Supplementary material to "On the theory of body motion in confined Stokesian fluids"

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Abstract

This supplementary material reports the Faxén operators of a sphere up to the second-order with Navier-slip boundary conditions and the regular part of the Stokesian multipoles in the semispace domain, useful for constructing the $[N]$-matrix for a sphere near a plane wall.

I. FAXÉN OPERATORS FOR A SPHERE WITH NAVIER SLIP BOUNDARY CONDITIONS

In the following, Faxén operators for a sphere with Navier-slip boundary conditions, evaluated in [2], are reported.
The zeroth-order Faxén operator is

$$\mathcal{F}_{\beta\alpha} = -\left(\frac{1+2\hat{\lambda}}{1+3\hat{\lambda}}\right)\left(\frac{3}{4}R_p + \frac{1}{8}\frac{R_p^3}{(1+2\hat{\lambda})}\Delta_\xi\right)\delta_{\alpha\beta} \tag{1}$$

where Δ_ξ is the Laplacian operator acting on the coordinate of the center of the sphere. The first-order Faxén operator reads

$$\mathcal{F}_{\alpha\beta\beta_1} = -\frac{R_p^3}{6(1+5\hat{\lambda})(1+3\hat{\lambda})}\Bigg\{\left[(4+20\hat{\lambda}+15\hat{\lambda}^2)\delta_{\alpha\beta}\nabla_{\beta_1} + (1+5\hat{\lambda}+15\hat{\lambda}^2)\delta_{\alpha\beta_1}\nabla_\beta\right]$$
$$+\frac{R_p^2}{10}\left[(4+12\hat{\lambda}-15\hat{\lambda}^2)\Delta_\xi\nabla_{\beta_1}\delta_{\alpha\beta} + (1+3\hat{\lambda}+15\hat{\lambda}^2)\Delta_\xi\nabla_\beta\delta_{\alpha\beta_1}\right]\Bigg\} \tag{2}$$

by which it is possible to obtain the Faxén operator for the torque on the sphere

$$\mathcal{T}_{\gamma\alpha} = \frac{\varepsilon_{\alpha\gamma_1\gamma}R_p^3\nabla_{\gamma_1}}{2(1+3\hat{\lambda})} \tag{3}$$

and for the stresses

$$\mathcal{E}_{\alpha\beta\beta_1} = -\frac{\mathcal{F}_{\alpha\beta\beta_1}+\mathcal{F}_{\alpha\beta_1\beta}}{2} = \left(\frac{5+10\hat{\lambda}}{6+30\hat{\lambda}} + \frac{\Delta_\xi}{12(1+5\hat{\lambda})}\right)\left(\frac{\nabla_\beta\delta_{\alpha\beta_1}+\nabla_{\beta_1}\delta_{\alpha\beta}}{2}\right) \tag{4}$$

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Finally, the second-order Faxén operator is

$$\mathcal{F}_{\alpha\beta\beta_1\beta_2} = -\frac{R_p^3}{4(1+4\hat{\lambda})(1+7\hat{\lambda})}\left\{\frac{(1+4\hat{\lambda})(1+7\hat{\lambda})}{1+3\hat{\lambda}}\left[\delta_{\alpha\beta}\delta_{\beta_1\beta_2}+\hat{\lambda}(\delta_{\beta\beta_1}\delta_{\alpha\beta_2}+\delta_{\beta\beta_2}\delta_{\alpha\beta_1})\right]\right.$$
$$+\frac{R_p^2}{6}\left[-4\hat{\lambda}^2\left(\frac{4+21\hat{\lambda}}{1+3\hat{\lambda}}\right)\Delta_\xi\delta_{\beta_1\beta_2}\delta_{\alpha\beta}+5(1+6\hat{\lambda})\nabla_{\beta_1\beta_2}\delta_{\alpha\beta}+(1+6\hat{\lambda}+28\hat{\lambda}^2)(\nabla_{\beta\beta_1}\delta_{\alpha\beta_2}+\nabla_{\beta\beta_2}\delta_{\alpha\beta_1})\right.$$
$$\left.(1+12\hat{\lambda}+56\hat{\lambda}^2)(\delta_{\alpha\beta_1}\delta_{\beta\beta_2}+\delta_{\alpha\beta_2}\delta_{\beta\beta_1})\Delta_\xi\right]$$
$$\left.+\frac{R_p^4}{84}\left[(5+20\hat{\lambda}-56\hat{\lambda}^2)\nabla_{\beta_1\beta_2}\delta_{\alpha\beta}+(1+4\hat{\lambda}+28\hat{\lambda}^2)(\nabla_{\beta\beta_1}\delta_{\alpha\beta_2}+\nabla_{\beta\beta_2}\delta_{\alpha\beta_1})\right]\Delta_\xi\right\} \quad (5)$$

II. REGULAR PART OF THE MULTIPOLES OF THE GREEN FUNCTION FOR A FLUID BOUNDED BY A PLANE WALL

In this Section, the regular part of multipoles of the Stokes flow, defined in the semispace with no-slip boundary conditions and evaluated at the center of a sphere with radius R_p, at distance h from the plane boundary, are reported.

The single-pole is obtained by evaluating the regular part of the Green function with both field $\boldsymbol{x}$ and pole point $\boldsymbol{\xi}$ at the position of the center of the sphere $(0,0,h)$. The regular part of the Green function bounded by a plane wall, expressed in the invariant form [1], reads

$$W_{a\alpha}(\boldsymbol{x},\boldsymbol{\xi}) = S_{a\alpha}(\boldsymbol{x}-\boldsymbol{\xi}_r)-(\boldsymbol{\xi}_r-\boldsymbol{\xi})\cdot\boldsymbol{n}\,J_{\alpha\beta'}[\nabla_{\beta'}S_{a3}(\boldsymbol{x}-\boldsymbol{\xi}_r)-\frac{(\boldsymbol{\xi}_r-\boldsymbol{\xi})\cdot\mathbf{n}}{2}\Delta_{\xi_r}S_{a\beta'}(\mathbf{x}-\boldsymbol{\xi}_r)] \quad (6)$$

where $\boldsymbol{J} = I - 2\boldsymbol{n}\otimes\boldsymbol{n}$ is the mirror operator and $\boldsymbol{n}$ is the unit vector normal to the plane wall inward into the fluid, and $\boldsymbol{\xi}_r = \boldsymbol{\xi} - 2h\boldsymbol{n}$ is the reflection point by the plane of the pole $\boldsymbol{\xi}$. Therefore, the non-vanishing entries are

$$W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{4h}$$
$$W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{2h} \quad (7)$$

Non-vanishing entries of the gradient of eq. (6) are

$$
\begin{aligned}
\partial_{\xi'_1} W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2} W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{8h^2} \\
\partial_{\xi'_1} W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1} W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2} W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2} W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_3} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1} W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_3} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2} W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{8h^2} \\
\partial_{\xi'_3} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{4h^2}
\end{aligned}
$$

Non-vanishing entries of the Laplacian of eq. (6) are

$$
\begin{aligned}
\Delta_{\xi'} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \Delta_{\xi} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\Delta_{\xi'} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \Delta_{\xi} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{1}{2h^3} \\
\Delta_{\xi'} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \Delta_{\xi} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{2}{h^3}
\end{aligned}
\tag{8}
$$

Non-vanishing entries of the second derivatives of eq. (6) performed at the field or at the pole point are

$$
\begin{aligned}
\partial_{\xi'_1}\partial_{\xi'_1} W_{11}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_1} W_{11}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2}\partial_{\xi'_2} W_{22}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_2} W_{22}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{7}{16h^3} \\
\partial_{\xi'_1}\partial_{\xi'_1} W_{22}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_1} W_{22}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2}\partial_{\xi'_2} W_{11}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_2} W_{11}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{5}{16h^3} \\
\partial_{\xi'_1}\partial_{\xi'_1} W_{33}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_1} W_{33}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2}\partial_{\xi'_2} W_{33}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_2} W_{33}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{4h^3} \\
\partial_{\xi'_3}\partial_{\xi'_3} W_{11}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_3} W_{11}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_3}\partial_{\xi'_3} W_{22}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_3} W_{22}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{5}{4h^3} \\
\partial_{\xi'_3}\partial_{\xi'_3} W_{33}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_3} W_{33}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{2h^3} \\
\partial_{\xi'_1}\partial_{\xi'_2} W_{12}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_2} W_{12}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_1}\partial_{\xi'_2} W_{21}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_2} W_{21}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2}\partial_{\xi'_1} W_{12}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_1} W_{12}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_2}\partial_{\xi'_1} W_{21}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_1} W_{21}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{1}{16h^3} \\
\partial_{\xi'_1}\partial_{\xi'_3} W_{13}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_1} W_{31}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi'_1}\partial_{\xi'_3} W_{31}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_1} W_{13}(\boldsymbol{\xi}', \boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =
\end{aligned}
$$

$$
\begin{aligned}
\partial_{\xi_2'}\partial_{\xi_3'}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_2}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi_2'}\partial_{\xi_3'}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_3}\partial_{\xi_2}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{2h^3} \\
\partial_{\xi_3'}\partial_{\xi_1'}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_3}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi_3'}\partial_{\xi_1'}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_1}\partial_{\xi_3}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi_3'}\partial_{\xi_2'}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_3}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\partial_{\xi_3'}\partial_{\xi_2'}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \partial_{\xi_2}\partial_{\xi_3}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{1}{4h^3} \qquad (9)
\end{aligned}
$$

Non-vanishing entries of the second derivatives of eq. (6) performed at the field and pole points are

$$\partial_{\xi_1'}\partial_{\xi_1} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_2} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{7}{16h^3}$$

$$\partial_{\xi_1'}\partial_{\xi_1} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_2} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{5}{16h^3}$$

$$\partial_{\xi_1'}\partial_{\xi_1} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_2} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{2h^3}$$

$$\partial_{\xi_3'}\partial_{\xi_3} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\partial_{\xi_3} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{5}{4h^3}$$

$$\partial_{\xi_3'}\partial_{\xi_3} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{h^3}$$

$$\partial_{\xi_1'}\partial_{\xi_2} W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_1'}\partial_{\xi_2} W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_1} W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_1} W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{16h^3}$$

$$\partial_{\xi_1'}\partial_{\xi_3} W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\partial_{\xi_1} W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_3} W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\partial_{\xi_2} W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{1}{4h^3}$$

$$\partial_{\xi_1'}\partial_{\xi_3} W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\partial_{\xi_1} W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\partial_{\xi_3} W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\partial_{\xi_2} W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{1}{2h^3}$$

Non-vanishing entries of the gradient at the field point of the Laplacian of eq. (6) are

$$\partial_{\xi_1'}\Delta_{\xi'} W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\Delta_{\xi} W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\Delta_{\xi'} W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\Delta_{\xi} W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{15}{8h^4}$$

$$\partial_{\xi_1'}\Delta_{\xi'} W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\Delta_{\xi} W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2'}\Delta_{\xi'} W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\Delta_{\xi} W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\Delta_{\xi'} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\Delta_{\xi} W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3'}\Delta_{\xi'} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\Delta_{\xi} W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{9}{8h^4}$$

$$\partial_{\xi_3'}\Delta_{\xi'} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\Delta_{\xi} W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{15}{4h^4}$$

Non-vanishing entries of the gradient at the pole point of the Laplacian of eq. (6) are

$$\partial_{\xi_1}\Delta_{\xi'}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_1}\Delta_{\xi}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2}\Delta_{\xi'}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_2}\Delta_{\xi}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{15}{8h^4}$$

$$\partial_{\xi_1}\Delta_{\xi'}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_1}\Delta_{\xi}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2}\Delta_{\xi'}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_2}\Delta_{\xi}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{8h^4}$$

$$\partial_{\xi_3}\Delta_{\xi'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\Delta_{\xi}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\Delta_{\xi'}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\Delta_{\xi}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{9}{8h^4}$$

$$\partial_{\xi_3}\Delta_{\xi'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\Delta_{\xi}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{9}{4h^4}$$

Non-vanishing entries of the second derivatives of the Laplacian of eq. (6) are

$$\partial_{\xi_1}\partial_{\xi_1'}\Delta_{\xi'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_1'}\Delta_{\xi}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_2}\partial_{\xi_2'}\Delta_{\xi'}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\partial_{\xi_2'}\Delta_{\xi}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{9}{4h^5}$$

$$\partial_{\xi_1}\partial_{\xi_1'}\Delta_{\xi'}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_1'}\Delta_{\xi}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_2}\partial_{\xi_2'}\Delta_{\xi'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\partial_{\xi_2'}\Delta_{\xi}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_1}\partial_{\xi_2'}\Delta_{\xi'}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_2'}\Delta_{\xi}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_1}\partial_{\xi_2'}\Delta_{\xi'}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_2'}\Delta_{\xi}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{4h^5}$$

$$\partial_{\xi_1}\partial_{\xi_1'}\Delta_{\xi'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_1'}\Delta_{\xi}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_2}\partial_{\xi_2'}\Delta_{\xi'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\partial_{\xi_2'}\Delta_{\xi}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_1}\partial_{\xi_3'}\Delta_{\xi'}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_3'}\Delta_{\xi}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_2}\partial_{\xi_3'}\Delta_{\xi'}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\partial_{\xi_3'}\Delta_{\xi}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{9}{2h^5}$$

$$\partial_{\xi_1}\partial_{\xi_3'}\Delta_{\xi'}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1}\partial_{\xi_3'}\Delta_{\xi}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_2}\partial_{\xi_3'}\Delta_{\xi'}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2}\partial_{\xi_3'}\Delta_{\xi}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_3}\partial_{\xi_1'}\Delta_{\xi'}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\partial_{\xi_1'}\Delta_{\xi}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$
$$\partial_{\xi_3}\partial_{\xi_2'}\Delta_{\xi'}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\partial_{\xi_2'}\Delta_{\xi}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{h^5}$$
$$\partial_{\xi_3}\partial_{\xi_1'}\Delta_{\xi'}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\partial_{\xi_1'}\Delta_{\xi}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_2'}\Delta_{\xi'}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\partial_{\xi_2'}\Delta_{\xi}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_3'}\Delta_{\xi'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\partial_{\xi_3'}\Delta_{\xi}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_3'}\Delta_{\xi'}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3}\partial_{\xi_3'}\Delta_{\xi}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{2h^5}$$

$$\partial_{\xi_3}\partial_{\xi_3'}\Delta_{\xi'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{6}{h^5}$$

Non-vanishing entries of the third order derivatives of eq. (6) performed at the field and pole points are

$$
\begin{aligned}
\left.\partial_{\xi_1}\partial_{\xi'_1}\partial_{\xi'_1}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_1}\partial_{\xi_1}\partial_{\xi_1}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_2}\partial_{\xi'_2}\partial_{\xi'_2}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_2}\partial_{\xi_2}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{45}{32h^4} \\
\left.\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_3}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_3}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{2h^4} \\
\left.\partial_{\xi_1}\partial_{\xi'_1}\partial_{\xi'_1}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_1}\partial_{\xi_1}\partial_{\xi_1}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_2}\partial_{\xi'_2}\partial_{\xi'_2}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_2}\partial_{\xi_2}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{45}{32h^4} \\
\left.\partial_{\xi_2}\partial_{\xi'_1}\partial_{\xi'_1}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_1}\partial_{\xi_1}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_1}\partial_{\xi'_2}\partial_{\xi'_1}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_1}\partial_{\xi_2}\partial_{\xi_1}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_2}\partial_{\xi'_2}\partial_{\xi'_1}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_2}\partial_{\xi_1}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_1}\partial_{\xi'_3}\partial_{\xi'_1}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_1}\partial_{\xi_3}\partial_{\xi_1}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_1}\partial_{\xi'_2}\partial_{\xi'_2}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_1}\partial_{\xi_2}\partial_{\xi_2}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_2}\partial_{\xi'_3}\partial_{\xi'_2}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_3}\partial_{\xi_2}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{15}{32h^4} \\
\left.\partial_{\xi_2}\partial_{\xi'_1}\partial_{\xi'_1}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_1}\partial_{\xi_1}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_3}\partial_{\xi'_1}\partial_{\xi'_1}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_3}\partial_{\xi_1}\partial_{\xi_1}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_1}\partial_{\xi'_2}\partial_{\xi'_1}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_1}\partial_{\xi_2}\partial_{\xi_1}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \\
\left.\partial_{\xi_2}\partial_{\xi'_2}\partial_{\xi'_1}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} &= \left.\partial_{\xi'_2}\partial_{\xi_2}\partial_{\xi_1}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\right|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =
\end{aligned}
$$

$$\partial_{\xi_1}\partial_{\xi_2'}\partial_{\xi_2'}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1'}\partial_{\xi_2}\partial_{\xi_2}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_2'}\partial_{\xi_2'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_2}\partial_{\xi_2}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{15}{32h^4}$$

$$\partial_{\xi_3}\partial_{\xi_1'}\partial_{\xi_1'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_1}\partial_{\xi_1}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_2'}\partial_{\xi_2'}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_2}\partial_{\xi_2}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{21}{32h^4}$$

$$\partial_{\xi_1}\partial_{\xi_3'}\partial_{\xi_1'}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1'}\partial_{\xi_3}\partial_{\xi_1}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2}\partial_{\xi_3'}\partial_{\xi_2'}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2'}\partial_{\xi_3}\partial_{\xi_2}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{21}{32h^4}$$

$$\partial_{\xi_3}\partial_{\xi_1'}\partial_{\xi_1'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_1}\partial_{\xi_1}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_2'}\partial_{\xi_2'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_2}\partial_{\xi_2}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{15}{8h^4}$$

$$\partial_{\xi_1}\partial_{\xi_3'}\partial_{\xi_1'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1'}\partial_{\xi_3}\partial_{\xi_1}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2}\partial_{\xi_3'}\partial_{\xi_2'}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2'}\partial_{\xi_3}\partial_{\xi_2}W_{33}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{15}{8h^4}$$

$$\partial_{\xi_3}\partial_{\xi_2'}\partial_{\xi_1'}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_2}\partial_{\xi_1}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi_2'}\partial_{\xi_1'}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_3'}\partial_{\xi_2}\partial_{\xi_1}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{32h^4}$$

$$\partial_{\xi_2}\partial_{\xi_3'}\partial_{\xi_1'}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2'}\partial_{\xi_3}\partial_{\xi_1}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_2}\partial_{\xi_3'}\partial_{\xi_1'}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_2'}\partial_{\xi_3}\partial_{\xi_1}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_1}\partial_{\xi_3'}\partial_{\xi_2'}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1'}\partial_{\xi_3}\partial_{\xi_2}W_{12}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_1}\partial_{\xi_3'}\partial_{\xi_2'}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi_1'}\partial_{\xi_3}\partial_{\xi_2}W_{21}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{32h^4}$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_1}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_1}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_2}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_2}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_3}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_3}W_{11}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_3}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_3}W_{22}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \frac{3}{4h^4}$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_1}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_1}W_{31}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_2}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_2}W_{32}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_1}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_1}W_{13}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} =$$

$$\partial_{\xi_3}\partial_{\xi'_3}\partial_{\xi'_2}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = \partial_{\xi'_3}\partial_{\xi_3}\partial_{\xi_2}W_{23}(\boldsymbol{\xi}',\boldsymbol{\xi})\Big|_{\boldsymbol{\xi}'=\boldsymbol{\xi}=(0,0,h)} = -\frac{3}{4h^4}$$

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