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Multi-scale invariant solutions in plane Couette flow: a reduced-order model approach (Supplemental Material)

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This document contains supplemental information to the main paper. In §1, we discuss the spatially averaged temporal dynamics of the model, comparing the results to the DNS results of Doohan *et al.* (2021). This helps affirm that the model captures physically meaningful dynamics.

1. Temporal Dynamics

Several important multi-scale processes have been numerically identified in the minimal multi-scale flow unit by (Doohan *et al.* 2021): 1) large and small scale self-sustaining processes; 2) energy cascade via the streak instability and breakdown; 3) energy transport from the large to small scales which drives small-scale turbulent production; 4) the feeding of energy from small to large scales. In a similar manner, we have identified the same processes in our reduced order model.

1.1. *Self-Sustaining Process*

In the main text, we show the temporal cross-correlation between the energy observables of the different structural components of the large- and small-scale self-sustaining processes (SSP), verifying their existence in the model. Further, we now examine the relationship between the large- and small-scale SSPs and the terms of the energy budget has also been examined. As expected, straight and wavy streaks are seen to be correlated to the production terms (figure 1), showing that the SSPs are being driven by the mean shear which transfers energy into the streamwise energy components in the form of high-speed streaks, in line with the lift-up effect (Ellingsen & Palm 1975; Brandt 2014). As the SSPs advance further in their cycle, a correlation between the structural components of the SSPs and the dissipation is observed (figure 1). In the streamwise direction, dissipation is seen to occur in response to the wavy streaks, whereas the spanwise and wall-normal dissipation is related to the wavy rolls. Wavy streaks at both scales are correlated with the streamwise pressure-strain terms

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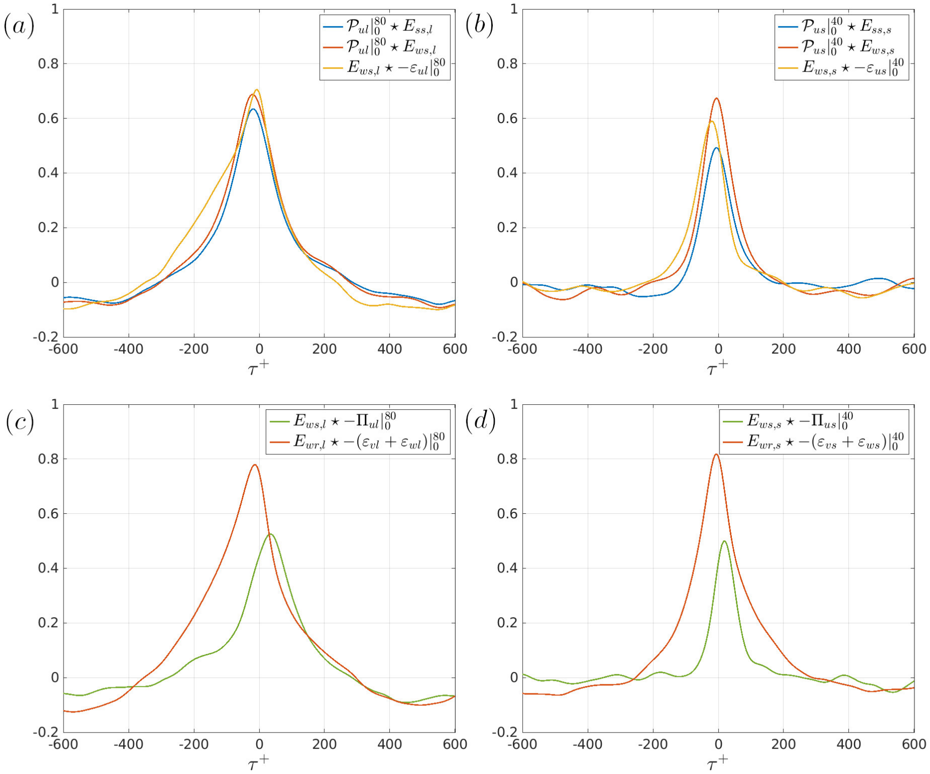

Figure 1: Temporal cross-correlation of the energy observables associated with the structural components of the self-sustaining process (SSP) with various terms of the energy budget.

which are responsible for the redistribution of turbulent kinetic energy from the streamwise to the spanwise and wall-normal directions. These results are similar to those of Doohan *et al.* (2021) and we thus conclude that the essential mechanisms of the self-sustaining process at both scales are captured by the reduced order model.

1.2. *Energy Cascade*

As noted in §3 of the main text, energy is seen to be directly transferred from large to small scales in the log layer in the spanwise and wall-normal directions, and thus, we now examine the relationship between the energy budget and self-sustaining process terms with respect to the energy cascade.

We observe a correlation between the structural components of the large-scale SSP and the spanwise and wall-normal inter-scale turbulent transport (figure 2), very similar to that found by Doohan *et al.* (2021). Thus, energy is seen to be transferred to the small scales as a result of the large-scale SSP, particularly in response to the breakdown of large-scale wavy streaks and the formation of wavy rolls. Also observed, is a correlation between the large-scale production and the wall-normal inter-scale transport (figure 2), in particular, one of the terms of the inter-scale turbulent transport appears to be responsible for this cascade, namely,

$$T_{v,\updownarrow c} = \langle v_l(\boldsymbol{u}_l \cdot \nabla v_s)\rangle_{x,z}. \tag{1.1}$$

This correlation is left-shifted indicating that this turbulent transport increases in delayed

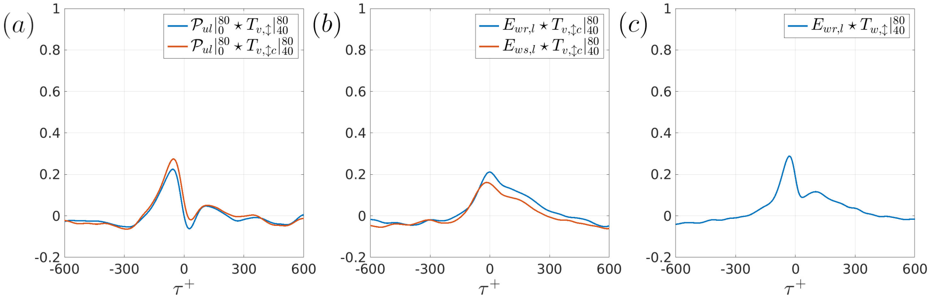

Figure 2: Cross-correlations relating the large-scale production and large-scale SSP to the (a, b) wall-normal and (c) spanwise inter-scale turbulent transport.

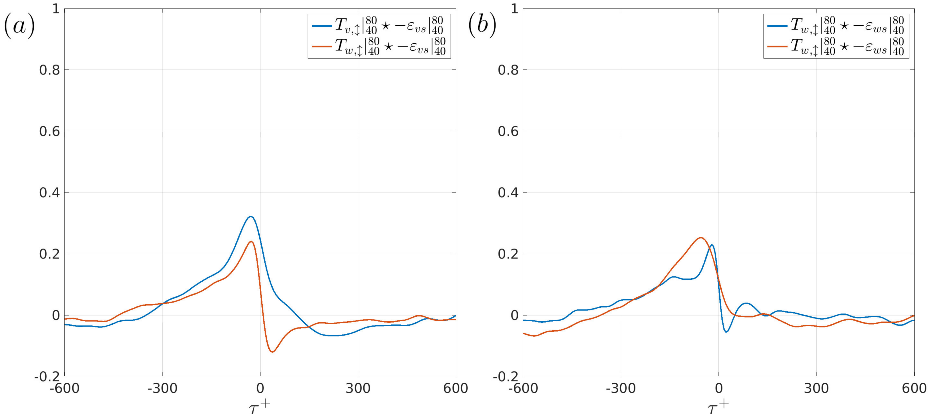

Figure 3: Cross-correlations relating the wall-normal and spanwise inter-scale turbulent transport to the (a, b) wall-normal and (c) spanwise small-scale dissipation terms.

response to the large-scale production. Since the large-scale production is seen to be correlated with the large-scale SSP, this is a further indication that this is the mechanism for which energy is transported to the small scales.

This increase in intra-scale transport is seen to also correlate with increases in small-scale wall-normal and spanwise energy dissipation in the log layer i.e. detached-eddy dissipation (figure 3). Overall, it appears that the energy cascade does exist with correlations existing in agreement with Doohan *et al.* (2021), although the overall magnitude of the small-scale dissipation in this region is smaller than expected, as shown in section 3 of the main text. This seems appropriate given the truncated nature of the model since additional small-scale structures are neglected.

1.3. *Driving Small Scale Production*

Large amounts of turbulent kinetic energy is seen to be produced in the near wall region at the small scales (figure 2 of main text). Considering the inter-scale turbulent transport terms in figure 3 of the main text, it is observed that energy is transferred from the large to the small scales in the wall-normal direction across the entirety of the wall-normal domain. This is in contrast to the streamwise and spanwise directions which both feature energy transport from the small to large scales in the near wall region. Inter-scale turbulent transport in the

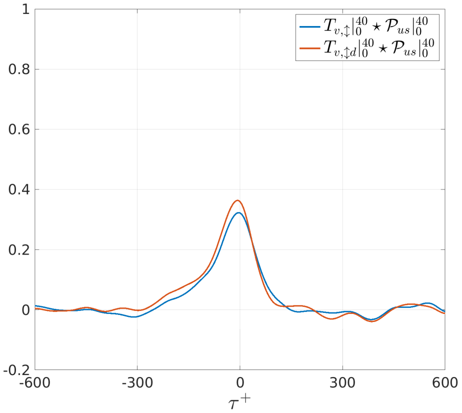

Figure 4: Cross-correlation between the wall-normal inter-scale turbulent transport and the small-scale production.

spanwise near-wall region was seen to drive the small-scale production term by Doohan *et al.* (2021), and thus we wish to confirm if this is also true of our reduced order model.

Following the method of Doohan *et al.* (2021), we split the inter-scale turbulent transport term by term to define,

$$T_{v,\updownarrow d} = -\langle v_s(\boldsymbol{u}_s \cdot \nabla v_l)\rangle_{x,z}. \tag{1.2}$$

This part of the inter-scale turbulent transport is seen to contribute to nearly all of the energy transport to the small-scale production term (figure 4), expected since this part of the inter-scale turbulent transport is directly proportional to the $-u_s v_s$ Reynolds stress.

Importantly, considering the previous definition of equation (1.1), the two terms of wall-normal inter-scale turbulent transport appear to have two independently acting terms i.e. $T_{v,\updownarrow} = T_{v,\updownarrow c} + T_{v,\updownarrow d}$, one that contributes to the energy cascade $T_{v,\updownarrow c}$, particularly active in the log layer, and the other which results in the driving of small-scale production $T_{v,\updownarrow d}$, particularly active in the near-wall region.

An examination of maximal small-scale production events reveals that a mechanism similar to the Orr mechanism (Orr 1907; Jiao *et al.* 2021) is responsible for the energy amplification seen in the small-scale production. However, the Orr mechanism is purely associated to the linearised Navier-Stokes operator leading only to transient growth, and thus we examine the correlation between the driving of small scale production and the small-scale self-sustaining process. Similar to Doohan *et al.* (2021), we see a slightly left-shifted correlation between the inter-scale energy transport and wavy/straight rolls (figure 5). This suggests that the wall-normal inter-scale energy transport is responsible for driving the small-scale self-sustaining process by energy injection to the v dominant structures of the SSP i.e. straight and wavy rolls, as well as directly through the small-scale production term.

A correlation between the wall-normal inter-scale energy transport and the small-scale spanwise and wall-normal dissipation terms is also observed, with approximately the same left shift that was observed with the small-scale wavy/straight rolls. Again, this suggests that energy is transferred to straight and wavy rolls by the inter-scale transport, which subsequently results in small-scale dissipation in the near-wall region.

Finally, we search for the mechanism in the large-scale SSP which drives the wall-normal inter-scale turbulent transport. Although weaker than Doohan *et al.* (2021), a correlation between large-scale wavy streaks and the small scale turbulent production is observed, suggesting that inter-scale turbulent transport is connected to the breakdown of large-scale streaks.

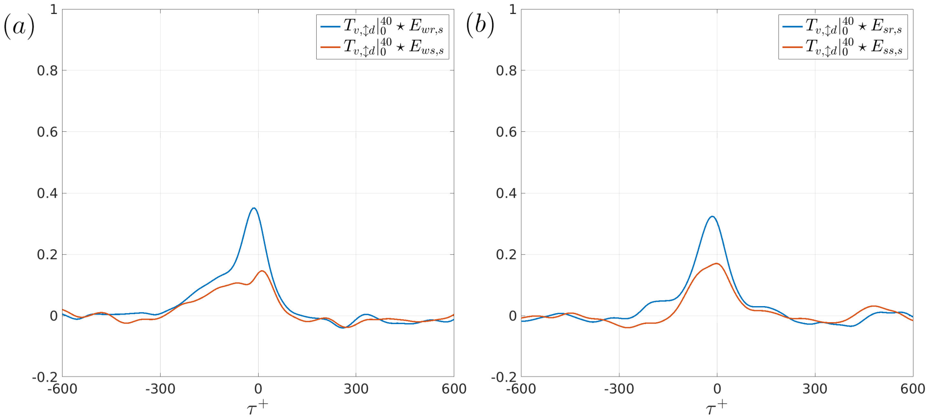

Figure 5: Cross-correlation between the wall-normal inter-scale turbulent transport and the energy observables associated with the structural components of the small-scale self-sustaining process.

1.4. *Feeding from Small to Large Scales*

Again, referring to figure 3 of the main text, it is noted that large amounts of energy are transported from the small to the large scales in the near wall region through the streamwise and spanwise inter-scale turbulent transport.

A similar correlation to that of Doohan *et al.* (2021) is observed between the inter-scale turbulent transport from small to large scales and the small-scale production, pressure strains and wavy streaks/rolls (figure 6). In the streamwise direction, inter-scale transport appears to be a consequence of both small-scale production and the breakdown of wavy streaks, with the latter seen through the slightly left-shifted peak of the correlation between the small-scale wavy streaks and the inter-scale transport. The influence of the small-scale production on the inter-scale transport appears to be slightly more complex, with the slightly longer timescale but slightly more rightward peak of this correlation suggesting that the production influences both the small-scale wavy streaks through the usual SSP, and the inter-scale transport directly. Spanwise inter-scale transport appears to be a consequence of wavy rolls, with energy being transferred through the spanwise pressure strain term.

As a consequence of this feeding process, a direct transfer of energy to the large-scale dissipation terms is observed, not occurring through the large-scale SSP. This is apparent by the observation that these dissipation events occur at a much shorter timescale than those of the large-scale SSP, as well as a direct lack of correlation between the inter-scale transport terms and the structural components of the large-scale SSP.

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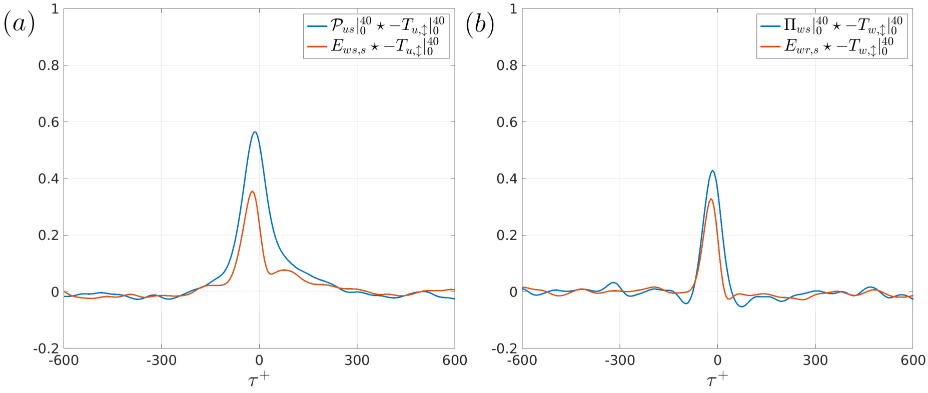

Figure 6: Cross-correlation of various small-scale energy budget and SSP terms with the (*a*) streamwise and (*b*) spanwise inter-scale turbulent transport. The negative sign of the inter-scale turbulent transport represents energy transport from small to large scales.

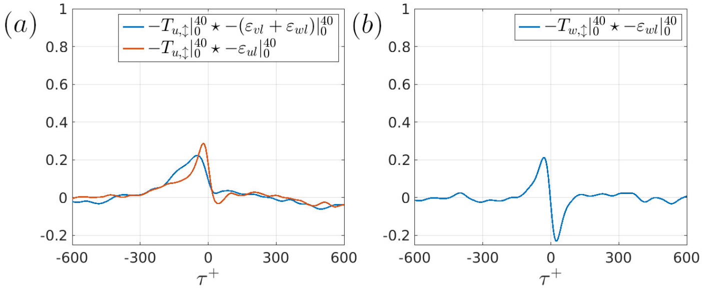

Figure 7: Cross-correlation between the (*a*) streamwise and (*b*) spanwise inter-scale turbulent transport with the large-scale dissipation terms.