Supplementary material

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This supplementary material explains the derivation of the analytic eigenvalue bound used in the paper entitled 'A sufficient condition for inviscid shear instability: Hurdle theorem and its application to alternating jets'.

Here we derive the bound of the unstable eigenvalues c in the complex plane following the inner envelope theory by Deguchi (2021). The first step is to write $\eta = \psi/(U-c)$ and $B = Q' + U''$ in the governing equation

$$\psi'' - (k^2 + L_d^{-2})\psi + \frac{Q'}{U-c}\psi = 0\,, \qquad y \in \Omega\,, \tag{0.1}$$

used in the paper. Here $\Omega = (-L/2, L/2)$ and Dirichlet conditions are imposed at the end points. Then multiplying η^* to the resultant equation

$$(U-c)(2U'\eta' + B\eta) + (U-c)^2\{\eta'' - (k^2 + L_d^{-2})\eta\} = 0 \tag{0.2}$$

and integrating over Ω, we get

$$\int_\Omega (U-c)^2 W dy = \int_\Omega (U-c)B|\eta|^2 dy\,, \tag{0.3}$$

where $W \equiv |\eta'|^2 + (k^2 + L_d^{-2})|\eta|^2$. The real and imaginary parts of (0.3) are

$$\int_\Omega \{(U-c_r)^2 - c_i^2\} W dy = \int_\Omega (U-c_r)B|\eta|^2 dy\,, \tag{0.4}$$

$$-c_i \int_\Omega 2(U-c_r) W dy = -c_i \int_\Omega B|\eta|^2 dy\,. \tag{0.5}$$

Using the identity

$$\begin{aligned}
&\int_\Omega \{(U-r_c)^2 - (c_r - r_c)^2 - c_i^2\} W dy \\
&= \int_\Omega \{(U-c_r)^2 + 2(U-c_r)(c_r-r_c) - c_i^2\} W dy
\end{aligned} \tag{0.6}$$

that holds for any $r_c \in \mathbb{R}$ to (0.4),

$$\begin{aligned}
&\int_\Omega \{(U-r_c)^2 - (c_r - r_c)^2 - c_i^2\} W dy \\
&= \int_\Omega (U-c_r)B|\eta|^2 dy + (c_r - r_c)\int_\Omega 2(U-c_r) W dy\,.
\end{aligned} \tag{0.7}$$

The right hand side can be simplified using (0.5),

$$\int_\Omega \{(U-r_c)^2 - (c_r - r_c)^2 - c_i^2\} W dy = \int_\Omega (U-r_c)B|\eta|^2 dy\,. \tag{0.8}$$

This integral equation implies that if we can find $R(r_c) > 0$ such that

$$\int_\Omega (U-r_c)^2 W dy - \int_\Omega (U-r_c)B|\eta|^2 dy \le R^2 \int_\Omega W dy\,, \tag{0.9}$$

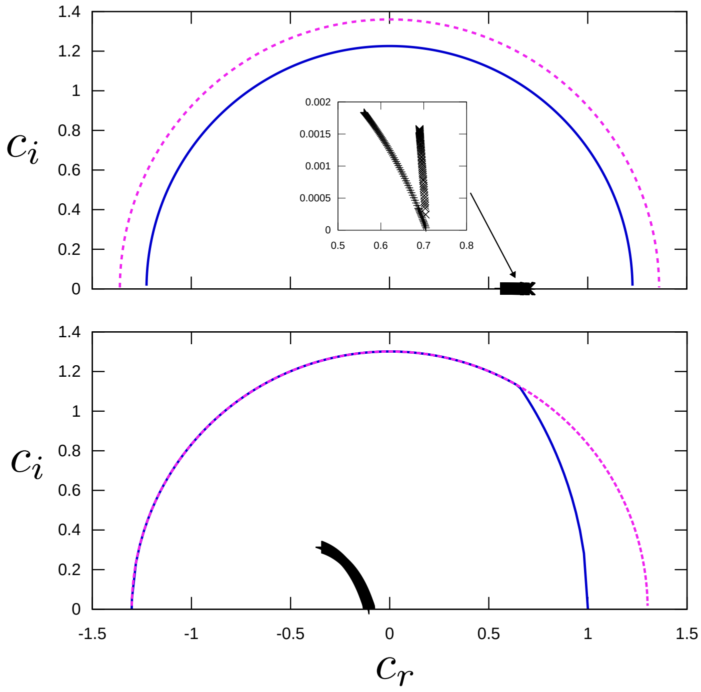

FIGURE 1. The points are eigenvalues of (0.1) in the complex plane, obtained at various k. The blue solid curve is the inner envelope bound. The magenta dashed curve is the Pedlosky semicircle bound. Top panel: The same set up as figure 12 of the paper. Bottom panel: $L = 16, L_d = 2, U = \tanh(y), Q' = 0.2 - U''$.

then we have the semicircle bound

$$(c_r - r_c)^2 + c_i^2 \leq R^2(r_c)\,. \tag{0.10}$$

The radius R depends on the centre of the semicircle r_c. The tightest possible bound can therefore be established by plotting families of semicircles on the complex plane with different r_c and taking their inner envelope.

To run this algorithm we still need to find $R(r_c)$ satisfying the inequality (0.9). We decompose $\overline{\Omega} = [-1, 1]$ into two parts, $\Omega_1 = \{y \in \overline{\Omega} | (r_c - U)B \geq 0\}$ and $\Omega_2 = \{y \in \overline{\Omega} | (U - r_c)B > 0\}$. Then using

$$\int_\Omega W dy \geq (\kappa_0^2 + k^2) \int_\Omega |\eta|^2 dy\,, \tag{0.11}$$

which can be deduced by Poincare's inequality, the left hand side of (0.9) can be estimated as

$$\int_\Omega (U-r_c)^2 W dy - \int_{\Omega_1} (U-r_c) B|\eta|^2 dy - \int_{\Omega_2} (U-r_c) B|\eta|^2 dy$$
$$\le \left\{ \max_{\overline{\Omega}} (U-r_c)^2 + \frac{1}{\kappa_0^2+k^2} \max_{\Omega_1} (r_c - U) B \right\} \int_\Omega W dy \,. \qquad (0.12)$$

Therefore the radius of the inner envelope theory bound, appeared in (0.10), can be found as

$$R(r_c) = \sqrt{\max_{\overline{\Omega}} (U-r_c)^2 + \frac{1}{\kappa_0^2+k^2} \max\left(\max_{\overline{\Omega}} (r_c - U)B, 0 \right)} \,. \qquad (0.13)$$

Let us consider the special case $r_c = (U_{\max} + U_{\min})/2$, where $U_{\max} = \max_{\overline{\Omega}} U$ and $U_{\min} = \min_{\overline{\Omega}} U$. Then we can show that the right hand side of (0.13) is larger than or equal to the Howard semicircle radius $(U_{\max} - U_{\min})/2$. However (0.13) is smaller than or equal to the well-known Pedlosky semicircle radius,

$$\sqrt{\left(\frac{U_{\max} + U_{\min}}{2} \right)^2 + \frac{1}{\kappa_0^2+k^2} \left(\frac{U_{\max} + U_{\min}}{2} \right) \max_\Omega |B|} \,, \qquad (0.14)$$

generalised for non-constant B and finite L_d^{-1}.

Figure 1 top panel shows the comparison of the eigenvalues obtained in figure 12 of the paper and the bounds obtained above with $k=0$. The magenta dashed curve is the Pedlosky semicircle bound, which can be found by (0.10) with the centre $r_c = 0$ and the radius (0.14). The blue solid curve is the inner envelope bound. In this example, the inner envelope of $(c_r - r_c)^2 + c_i^2 = R^2(r_c)$ is merely a circle centred at the origin, but its radius is smaller than that of Pedlosky's. Figure 1 bottom panel is another example. The zonal flow is also $U = \tanh(y)$, but B is set to a constant of 0.2. In this case the inner envelope bound is not a circle and part of it overlaps the Pedlosky bound. Finally, we note that if $(\frac{U_{\max}+U_{\min}}{2} - U)B$ is negative for all $y \in \Omega$, then the inner envelope bound matches to Howard's semicircle. This occurs, for example, in the situation shown in figure 8b of the paper.