger}\right)/2$, which approximates the value of $a\left(\hat{t}\right)$ at time $\hat{t}_n+\Delta/2$. Thus,

$$a\left(\hat{t}_{n+1}\right) = a\left(\hat{t}_n\right)\exp\left(-\hat{D}_{00}\Delta\right) + \tilde{\gamma}^{\dagger}\frac{1-\exp\left(-\hat{D}_{00}\Delta\right)}{\hat{D}_{00}} \tag{F.16}$$

completes the time step. The advantage of the above scheme, compared with more traditional ones for integrating general differential equations, is that it only assumes that the time scale for significant changes in $a\left(\hat{t}\right)$ is long compared with Δ. As k increases, so does $\hat{D}_{00}\left(\mathbf{k}\right)$, and, when $\hat{D}_{00}\left(\mathbf{k}\right)\Delta$ becomes of $O\left(1\right)$, visco-diffusive effects are significant within a time step. At this point, the precision of traditional schemes is degraded, whereas it is maintained for the above scheme.

Using (F.5) and the fact that $\hat{u}_i^{v}$ and $\hat{\eta}^{v}$ are real, the averaged, scaled energy density of the NP component in physical space is

$$\frac{1}{2}\left(\overline{\hat{u}_i^{NP}\hat{u}_i^{NP}} + \overline{\hat{\eta}^{NP2}}\right) = \frac{1}{2}\sum_{\substack{l_1,l_2,l_3=-\infty \\ l_1',l_2',l_3'=-\infty}}^{\infty}\left(\overline{\tilde{u}_i^{*}\left(l_1,l_2,l_3\right)\tilde{u}_i\left(l_1',l_2',l_3'\right)} + \overline{\tilde{\eta}^{*}\left(l_1,l_2,l_3\right)\tilde{\eta}\left(l_1',l_2',l_3'\right)}\right)\exp\left(i\left(\mathbf{k}'-\mathbf{k}\right).\mathbf{x}\right) . \tag{F.17}$$

Statistical homogeneity implies that only terms with $l_i' = l_i$ are nonzero. Using (F.7) and (F.8) for $\mathbf{k}\neq 0$ and $\tilde{u}_i\left(0,0,0\right) = \tilde{\eta}\left(0,0,0\right) = 0$,

$$\frac{1}{2}\left(\overline{\hat{u}_i^{NP}\hat{u}_i^{NP}} + \overline{\hat{\eta}^{NP2}}\right) = \frac{1}{2}\sum_{l_1,l_2,l_3=-\infty}^{\infty}\overline{\left|a\left(l_1,l_2,l_3\right)\right|^2}, \tag{F.18}$$

where, here and henceforth, $a\left(0\right)=0$. On the other hand, prior to discretization,

$$\frac{1}{2}\left(\overline{\hat{u}_i^{NP}\hat{u}_i^{NP}} + \overline{\hat{\eta}^{NP2}}\right) = \int \hat{e}_v\left(\mathbf{k}\right)d^3\mathbf{k} = \frac{1}{2}\int \hat{A}_{00}\left(\mathbf{k}\right)d^3\mathbf{k}, \tag{F.19}$$

where $\hat{A}_{00} = A_{00} / \varepsilon_{NP}^2$ and $\hat{e}_{NP} = e_{NP} / \varepsilon_{NP}^2$, expresses the scaled NP energy as an integral over spectral space. (F.18) is a numerical approximation of (F.19). Since the density of discrete $\mathbf{k}$ in spectral space is $L_1 L_2 L_3 / 8\pi^3$,

$$\hat{A}_{00}(\mathbf{k}) = \frac{L_1 L_2 L_3}{8\pi^3} \overline{\left| a(l_1, l_2, l_3) \right|^2} \tag{F.20}$$

provides an approximation of $\hat{A}_{00}(\mathbf{k})$.

The scheme is initialised as follows. The coefficients $a(l_1, l_2, l_3)$ are of zero mean and must satisfy $\overline{a^*(l_1, l_2, l_3) a(l_1', l_2', l_3')} = 0$ unless $l_i' = l_i$, as well as $a(-\mathbf{k}) = a^*(\mathbf{k})$. Assuming the initial $\hat{A}_{00}(\mathbf{k})$ is given and such that $\hat{A}_{00}(-\mathbf{k}) = \hat{A}_{00}(\mathbf{k})$, (F.20) determines $\overline{\left| a(l_1, l_2, l_3) \right|^2}$. These conditions are satisfied by taking the initial coefficients

$$a(\mathbf{k}) = \left(\frac{8\pi^3}{L_1 L_2 L_3} \hat{A}_{00}(\mathbf{k}) \right)^{1/2} \exp\left(i\phi(\mathbf{k}) \right), \tag{F.21}$$

where the phases $\phi(\mathbf{k})$ are random variables, uniformly distributed between 0 and 2π and independent apart from the constraint $\phi(-\mathbf{k}) = -\phi(\mathbf{k})$.

Having described the methods used to initialise and integrate (F.9), we now want to extract information from the results. The problem with using (F.20) to determine $\hat{A}_{00}(\mathbf{k})$ is that it contains an ensemble average. To calculate such an average numerically would require very many DNS runs, which, given the computational cost, is not attempted. We instead extract information from a single run. In so doing, we assume that, prior to discretization, the flow is statistically axisymmetric. In particular, $\hat{A}_{00}(\mathbf{k})$ is axisymmetric and the aim here is results which are representative of the scaled energy density in spectral space, $\hat{e}_v = \hat{A}_{00}(k_\perp, k_\parallel)/2$, without using ensemble averaging. Note that $\hat{A}_{00}(-\mathbf{k}) = \hat{A}_{00}(\mathbf{k})$ and axisymmetry give $\hat{A}_{00}(k_\perp, -k_\parallel) = \hat{A}_{00}(k_\perp, k_\parallel)$. Thus, we restrict attention to determination of $\hat{A}_{00}$ for $k_\parallel \geq 0$. For simplicity's sake, from here on we assume $L_1 = L_2 = L_3 = L$ and $K_1 = K_2 = K_3 = K$. Let

$$k_m = \frac{\pi}{L}(2m - 1) \qquad 0 \leq m \leq K + 1. \tag{F.22}$$

Define axisymmetric regions $D(m_\perp, m_\parallel)$ in spectral space by $k_{m_\perp} \leq k_\perp < k_{m_\perp + 1}$ and $k_{m_\parallel} \leq k_\parallel < k_{m_\parallel + 1}$, where $0 \leq m_\perp \leq K$ and $0 \leq m_\parallel \leq K$. Let

$$\hat{A}^{m_\perp m_\parallel} = \frac{L^3}{8\pi^3 N_{m_\perp m_\parallel}} \sum_{\mathbf{k}\in D(m_\perp, m_\parallel)} \left|a(\mathbf{k})\right|^2 , \tag{F.23}$$

where the sum is over discrete wave vectors inside $D(m_\perp, m_\parallel)$ and $N_{m_\perp m_\parallel}$ is the number of such wave vectors (which is non-zero given the construction of $D(m_\perp, m_\parallel)$). Taking the average of (F.23) and using (F.20),

$$\overline{\hat{A}^{m_\perp m_\parallel}} = \frac{1}{N_{m_\perp m_\parallel}} \sum_{\mathbf{k}\in D(m_\perp, m_\parallel)} \hat{A}_{00}(\mathbf{k}) . \tag{F.24}$$

If the region is small in the $k_\perp$-$k_\parallel$ plane (large enough L), variations of $\hat{A}_{00}(\mathbf{k})$ across $D(m_\perp, m_\parallel)$ can be neglected and the right-hand side of (F.24) approximates $\hat{A}_{00}(\mathbf{k})$ throughout the given region. The problem is that, as noted earlier, it is not feasible to determine the ensemble average using DNS. Thus, we take the right-hand side of (F.23) as representative of $\hat{A}_{00}(k_\perp, k_\parallel)$ at

$$k_\perp = \frac{2\pi}{L} m_\perp \qquad 0 \le m_\perp \le K , \tag{F.25}$$

$$k_\parallel = \frac{2\pi}{L} m_\parallel \qquad 0 \le m_\parallel \le K . \tag{F.26}$$

One can also regard $\hat{A}_{00}(\mathbf{k})$ as a function of $k = |\mathbf{k}|$ and $\theta_\mathbf{k}$, the angle between the vectors $\mathbf{k}$ and $\mathbf{e}$. For given $\theta_\mathbf{k}$, k runs over the values

$$k = \frac{2\pi}{L} m \qquad 0 \le m \le K . \tag{F.27}$$

The associated value of $\hat{A}_{00}$ is taken from the nearest point in the $k_\perp$-$k_\parallel$ plane at which it has been calculated as described above.

The spherically-averaged energy spectrum of the NP component can be obtained as follows. Let $S(m)$ be the region $k_m \le k < k_{m+1}$ in spectral space, where $0 \le m \le K$. The quantity

$$\hat{E}_m = \frac{L}{4\pi} \sum_{\mathbf{k}\in S(m)} \left|a(\mathbf{k})\right|^2 \tag{F.28}$$

is taken as representative of the scaled spectrum, $\hat{E}_{NP}(k) = E_{NP}(k)/\varepsilon_{NP}^2$, at k given by (F.27).

Finally, the total (unaveraged) scaled NP energy is

$$\frac{1}{2}\sum_{\mathbf{k}}\left|a\left(\mathbf{k}\right)\right|^2 , \tag{F.29}$$

where the sum is over all wavenumbers.

The numerical parameters used to obtain the results described in section 4.1 were $N_i = 512$, $K_i = 255$, $L_i = 60$, $\Delta = 5\times10^{-3}$ and $d_0 = 1.5\times10^{-7}$, where the hyperviscous damping factor is $\hat{D}_{00} = d_0 k^6$.

Appendix G: Numerical scheme for (3.13)

(3.13) yields

$$\begin{aligned}&\frac{\partial \hat{A}_{ss}\left(\mathbf{k}\right)}{\partial T} + 2\hat{D}\left(\mathbf{k}\right)\hat{A}_{ss}\left(\mathbf{k}\right) = \\ &2\pi\sum_{s_p,s_q=\pm1}\int \hat{A}_{s_p s_p}\left(\mathbf{p}\right)\delta\left(F_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)\right) \qquad , \\ &\left(\lambda_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)\hat{A}_{s_q s_q}\left(-\mathbf{k}-\mathbf{p}\right) + \Re\left(\zeta_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)\right)\hat{A}_{ss}\left(\mathbf{k}\right)\right)d^3\mathbf{p}\end{aligned} \tag{G.1}$$

where we have used the Dirac function $\delta\left(F_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)\right)$ to turn the surface integral into a volume integral.

Here we assume axisymmetry of $\hat{A}_{ss}\left(\mathbf{k}\right)$ and, before discussing numerical methods, (G.1) is recast in a form which is better suited for numerical purposes. From a numerical point of view, a problem with (G.1) is that the coefficients $\lambda_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)$ and $\zeta_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)$ are mildly singular at both $\mathbf{p}=0$ and $\mathbf{p}=-\mathbf{k}$, points which may lie on or near the surface $F_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)=0$, a surface which is significant given the Dirac function in (G.1). By mildly singular we mean that the coefficients have different limiting values depending on the direction from which the singular point is approached. The following analysis is intended to remove the point $\mathbf{p}=-\mathbf{k}$ from consideration, thus reducing from two to one the number of singular points.

First consider

$$\sum_{s_p,s_q=\pm1}\int \Re\left\{\zeta_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)\right\}\hat{A}_{s_p s_p}\left(\mathbf{p}\right)\delta\left(F_{ss_p s_q}\left(\mathbf{k},\mathbf{p}\right)\right)d^3\mathbf{p}, \tag{G.2}$$

which forms one of the components of (G.1). $\mathbf{p}$-space is divided into two parts: $|\mathbf{p}|<|\mathbf{p}+\mathbf{k}|$ and $|\mathbf{p}|>|\mathbf{p}+\mathbf{k}|$, which are separated by the planar boundary $|\mathbf{p}|=|\mathbf{p}+\mathbf{k}|$. The integral in (G.2) is the sum of contributions from these two parts of $\mathbf{p}$-space, i.e.

$$\begin{aligned}&\sum_{s_p,s_q=\pm1}\int \Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}=\\&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}|}\Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}+\\&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|>|\mathbf{k}+\mathbf{p}|}\Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}\end{aligned}. \qquad \text{(G.3)}$$

Changing the integration variable to $\mathbf{q}=-\mathbf{k}-\mathbf{p}$ and permuting the summation indices according to $s_p\leftrightarrow s_q$,

$$\begin{aligned}&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|>|\mathbf{k}+\mathbf{p}|}\Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}=\\&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{q}|<|\mathbf{k}+\mathbf{q}|}\Re\left\{\zeta_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{q})\right\}\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{q})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{q})\right)d^3\mathbf{q}\end{aligned}, \qquad \text{(G.4)}$$

where we have used $F_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{q})=F_{ss_ps_q}(\mathbf{k},\mathbf{q})$. Renaming the integration variable on the right-hand side of (G.4) and using the result in (G.3),

$$\begin{aligned}&\sum_{s_p,s_q=\pm1}\int \Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}=\\&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}|}\Big[\Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})+\\&\Re\left\{\zeta_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{p})\right\}\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\Big]\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}\end{aligned}. \qquad \text{(G.5)}$$

The quantity

$$\sum_{s_p,s_q=\pm1}\int \lambda_{ss_ps_q}(\mathbf{k},\mathbf{p})\hat{A}_{s_ps_p}(\mathbf{p})\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p} \qquad \text{(G.6)}$$

also appears in (G.1) and can be split into contributions from $|\mathbf{p}|<|\mathbf{p}+\mathbf{k}|$ and $|\mathbf{p}|>|\mathbf{p}+\mathbf{k}|$ in the manner of (G.3). Changing the integration variable to $\mathbf{q}=-\mathbf{k}-\mathbf{p}$ and permuting the summation indices $s_p\leftrightarrow s_q$,

$$\begin{aligned}&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|>|\mathbf{k}+\mathbf{p}|}\lambda_{ss_ps_q}(\mathbf{k},\mathbf{p})\hat{A}_{s_ps_p}(\mathbf{p})\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}=\\&\sum_{s_p,s_q=\pm1}\int_{|\mathbf{q}|<|\mathbf{k}+\mathbf{q}|}\lambda_{ss_ps_q}(\mathbf{k},\mathbf{q})\hat{A}_{s_ps_p}(\mathbf{q})\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{q})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{q})\right)d^3\mathbf{q}\end{aligned}, \qquad \text{(G.7)}$$

where we have used $F_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{q})=F_{ss_ps_q}(\mathbf{k},\mathbf{q})$ and $\lambda_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{q})=\lambda_{ss_ps_q}(\mathbf{k},\mathbf{q})$, which is a consequence of (3.16) and $M_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{q})=M_{ss_ps_q}(\mathbf{k},\mathbf{q})$. Renaming the integration variable on the right-hand side of (G.7),

$$\begin{aligned}&\sum_{s_p,s_q=\pm1}\int\lambda_{ss_ps_q}(\mathbf{k},\mathbf{p})\hat{A}_{s_ps_p}(\mathbf{p})\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}=\\&2\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}|}\lambda_{ss_ps_q}(\mathbf{k},\mathbf{p})\hat{A}_{s_ps_p}(\mathbf{p})\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}\end{aligned}. \quad \text{(G.8)}$$

The integrals on the right-hand sides of (G.5) and (G.8) do not contain the point $\mathbf{p}=-\mathbf{k}$, which was the motivation for the above analytical manipulations.

Using (G.5) and (G.8) in (G.1),

$$\frac{\partial\hat{A}_{ss}(\mathbf{k})}{\partial T}=K_s(\mathbf{k})\hat{A}_{ss}(\mathbf{k})+J_s(\mathbf{k}), \quad \text{(G.9)}$$

where

$$\begin{aligned}K_s(\mathbf{k})=-2\hat{D}(\mathbf{k})+2\pi\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}|}\Big[&\Re\left\{\zeta_{ss_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})+\\&\Re\left\{\zeta_{ss_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{p})\right\}\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\Big]\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}\end{aligned} \quad \text{(G.10)}$$

and

$$\begin{aligned}&J_s(\mathbf{k})=\\&4\pi\sum_{s_p,s_q=\pm1}\int_{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}|}\lambda_{ss_ps_q}(\mathbf{k},\mathbf{p})\hat{A}_{s_ps_p}(\mathbf{p})\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}\end{aligned}. \quad \text{(G.11)}$$

Some remarks concerning (G.9)-(G.11) are in order. Firstly, bearing in mind that $\hat{A}_{-s,-s}(\mathbf{k})=\hat{A}_{ss}(-\mathbf{k})$, we restrict (G.9) to $s=1$ in what follows. Secondly, because the resonant surface does not exist when $s_p=s_q=s$, the term $s_p=s_q=1$ is zero and is dropped.

Consider a Cartesian coordinate system such that $k_1\geq0$, $k_2=0$ and $\mathbf{e}=(0,0,1)$. Given axisymmetry, the integrands in (G.10) and (G.11) are symmetric under the reflection $p_2\leftrightarrow-p_2$. Thus, specializing to $s=1$, (G.10) and (G.11) give

$$\begin{aligned}K_1(\mathbf{k})=-2\hat{D}(\mathbf{k})+4\pi\sum_{s_p,s_q=\pm1}\int_{\substack{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}|\\p_2>0}}\Big[&\Re\left\{\zeta_{1s_ps_q}(\mathbf{k},\mathbf{p})\right\}\hat{A}_{s_ps_p}(\mathbf{p})+\\&\Re\left\{\zeta_{1s_qs_p}(\mathbf{k},-\mathbf{k}-\mathbf{p})\right\}\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})\Big]\delta\left(F_{1s_ps_q}(\mathbf{k},\mathbf{p})\right)d^3\mathbf{p}\end{aligned}, \quad \text{(G.12)}$$

$$J_1(\mathbf{k}) = 8\pi \sum_{s_p, s_q = \pm 1} \int_{\substack{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}| \\ p_2>0}} \lambda_{1 s_p s_q}(\mathbf{k},\mathbf{p}) \hat{A}_{s_p s_p}(\mathbf{p}) \hat{A}_{s_q s_q}(-\mathbf{k}-\mathbf{p}) \delta\left(F_{1 s_p s_q}(\mathbf{k},\mathbf{p})\right) d^3\mathbf{p}, \quad \text{(G.13)}$$

where the term $s_p = s_q = 1$ is omitted from the sums. The identities $\hat{A}_{-1,-1}(\mathbf{p}) = \hat{A}_{11}(-\mathbf{p})$ and $\hat{A}_{-1,-1}(-\mathbf{k}-\mathbf{p}) = \hat{A}_{11}(\mathbf{k}+\mathbf{p})$ provide the spectra in (G.12) and (G.13) in terms of $\hat{A}_{11}$ when $s_p = -1$ or $s_q = -1$. (G.9) with $s = 1$ and (G.12), (G.13) give the wave-turbulence equation which is treated numerically.

The integrals in (G.12) and (G.13) are evaluated numerically. This requires a numerical representation of the spectrum $\hat{A}_{11}$. Given axisymmetry, $\hat{A}_{11}(\mathbf{k})$ depends on $k = |\mathbf{k}|$ and $0 \le \theta_{\mathbf{k}} \le \pi$, the angle between $\mathbf{k}$ and the axial vector $\mathbf{e}$. k is discretized and truncated according to

$$\log k = \log k_{\min} + n_k \Delta_k, \quad 0 \le n_k \le N_k, \quad \text{(G.14)}$$

where $\Delta_k = N_k^{-1} \log(k_{\max}/k_{\min})$. Here, N_k, $k_{\min}$ and $k_{\max}$ are numerical parameters. N_k is large, while $0 < k_{\min} < k_{\max}$ define the range of k which is treated numerically. The motivation for the logarithmic distribution (G.14), compared to a linear one, is to widen the range of k for a given N_k. $\theta_{\mathbf{k}}$ is also discretized:

$$\theta_{\mathbf{k}} = \left(n_\theta + \frac{1}{2}\right)\Delta_\theta, \quad 0 \le n_\theta < 2N_\theta, \quad \text{(G.15)}$$

where $\Delta_\theta = \pi/(2N_\theta)$ and N_θ is large. (G.9) (with $s = 1$), (G.12) and (G.13) are applied for all $\mathbf{k}$ defined by (G.14), (G.15) and $k_2 = 0$. For each such $\mathbf{k}$, the integrals in (G.12) and (G.13) require numerical evaluation, which is carried out as follows.

The integrals in (G.12) and (G.13) have the form

$$\int_{\substack{|\mathbf{p}|<|\mathbf{k}+\mathbf{p}| \\ p_2>0}} f(\mathbf{k},\mathbf{p}) \delta\left(F_{s s_p s_q}(\mathbf{k},\mathbf{p})\right) d^3\mathbf{p}. \quad \text{(G.16)}$$

With the aim of evaluating such integrals, we introduce spherical coordinates, $p = |\mathbf{p}|$, $0 \le \theta \le \pi$ and $0 \le \psi \le \pi$ (because $p_2 \ge 0$ in (G.16)), in $\mathbf{p}$-space such that the Cartesian system defined above is $p_1 = p\sin\theta\cos\psi$, $p_2 = p\sin\theta\sin\psi$, $p_3 = p\cos\theta$. We next introduce $\rho = \ln(p/k)$ to replace the radial coordinate, p. Thus, (G.16) becomes

$$k^3 \int_{\substack{p<|\mathbf{k}+\mathbf{p}| \\ 0<\psi<\pi}} f(\mathbf{k},\mathbf{p}) \delta\left(F_{s s_p s_q}(\mathbf{k},\mathbf{p})\right) e^{3\rho} \sin\theta \, d\rho \, d\theta \, d\psi. \quad \text{(G.17)}$$

ρ is truncated to $-(N_k+1/2)\Delta_k \le \rho \le (N_k+1/2)\Delta_k$, a range which is divided into $2N_k+1$ intervals of length Δ_k. Likewise, $0 \le \theta \le \pi$ is divided into $2N_\theta$ intervals of length Δ_θ and $0 \le \psi \le \pi$ into a large number, N_ψ, of intervals of length $\Delta_\psi = \pi / N_\psi$. Taken together, these intervals divide (ρ,θ,ψ)-space into many small rectangular blocks. The numerical contribution of each block to the integral in (G.17) is determined separately and the sum over blocks taken to give the total value. Numerically, the value of $f(\mathbf{k},\mathbf{p})e^{3\rho}\sin\theta$ is approximated by its value at the block centre. Blocks whose centre point are such that $p < k_{\min}$, $p > k_{\max}$, $|\mathbf{k}+\mathbf{p}| < k_{\min}$ or $|\mathbf{k}+\mathbf{p}| > k_{\max}$ are ignored, because they involve values outside the numerical range of $\hat{A}_{11}$. Blocks whose centre point does not satisfy $p < |\mathbf{k}+\mathbf{p}|$ are also ignored because their centre points are excluded by the integral in (G.17). For each of the remaining blocks, the contribution to (G.17) is

$$k^3 f(\mathbf{k},\mathbf{p})e^{3\rho}\sin\theta \int \delta\left(F_{ss_ps_q}(\mathbf{k},\mathbf{p})\right) d\rho d\theta d\psi \,, \qquad \text{(G.18)}$$

where the factor multiplying the integral has the value at the block centre. Its calculation involves the determination of $\hat{A}_{s_ps_p}(\mathbf{p})$ and $\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})$ in (G.12) and (G.13). Whereas $\hat{A}_{s_ps_p}(\mathbf{p})$ follows directly from one of the values in the numerical representation of $\hat{A}_{11}$, this is not generally true of $\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})$. $\hat{A}_{s_qs_q}(-\mathbf{k}-\mathbf{p})$ is approximated using the value of $\hat{A}_{11}$ at the nearest available point in the $\log k$ - $\theta_{\mathbf{k}}$ plane.

It remains to determine the integral in (G.18). Let $X_\rho = \rho/\Delta_k$, $X_\theta = \theta/\Delta_\theta$ and $X_\psi = \psi/\Delta_\psi$, so the rectangular blocks in (ρ,θ,ψ)-space become unit cubes in the new $(X_\rho, X_\theta, X_\psi)$-space. For each cube, $F_{ss_ps_q}(\mathbf{k},\mathbf{p})$ is approximated using a Taylor's expansion about the cube centre, neglecting terms of order two and above. Thus,

$$F_{ss_ps_q}(\mathbf{k},\mathbf{p}) = F_0 + \alpha_\rho\left(X_\rho - X_\rho^0\right) + \alpha_\theta\left(X_\theta - X_\theta^0\right) + \alpha_\psi\left(X_\psi - X_\psi^0\right), \quad \text{(G.19)}$$

where X_ρ^0, X_θ^0 and X_ψ^0 denote the cube centre, F_0 is the value of $F_{ss_ps_q}(\mathbf{k},\mathbf{p})$ at the centre and

$$\alpha_\rho = \Delta_k \frac{\partial F_{ss_ps_q}}{\partial\rho}, \qquad \alpha_\theta = \Delta_\theta \frac{\partial F_{ss_ps_q}}{\partial\theta}, \qquad \alpha_\psi = \Delta_\psi \frac{\partial F_{ss_ps_q}}{\partial\psi}, \qquad \text{(G.20)}$$

the derivatives being also evaluated at the centre. The resulting approximation of the integral in (G.18) is

$$\int \delta\left(F_{ss_p s_q}(\mathbf{k},\mathbf{p})\right) d\rho d\theta d\psi = \Delta_k \Delta_\theta \Delta_\psi \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} \delta\left(F_0 + \alpha_\rho Y_\rho + \alpha_\theta Y_\theta + \alpha_\psi Y_\psi\right) dY_\rho dY_\theta dY_\psi \,. \tag{G.21}$$

The integral on the right-hand side of (G.21), henceforth denoted I, can be evaluated analytically. The results can be expressed in terms of $\alpha_{\min}$, α_{mid} and $\alpha_{\max}$, which are the values of $|\alpha_\rho|$, $|\alpha_\theta|$ and $|\alpha_\psi|$, ordered according to $\alpha_{\min} \le \alpha_{\mathrm{mid}} \le \alpha_{\max}$. There are a number of different cases:

i) $2|F_0| \ge \alpha_{\min} + \alpha_{\mathrm{mid}} + \alpha_{\max}$. In that case $I = 0$ and the given cube is ignored.

ii) $\left|2|F_0| - \alpha_{\mathrm{mid}} - \alpha_{\max}\right| < \alpha_{\min}$ gives

$$I = \frac{1}{8\alpha_{\min}\alpha_{\mathrm{mid}}\alpha_{\max}}\left(2|F_0| - \alpha_{\min} - \alpha_{\mathrm{mid}} - \alpha_{\max}\right)^2 . \tag{G.22}$$

iii) $\alpha_{\min} - \alpha_{\mathrm{mid}} < 2|F_0| - \alpha_{\max} \le \alpha_{\mathrm{mid}} - \alpha_{\min}$ yields

$$I = \frac{1}{2\alpha_{\mathrm{mid}}\alpha_{\max}}\left(\alpha_{\mathrm{mid}} + \alpha_{\max} - 2|F_0|\right). \tag{G.23}$$

iv) $\left|\alpha_{\min} + \alpha_{\mathrm{mid}} - \alpha_{\max}\right| < 2|F_0| \le \alpha_{\min} - \alpha_{\mathrm{mid}} + \alpha_{\max}$ leads to

$$I = \frac{1}{\alpha_{\max}}\left(1 - \frac{1}{8\alpha_{\min}\alpha_{\mathrm{mid}}}\left(2|F_0| + \alpha_{\min} + \alpha_{\mathrm{mid}} - \alpha_{\max}\right)^2\right). \tag{G.24}$$

v) $\alpha_{\max} \ge \alpha_{\min} + \alpha_{\mathrm{mid}}$ and $2|F_0| \le \alpha_{\max} - \alpha_{\min} - \alpha_{\mathrm{mid}}$ gives $I = 1/\alpha_{\max}$.

vi) $\alpha_{\max} < \alpha_{\min} + \alpha_{\mathrm{mid}}$ and $2|F_0| \le \alpha_{\min} + \alpha_{\mathrm{mid}} - \alpha_{\max}$ yields

$$I = \frac{1}{\alpha_{\max}}\left(1 - \frac{1}{4\alpha_{\min}\alpha_{\mathrm{mid}}}\left(4|F_0|^2 + \left(\alpha_{\min} + \alpha_{\mathrm{mid}} - \alpha_{\max}\right)^2\right)\right). \tag{G.25}$$

Having described the numerical methods employed to determine the integrals in (G.12) and (G.13), we turn attention to the scheme used to integrate (G.9) with respect to time. As usual, it involves stepping forwards in time, the time step being denoted by Δ. Each step involves two stages. In the first stage, K_1 and J_1 are approximated as time-independent and equal to their values at the beginning of the step, values which are determined as described above. Given time-independence of K_1 and J_1, (G.9) gives

$$\hat{A}_{11}^{\dagger}(\mathbf{k}) = \hat{A}_{11}(\mathbf{k},T)\exp\left(\frac{1}{2}K_1(\mathbf{k},T)\Delta\right) + J_1(\mathbf{k},T)\frac{\exp\left(K_1(\mathbf{k},T)\Delta/2\right)-1}{K_1(\mathbf{k},T)} \quad \text{(G.26)}$$

as an approximation of $\hat{A}_{11}(\mathbf{k},T+\Delta/2)$. The second stage determines K_1 and J_1 using $\hat{A}_{11}^{\dagger}$ and the resulting time-independent approximations, $K_1 = K_1^{\dagger}$, $J_1 = J_1^{\dagger}$, are employed in (G.9), yielding

$$\hat{A}_{11}(\mathbf{k},T+\Delta) = \hat{A}_{11}(\mathbf{k},T)\exp\left(K_1^{\dagger}(\mathbf{k})\Delta\right) + J_1^{\dagger}(\mathbf{k})\frac{\exp\left(K_1^{\dagger}(\mathbf{k})\Delta\right)-1}{K_1^{\dagger}(\mathbf{k})}, \quad \text{(G.27)}$$

which is the final result of the time step. An advantage of this scheme, compared to more conventional ones, is that, since $\left(\exp\left(K_1\Delta/2\right)-1\right)/K_1 > 0$ and $\left(\exp\left(K_1\Delta\right)-1\right)/K_1 > 0$, no matter what the sign of K_1, and $J_1 \geq 0$ according to (G.13), it guarantees that non-negative A_{11} is maintained.

The numerical parameters used to obtain the results of section 4.2 were $k_{\min} = 10^{-1}$, $k_{\max} = 10^{3}$, $N_k = 400$, $N_\theta = 200$, $N_\psi = 100$, $\Delta = 5\times10^{-3}$ and $d = 10^{-5}$, where the hyperviscous damping factor is $\hat{D}(\mathbf{k}) = dk^4$.