A. Supplementary Material

This supplementary material provides the proofs of the main theoretical results and conclusions in Examples 7-11. Specifically, Section A.1 provides the proofs of Propositions 1 and 2, while the identifiability results for the GPDINA model and the Sequential DINA model are presented in Sections A.2 and A.3, respectively. Additionally, the proofs of the examples are provided in Section A.4.

A.1 Proofs of Propositions 1 and 2

This section deals with the zero $\mathbf{q}$-vectors ($\mathbf{q} = \mathbf{0}$) in $\mathbf{Q}$-matrix. Our propositions show that, for both the GPDINA model and the Sequential DINA model, excluding items or categories whose corresponding $\mathbf{q}$-vectors are all zero does not affect the identifiability results.

Proposition 1. *Let $\Delta = \{j \in [J] : \mathbf{q}_j = \mathbf{0}\}$ denote the set of items whose $\mathbf{q}$-vectors are zero, then the GPDINA model parameters with $\mathbf{Q}$-matrix are identifiable if and only if the GPDINA model parameters with $\mathbf{Q}_{-\Delta}$-matrix are identifiable, where $\mathbf{Q}_{-\Delta}$ is obtained by removing the $\mathbf{q}$-vectors in $\mathbf{Q}$ corresponding to the items in Δ.*

Proof. According to Lemma 1, it suffices to show that the GPDINA models with $\mathbf{Q}$-matrix and $\mathbf{Q}_{\Delta}$-matrix yield the same equation system $\mathbf{T}\boldsymbol{p} = \bar{\mathbf{T}}\bar{\boldsymbol{p}}$. Let $\mathbf{T}$ and $\mathbf{T}'$ denote the $\mathbf{T}$-matrix under the $\mathbf{Q}$-matrix and $\mathbf{Q}_{-\Delta}$-matrix separately, and let $\boldsymbol{r}_{\Delta}$ denote the Δ-coordinates of $\boldsymbol{r}$. Then $\mathbf{T}'$ is a submatrix of $\mathbf{T}$ which excludes vectors $\mathbf{T}_{\boldsymbol{r}}$ in $\mathbf{T}$ with $\boldsymbol{r}_{\Delta} \neq \mathbf{0}$. i.e., $\mathbf{T} = \mathbf{T}' \cup \{\mathbf{T}_{\boldsymbol{r}} : \boldsymbol{r}_{\Delta} \neq \mathbf{0}\}$. We now show that $\{\mathbf{T}_{\boldsymbol{r}} : \boldsymbol{r}_{\Delta} \neq \mathbf{0}\}$ does not add additional constraints to the equation system $\mathbf{T}'\boldsymbol{p} = \bar{\mathbf{T}}'\bar{\boldsymbol{p}}$. For $j \in \Delta$ and $l \in [H_j]$, recall that $\mathbf{1}^\top = (1, 1, \ldots, 1)$, since $\mathbf{q}_j = \mathbf{0}$ and $\mathbf{1}^\top \boldsymbol{p} = \mathbf{1}^\top \bar{\boldsymbol{p}} = 1$, we have

$$\mathbf{T}_{l\boldsymbol{e}_j}\boldsymbol{p} = \theta^+_{j,l}\mathbf{1}^\top\boldsymbol{p} = \theta^+_{j,l}, \quad \bar{\mathbf{T}}_{l\boldsymbol{e}_j}\bar{\boldsymbol{p}} = \bar{\theta}^+_{j,l}\mathbf{1}^\top\bar{\boldsymbol{p}} = \bar{\theta}^+_{j,l}.$$

So $\mathbf{T}_{l\boldsymbol{e}_j}\boldsymbol{p} = \bar{\mathbf{T}}_{l\boldsymbol{e}_j}\bar{\boldsymbol{p}}$ gives $\theta^+_{j,l} = \bar{\theta}^+_{j,l}$ for $l \in [H_j]$. Therefore, for $j \in \Delta$, parameters $\theta^+_{j,l}$ are all identifiable. Furthermore, for any $\boldsymbol{r}$ s.t. $\boldsymbol{r}_\Delta \neq 0$, write $\boldsymbol{r} = \sum_{j\in\Delta} r_j\boldsymbol{e}_j + \Big(\boldsymbol{r} - \sum_{j\in\Delta} r_j\boldsymbol{e}_j\Big)$, then $\mathbf{T}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)}$ is a vector in $\mathbf{T}'$-matrix, and

$$\mathbf{T}_{\boldsymbol{r}} = \big(\circ_{j\in\Delta}\mathbf{T}_{r_j\boldsymbol{e}_j}\big) \circ \mathbf{T}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)} = \prod_{j\in\Delta}\theta^+_{j,r_j}\mathbf{T}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)} = \prod_{j\in\Delta}\bar{\theta}^+_{j,r_j}\mathbf{T}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)}.$$

Similarly, we have $\bar{\mathbf{T}}_{\boldsymbol{r}} = \prod_{j\in\Delta}\bar{\theta}^+_{j,r_j}\bar{\mathbf{T}}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)}$. Therefore, $\mathbf{T}_{\boldsymbol{r}}\boldsymbol{p} = \bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}}$ is equivalent to $\mathbf{T}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)}\boldsymbol{p} = \bar{\mathbf{T}}_{(\boldsymbol{r}-\sum_{j\in\Delta} r_j\boldsymbol{e}_j)}\bar{\boldsymbol{p}}$. Therefore, the model with $\mathbf{Q}$-matrix and $\mathbf{Q}_\Delta$-matrix give the same equation system $\mathbf{T}\boldsymbol{p} = \bar{\mathbf{T}}\bar{\boldsymbol{p}}$. □

Proposition 2. *Let $\Delta^s = \{(j,l) : \mathbf{q}_{j,l} = \mathbf{0}\}$ denote the set of categories whose $\mathbf{q}$-vectors are zero, then the Sequential DINA model parameters with $\mathbf{Q}$-matrix are identifiable if and only if the Sequential DINA model parameters with $\mathbf{Q}_{-\Delta^s}$-matrix are identifiable, where $\mathbf{Q}_{-\Delta^s}$ is obtained by removing the $\mathbf{q}$-vectors in $\mathbf{Q}$ corresponding to the categories in Δ^s.*

Proof. Similar to the previous analysis, we can show that $\beta^+_{j,l} = \bar{\beta}^+_{j,l}$ for $(j,l) \in \Delta^s$. If $l = 1$, since $\mathbf{q}_{j,l} = \mathbf{0}$ and $\mathbf{1}^\top\boldsymbol{p} = \mathbf{1}^\top\bar{\boldsymbol{p}} = 1$, we have

$$\mathbf{T}_{\boldsymbol{e}_j}\boldsymbol{p} = \beta^+_{j,1}\mathbf{1}^\top\boldsymbol{p} = \beta^+_{j,1}, \quad \bar{\mathbf{T}}_{\boldsymbol{e}_j}\bar{\boldsymbol{p}} = \bar{\beta}^+_{j,1}\mathbf{1}^\top\bar{\boldsymbol{p}} = \bar{\beta}^+_{j,1}.$$

Thus, $\mathbf{T}_{\boldsymbol{e}_j}\boldsymbol{p} = \bar{\mathbf{T}}_{\boldsymbol{e}_j}\bar{\boldsymbol{p}}$ gives $\beta^+_{j,1} = \bar{\beta}^+_{j,1}$. For $l > 1$, since $\mathbf{q}_{j,l} = \mathbf{0}$, $\mathbf{T}_{l\boldsymbol{e}_j} = \beta^+_{j,l}\mathbf{T}_{(l-1)\boldsymbol{e}_j}$, and since

$\mathbf{T}_{(l-1)\boldsymbol{e}_j}\boldsymbol{p} = \bar{\mathbf{T}}_{(l-1)\boldsymbol{e}_j}\bar{\boldsymbol{p}}$, we have

$$\mathbf{T}_{l\boldsymbol{e}_j}\boldsymbol{p} = \beta_{j,l}^{+}\mathbf{T}_{(l-1)\boldsymbol{e}_j}\boldsymbol{p} = \beta_{j,l}^{+}\bar{\mathbf{T}}_{(l-1)\boldsymbol{e}_j}\bar{\boldsymbol{p}} = \bar{\beta}_{j,l}^{+}\bar{\mathbf{T}}_{(l-1)\boldsymbol{e}_j}\bar{\boldsymbol{p}} = \bar{\mathbf{T}}_{l\boldsymbol{e}_j}\bar{\boldsymbol{p}}.$$

Thus, $\beta_{j,l}^{+} = \bar{\beta}_{j,l}^{+}$. So the item parameters in Δ^s are all identifiable. Furthermore, write

$$\boldsymbol{r} = \sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j + \left(\boldsymbol{r} - \sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right),$$

then $\mathbf{T}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)}\boldsymbol{p} = \bar{\mathbf{T}}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)}\bar{\boldsymbol{p}}$, and

$$\mathbf{T}_{\boldsymbol{r}} = \left(\circ_{(j,l)\in\Delta^s}\mathbf{T}_{l\boldsymbol{e}_j}\right)\circ\mathbf{T}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)} = \prod_{(j,l)\in\Delta^s}\beta_{j,l}^{+}\mathbf{T}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)} = \prod_{(j,l)\in\Delta^s}\bar{\beta}_{j,l}^{+}\mathbf{T}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)},$$

similarly, we have $\bar{\mathbf{T}}_{\boldsymbol{r}} = \prod_{j\in\Delta^s}\bar{\theta}_{j,r_j}^{+}\bar{\mathbf{T}}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)}$. Therefore, $\mathbf{T}_{\boldsymbol{r}}\boldsymbol{p} = \bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}}$ gives

$$\mathbf{T}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)}\boldsymbol{p} = \bar{\mathbf{T}}_{\left(\boldsymbol{r}-\sum_{(j,l)\in\Delta^s} l\boldsymbol{e}_j\right)}\bar{\boldsymbol{p}}.$$

Therefore, the model with $\mathbf{Q}$-matrix and $\mathbf{Q}_{\Delta^s}$-matrix give exactly the same equation system $\mathbf{T}\boldsymbol{p} = \bar{\mathbf{T}}\bar{\boldsymbol{p}}$. □

A.2 Identifiability of GPDINA

Theorem 1. *Conditions C1-C3 are sufficient and necessary for the identifiability of the parameters of the GPDINA model.*

Proof of sufficiency. Suppose the $\mathbf{Q}$-matrix satisfies conditions C1-C3. Using Lemma 1, we show that $\mathbf{T}\boldsymbol{p} = \bar{\mathbf{T}}\bar{\boldsymbol{p}}$ will give $(\boldsymbol{\theta}^{+}, \boldsymbol{\theta}^{-}, \boldsymbol{p}) = (\bar{\boldsymbol{\theta}}^{+}, \bar{\boldsymbol{\theta}}^{-}, \bar{\boldsymbol{p}})$. Take one arbitrary non-zero category from each item, denoted by $c(j)$ $(1 \leq c(j) \leq H_j)$, for $j \in [J]$. We show that $\theta^+_{j,c(j)} = \bar{\theta}^+_{j,c(j)}$ and $\theta^-_{j,c(j)} = \bar{\theta}^-_{j,c(j)}$.

Our proof leverages the identifiability results of the binary DINA model. Consider constructing the following binary DINA model: if we focus on solely the category $c(j)$ of each item j, for $j \in [J]$. If we dichotomize each item j through category $c(j)$, and reframe the response as binary response $I(R_j = c(j))_{j\in[J]}$, then the model is reduced to a binary DINA model. According to the model construction, the $\mathbf{Q}$-matrix for the reduced model is equivalent to the $\mathbf{Q}$-matrix for the original polytomous model, since every non-zero category of the same item requires the same attributes. The $\mathbf{T}$-matrix for this reduced model, i.e., the marginal probability distribution for the dichotomized response $I(R_j = c(j))$, is simply a submatrix of the original $\mathbf{T}$-matrix. It is made up of the vectors that only involve category $c(j)$ of item j, i.e., $\left(\mathbf{T}_{c(j)\cdot \boldsymbol{e}_j}\right)_{j\in[J]}$ and their element-wise products. The parameters for this reduced model are $\left(\left\{\theta^+_{j,c(j)},\, \theta^-_{j,c(j)}\right\}_{j\in[J]},\, \boldsymbol{p}\right)$. So the reduced model is completely a binary DINA model. Since the $\mathbf{Q}$-matrix for the reduced model satisfies conditions C1-C3, as a direct result of Gu and Xu (2019b), $\bar{\mathbf{T}}\bar{\boldsymbol{p}} = \mathbf{T}\boldsymbol{p}$ will give $\bar{\boldsymbol{p}} = \boldsymbol{p}$, $\theta^+_{j,c(j)} = \bar{\theta}^+_{j,c(j)}$, $\theta^-_{j,c(j)} = \bar{\theta}^-_{j,c(j)}$, for $j \in [J]$. This holds for any $c(j) \in [H_j]$, thus $(\bar{\boldsymbol{\theta}}^+, \bar{\boldsymbol{\theta}}^-, \bar{\boldsymbol{p}}) = (\boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{p})$ and we complete the proof.

$\square$

Proof of necessity. We prove separately each condition is necessary.

Necessity of condition C1. Suppose the $\mathbf{Q}$-matrix is not complete, and WLOG, assume that $\boldsymbol{e}_1^\top \notin (\mathbf{q}_j)_{j=1}^J$. Then attributes profiles $\mathbf{0}$ and $\boldsymbol{e}_1$ have the same conditional response distributions. Therefore, the parameters $p_{\mathbf{0}}$ and $p_{\boldsymbol{e}_1}$ are exchangeable, and thus can not be identified. Therefore, condition C1 is necessary.

Necessity of condition C2. Suppose the $\mathbf{Q}$-matrix satisfies condition C1, but does not satisfy condition C2, i.e., there exists some attribute which is only required by at most two items. WLOG, assume this is the first attribute, and it is the first and second items that require the first attribute, so the $\mathbf{Q}$-matrix can be written as follows:

$$\mathbf{Q} = \begin{pmatrix} \text{item 1} & 1 & \mathbf{0}^\top \\ \hdashline \text{item 2} & 1 & \boldsymbol{v}^\top \\ \hdashline \text{item } (3:J) & \mathbf{0} & \mathbf{Q}' \end{pmatrix}. \tag{26}$$

We partition $\boldsymbol{\alpha}$ into two groups according to the first attribute:

$$\boldsymbol{g}^0 = \{\boldsymbol{\alpha} : \alpha_1 = 0\} = \{\boldsymbol{\alpha} = (0, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}\},$$
$$\boldsymbol{g}^1 = \{\boldsymbol{\alpha} : \alpha_1 = 1\} = \{\boldsymbol{\alpha} = (1, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}\},$$

so each group has 2^{K-1} attribute profiles, and we index the entries in each group by

$$\boldsymbol{g}_1^0 = (0, \mathbf{0}),\ \boldsymbol{g}_2^0 = (0, \boldsymbol{e}_1), \ldots, \boldsymbol{g}_K^0 = (0, \boldsymbol{e}_{K-1}),\ \boldsymbol{g}_{K+1}^0 = (0, \boldsymbol{e}_1 + \boldsymbol{e}_2), \ldots, \boldsymbol{g}_{2^{K-1}}^0 = \left(0, \sum_{k=1}^{K-1} \boldsymbol{e}_k\right),$$
$$\boldsymbol{g}_1^1 = (1, \mathbf{0}),\ \boldsymbol{g}_2^1 = (1, \boldsymbol{e}_1), \ldots, \boldsymbol{g}_K^1 = (1, \boldsymbol{e}_{K-1}),\ \boldsymbol{g}_{K+1}^1 = (1, \boldsymbol{e}_1 + \boldsymbol{e}_2), \ldots, \boldsymbol{g}_{2^{K-1}}^1 = \left(1, \sum_{k=1}^{K-1} \boldsymbol{e}_k\right),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{K-1} \in \{0,1\}^{K-1}$ have $K-1$ elements. Therefore, the k-th ($k \in [2^{K-1}]$) entry of $\boldsymbol{g}^0$ and $\boldsymbol{g}^1$, $\boldsymbol{g}_k^0$ and $\boldsymbol{g}_k^1$ share the same attributes except for the first one α_1. Index the population proportion parameters $\boldsymbol{p}$ in the following way:

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^0} \\ \boldsymbol{p}_{\boldsymbol{g}^1} \end{pmatrix}, \quad \text{where} \;\; \boldsymbol{p}_{\boldsymbol{g}^0} = \begin{pmatrix} p_{\boldsymbol{g}_1^0} \\ p_{\boldsymbol{g}_2^0} \\ \vdots \\ p_{\boldsymbol{g}^0_{2^{K-1}}} \end{pmatrix} \;\; \text{and} \;\; \boldsymbol{p}_{\boldsymbol{g}^1} = \begin{pmatrix} p_{\boldsymbol{g}_1^1} \\ p_{\boldsymbol{g}_2^1} \\ \vdots \\ p_{\boldsymbol{g}^1_{2^{K-1}}} \end{pmatrix}. \tag{27}$$

We now seek to construct $(\bar{\boldsymbol{\theta}}^+, \bar{\boldsymbol{\theta}}^-, \bar{\boldsymbol{p}}) \neq (\boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{p})$ such that (13) holds: take $\bar{\boldsymbol{\theta}}_j^+ = \boldsymbol{\theta}_j^+$ for $j > 2$ and $\bar{\boldsymbol{\theta}}_j^- = \boldsymbol{\theta}_j^-$ for $j \in [J]$. We claim that, with a simplified matrix $\boldsymbol{\mathcal{T}}$, for $\bar{\mathbf{T}}\bar{\boldsymbol{p}} = \mathbf{T}\boldsymbol{p}$ to hold, it suffices to have $\bar{\boldsymbol{\mathcal{T}}}\bar{\boldsymbol{p}} = \boldsymbol{\mathcal{T}}\boldsymbol{p}$. When the entries in $\boldsymbol{p}$ are indexed according to (27), the $\boldsymbol{\mathcal{T}}$ is given as follows:

$$\boldsymbol{\mathcal{T}} = \begin{pmatrix} 1 & 1 \\ \boldsymbol{\theta}_1^- & \boldsymbol{\theta}_1^+ \\ \boldsymbol{\theta}_2^- & \boldsymbol{\theta}_2^+ \\ \boldsymbol{\theta}_1^- \otimes \boldsymbol{\theta}_2^- & \boldsymbol{\theta}_1^+ \otimes \boldsymbol{\theta}_2^+ \end{pmatrix} \otimes \boldsymbol{\mathcal{I}}, \tag{28}$$

where $\boldsymbol{\mathcal{I}} = \boldsymbol{\mathcal{I}}_{2^{K-1}}$. We now prove the claim.

- For any $\boldsymbol{r}$ s.t. $r_1 = r_2 = 0$, $\mathbf{T}_{\boldsymbol{r}}$ does not involve $\boldsymbol{\theta}_1^+$, $\boldsymbol{\theta}_1^-$, $\boldsymbol{\theta}_2^+$, and $\boldsymbol{\theta}_2^-$. According to the construction of the $\mathbf{Q}$-matrix in (26), the response $\boldsymbol{r}$ does not require α_1, so the response distributions in both groups are the same, i.e., $t_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv t_{\boldsymbol{r},\boldsymbol{g}_k^1}$ for $k \in [2^{K-1}]$. Since $\bar{\boldsymbol{\theta}}_j^+ = \boldsymbol{\theta}_j^+$ and $\bar{\boldsymbol{\theta}}_j^- = \boldsymbol{\theta}_j^-$ for $j > 2$, we further have

$$t_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv t_{\boldsymbol{r},\boldsymbol{g}_k^1} \equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^1} \text{ for } k \in [2^{K-1}]. \tag{29}$$

If we denote $\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})}$ as the first half of the vector $\mathbf{T}_{\boldsymbol{r}}$, i.e., the response of group $\boldsymbol{g}^0$, then equation (29) indicates that $\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})}$ is also the response of group $\boldsymbol{g}^1$. Moreover, $\bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{2})} = \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})}$ and

$$\mathbf{T}_{\boldsymbol{r}} = \begin{pmatrix} \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})} & \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})} \end{pmatrix} = \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})} \left(\boldsymbol{\mathcal{I}} \;\; \boldsymbol{\mathcal{I}} \right) = \bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{2})} \left(\boldsymbol{\mathcal{I}} \;\; \boldsymbol{\mathcal{I}} \right) = \bar{\mathbf{T}}_{\boldsymbol{r}}.$$

Therefore, equations $\begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} \bar{\boldsymbol{p}} = \begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} \boldsymbol{p}$ in (28) gives

$$\bar{\mathbf{T}}_{\boldsymbol{r}} \bar{\boldsymbol{p}} = \bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{2})} \left(\boldsymbol{\mathcal{I}} \;\; \boldsymbol{\mathcal{I}} \right) \bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{2})} \begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} \boldsymbol{p} = \mathbf{T}_{\boldsymbol{r}} \boldsymbol{p}.$$

- For any $\boldsymbol{r}$ s.t. $r_1 \neq 0, r_2 = 0$, write $\boldsymbol{r} = r_1 \boldsymbol{e}_1 + (\boldsymbol{r} - r_1 \boldsymbol{e}_1)$, then $\mathbf{T}_{\boldsymbol{r}} = \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)} \circ \mathbf{T}_{r_1 \boldsymbol{e}_1}$, and the response $(\boldsymbol{r} - r_1 \boldsymbol{e}_1)$ belongs to the case we analyzed previously. Therefore, $\bar{\mathbf{T}}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})} = \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})}$ and

$$\bar{\mathbf{T}}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)} = \bar{\mathbf{T}}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})} \begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} = \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})} \begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} = \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)},$$

which gives

$$\begin{aligned} \bar{\mathbf{T}}_{\boldsymbol{r}} \bar{\boldsymbol{p}} = (\bar{\mathbf{T}}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)} \circ \bar{\mathbf{T}}_{r_1 \boldsymbol{e}_1})\, \bar{\boldsymbol{p}} &= \bar{\mathbf{T}}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)} \left(\bar{\mathbf{T}}_{r_1 \boldsymbol{e}_1} \circ \bar{\boldsymbol{p}} \right) \\ &= \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})} \begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} \left(\bar{\mathbf{T}}_{r_1 \boldsymbol{e}_1} \circ \bar{\boldsymbol{p}} \right) \\ &= \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})} \left(\begin{pmatrix} \boldsymbol{\mathcal{I}} & \boldsymbol{\mathcal{I}} \end{pmatrix} \circ \bar{\mathbf{T}}_{r_1 \boldsymbol{e}_1} \right) \bar{\boldsymbol{p}} \\ &= \mathbf{T}_{(\boldsymbol{r} - r_1 \boldsymbol{e}_1)}^{(\frac{1}{2})} \begin{pmatrix} \bar{\theta}_{1,r_1}^{-} \boldsymbol{\mathcal{I}} & \bar{\theta}_{1,r_1}^{+} \boldsymbol{\mathcal{I}} \end{pmatrix} \bar{\boldsymbol{p}}. \end{aligned}$$

Therefore, $\begin{pmatrix} \bar{\boldsymbol{\theta}}_1^{-} \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_1^{+} \otimes \boldsymbol{\mathcal{I}} \end{pmatrix} \bar{\boldsymbol{p}} = \begin{pmatrix} \boldsymbol{\theta}_1^{-} \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_1^{+} \otimes \boldsymbol{\mathcal{I}} \end{pmatrix} \boldsymbol{p}$ in equation (28) guarantees that for $\forall r_1 \in [H_1]$,

$$\begin{pmatrix}\bar{\theta}^-_{1,r_1}\boldsymbol{\mathcal{I}} & \bar{\theta}^+_{1,r_1}\boldsymbol{\mathcal{I}}\end{pmatrix}\bar{\boldsymbol{p}} = \begin{pmatrix}\theta^-_{1,r_1}\boldsymbol{\mathcal{I}} & \theta^+_{1,r_1}\boldsymbol{\mathcal{I}}\end{pmatrix}\boldsymbol{p}.$$

Therefore,

$$\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}^{(\frac{1}{2})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\begin{pmatrix}\bar{\theta}^-_{1,r_1}\boldsymbol{\mathcal{I}} & \bar{\theta}^+_{1,r_1}\boldsymbol{\mathcal{I}}\end{pmatrix}\bar{\boldsymbol{p}} = \mathbf{T}^{(\frac{1}{2})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\begin{pmatrix}\theta^-_{1,r_1}\boldsymbol{\mathcal{I}} & \theta^+_{1,r_1}\boldsymbol{\mathcal{I}}\end{pmatrix}\boldsymbol{p} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}.$$

- Similarly, for any $\boldsymbol{r}$ s.t. $r_1 = 0, r_2 \neq 0$, $\begin{pmatrix}\bar{\boldsymbol{\theta}}^-_2 \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}^+_2 \otimes \boldsymbol{\mathcal{I}}\end{pmatrix}\bar{\boldsymbol{p}} = \begin{pmatrix}\boldsymbol{\theta}^-_2 \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}^+_2 \otimes \boldsymbol{\mathcal{I}}\end{pmatrix}\boldsymbol{p}$ with $\bar{\boldsymbol{\theta}}^-_2 = \boldsymbol{\theta}^-_2$ guarantees that $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$.
- Similarly, for any $\boldsymbol{r}$ s.t. $r_1 \neq 0, r_2 \neq 0$, $\begin{pmatrix}\bar{\boldsymbol{\theta}}^-_1 \otimes \bar{\boldsymbol{\theta}}^-_2 \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}^+_1 \otimes \bar{\boldsymbol{\theta}}^+_2 \otimes \boldsymbol{\mathcal{I}}\end{pmatrix}\bar{\boldsymbol{p}} = \begin{pmatrix}\boldsymbol{\theta}^-_1 \otimes \boldsymbol{\theta}^-_2 \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}^+_1 \otimes \boldsymbol{\theta}^+_2 \otimes \boldsymbol{\mathcal{I}}\end{pmatrix}\boldsymbol{p}$ guarantees $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$.

Next we construct $(\bar{\boldsymbol{\theta}}^+_1, \bar{\boldsymbol{\theta}}^+_2, \bar{\boldsymbol{\theta}}^-_1, \bar{\boldsymbol{p}})$ s.t. $\bar{\boldsymbol{\mathcal{T}}}\bar{\boldsymbol{p}} = \boldsymbol{\mathcal{T}}\boldsymbol{p}$ holds. Let $\bar{\boldsymbol{p}}_{\boldsymbol{g}^1} = \rho \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$, $\boldsymbol{p}_{\boldsymbol{g}^0} = u \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$, and $\boldsymbol{p}_{\boldsymbol{g}^1} = v \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$. Then $\bar{\boldsymbol{\mathcal{T}}}\bar{\boldsymbol{p}} = \boldsymbol{\mathcal{T}}\boldsymbol{p}$ can be simplified to

$$\bar{\boldsymbol{\mathcal{T}}}\begin{pmatrix}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} \\ \rho \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}\end{pmatrix} = \boldsymbol{\mathcal{T}}\begin{pmatrix}u \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0} \\ v \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}\end{pmatrix},$$

i.e.,

$$\begin{cases} \boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + \rho\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} = u\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + v\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0}; \\ \bar{\theta}^-_{1,l_1}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + \rho\bar{\theta}^+_{1,l_1}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} = u\theta^-_{1,l_1}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + v\theta^+_{1,l_1}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0}, \ \ l_1 \in [H_1]; \\ \bar{\theta}^-_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + \rho\bar{\theta}^+_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} = u\theta^-_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + v\theta^+_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0}, \ \ l_2 \in [H_2]; \\ \bar{\theta}^-_{1,l_1}\bar{\theta}^-_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + \rho\bar{\theta}^+_{1,l_1}\bar{\theta}^+_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} = u\theta^-_{1,l_1}\theta^-_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0} + v\theta^+_{1,l_1}\theta^+_{2,l_2}\boldsymbol{\mathcal{I}}\bar{\boldsymbol{p}}_{\boldsymbol{g}^0}, \ \ l_1 \in [H_1], \ \ l_2 \in [H_2]. \end{cases}$$

Then it suffices to have

$$\begin{cases} 1 + \rho = u + v; \\ \bar{\theta}^-_{1,l_1} + \rho\bar{\theta}^+_{1,l_1} = u\theta^-_{1,l_1} + v\theta^+_{1,l_1}, \ \ l_1 \in [H_1]; \\ \bar{\theta}^-_{2,l_2} + \rho\bar{\theta}^+_{2,l_2} = u\theta^-_{2,l_2} + v\theta^+_{2,l_2}, \ \ l_2 \in [H_2]; \\ \bar{\theta}^-_{1,l_1}\bar{\theta}^-_{2,l_2} + \rho\bar{\theta}^+_{1,l_1}\bar{\theta}^+_{2,l_2} = u\theta^-_{1,l_1}\theta^-_{2,l_2} + v\theta^+_{1,l_1}\theta^+_{2,l_2}, \ \ l_1 \in [H_1], \ \ l_2 \in [H_2]. \end{cases} \tag{30}$$

Let some $\kappa \in (0, 1)$ s.t

$$\begin{pmatrix} \theta^-_{1,l_1} \\ \bar{\theta}^-_{1,l_1} \\ \theta^+_{1,l_1} \\ \bar{\theta}^+_{1,l_1} \end{pmatrix} = \kappa^{l_1-1} \begin{pmatrix} \theta^-_{1,1} \\ \bar{\theta}^-_{1,1} \\ \theta^+_{1,1} \\ \bar{\theta}^+_{1,1} \end{pmatrix}, \text{ and } \begin{pmatrix} \theta^-_{2,l_2} \\ \bar{\theta}^-_{2,l_2} \\ \theta^+_{2,l_2} \\ \bar{\theta}^+_{2,l_2} \end{pmatrix} = \kappa^{l_2-1} \begin{pmatrix} \theta^-_{2,1} \\ \bar{\theta}^-_{2,1} \\ \theta^+_{2,1} \\ \bar{\theta}^+_{2,1} \end{pmatrix} \quad \text{for } \ l_1 \in [H_1], \ l_2 \in [H_2], \tag{31}$$

then equations (30) are reduced to the following four equations:

$$\begin{cases} 1+\rho = u+v; \\ \bar{\theta}^-_{1,1} + \rho\bar{\theta}^+_{1,1} = u\theta^-_{1,1} + v\theta^+_{1,1}; \\ \bar{\theta}^-_{2,1} + \rho\bar{\theta}^+_{2,1} = u\theta^-_{2,1} + v\theta^+_{2,1}; \\ \bar{\theta}^-_{1,1}\bar{\theta}^-_{2,1} + \rho\bar{\theta}^+_{1,1}\bar{\theta}^+_{2,1} = u\theta^-_{1,1}\theta^-_{2,1} + v\theta^+_{1,1}\theta^+_{2,1}. \end{cases}$$

There are five parameters $(\rho, u, v,\ \bar{\theta}^+_{1,1},\ \bar{\theta}^+_{2,1})$ with four constraints, so there are infinite many solutions. Consequently, parameters $(\boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{p})$ are not identifiable and condition C2 is indeed necessary.

Necessity of condition C3. Suppose the $\mathbf{Q}$-matrix satisfies conditions C1 and C2, but does not satisfy condition C3. WLOG, we may write $\mathbf{Q}$ as

$$\mathbf{Q} = \begin{pmatrix} \boldsymbol{\mathcal{I}}_K \\ \mathbf{Q}^* \end{pmatrix}_{J\times K}, \quad \text{where } \mathbf{Q}^* = \begin{pmatrix} \boldsymbol{v} & \boldsymbol{v} & \vdots & \vdots & \vdots \end{pmatrix}.$$

We partition $\boldsymbol{\alpha}$ into four groups according to the first and the second attributes:

$$\boldsymbol{g}^{00} = \{\boldsymbol{\alpha} : \alpha_1 = 0, \alpha_2 = 0\} = \{\boldsymbol{\alpha} = (0, 0, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\},$$

$$\boldsymbol{g}^{10} = \{\boldsymbol{\alpha} : \alpha_1 = 1, \alpha_2 = 0\} = \{\boldsymbol{\alpha} = (1, 0, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\},$$

$$\boldsymbol{g}^{01} = \{\boldsymbol{\alpha} : \alpha_1 = 0, \alpha_2 = 1\} = \{\boldsymbol{\alpha} = (0, 1, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\},$$

$$\boldsymbol{g}^{11} = \{\boldsymbol{\alpha} : \alpha_1 = 1, \alpha_2 = 1\} = \{\boldsymbol{\alpha} = (1, 1, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\}.$$

So each group has 2^{K-2} attribute profiles. Index the entries in each group through the following: in group $\boldsymbol{g}^{00}$,

$$\boldsymbol{g}_1^{00} = (0, 0, \boldsymbol{0}),\ \boldsymbol{g}_2^{00} = (0, 0, \boldsymbol{e}_1), \ldots, \boldsymbol{g}_K^{00} = (0, 0, \boldsymbol{e}_{K-2}),\ \boldsymbol{g}_{K+1}^{00} = (0, 0, \boldsymbol{e}_1 + \boldsymbol{e}_2), \ldots, \boldsymbol{g}_{2^{K-2}}^{00} = \left(0, 0, \sum_{k=1}^{K-2} \boldsymbol{e}_k\right),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{K-2} \in \{0, 1\}^{K-2}$ have $K-2$ elements. Similarly we index the elements in $\boldsymbol{g}^{10}$, $\boldsymbol{g}^{01}$, $\boldsymbol{g}^{11}$, so that $\boldsymbol{g}_k^{00}$, $\boldsymbol{g}_k^{10}$, $\boldsymbol{g}_k^{01}$ and $\boldsymbol{g}_k^{11}$ for $k \in [2^{K-2}]$ share the same attributes except for the first and second attributes. Index the population proportion parameters $\boldsymbol{p}$ in the following way:

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^{00}} \\ \boldsymbol{p}_{\boldsymbol{g}^{10}} \\ \boldsymbol{p}_{\boldsymbol{g}^{01}} \\ \boldsymbol{p}_{\boldsymbol{g}^{11}} \end{pmatrix}, \quad \text{where } \boldsymbol{p}_{\boldsymbol{g}^{00}} = \begin{pmatrix} p_{\boldsymbol{g}_1^{00}} \\ p_{\boldsymbol{g}_2^{00}} \\ \vdots \\ p_{\boldsymbol{g}_{2^{K-2}}^{00}} \end{pmatrix}, \ \boldsymbol{p}_{\boldsymbol{g}^{10}} = \begin{pmatrix} p_{\boldsymbol{g}_1^{10}} \\ p_{\boldsymbol{g}_2^{10}} \\ \vdots \\ p_{\boldsymbol{g}_{2^{K-2}}^{10}} \end{pmatrix}, \ \boldsymbol{p}_{\boldsymbol{g}^{01}} = \begin{pmatrix} p_{\boldsymbol{g}_1^{01}} \\ p_{\boldsymbol{g}_2^{01}} \\ \vdots \\ p_{\boldsymbol{g}_{2^{K-2}}^{01}} \end{pmatrix}, \ \boldsymbol{p}_{\boldsymbol{g}^{11}} = \begin{pmatrix} p_{\boldsymbol{g}_1^{11}} \\ p_{\boldsymbol{g}_2^{11}} \\ \vdots \\ p_{\boldsymbol{g}_{2^{K-2}}^{11}} \end{pmatrix}.$$

Next, we seek to construct $(\bar{\boldsymbol{\theta}}^+, \bar{\boldsymbol{\theta}}^-, \bar{\boldsymbol{p}}) \neq (\boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{p})$ such that (13) holds: take $\bar{\boldsymbol{\theta}}_j^+ = \boldsymbol{\theta}_j^+$ for all j, $\bar{\boldsymbol{\theta}}_j^- = \boldsymbol{\theta}_j^-$ for $j > 2$, and let $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{11}} = \boldsymbol{p}_{\boldsymbol{g}^{11}}$. If we let $\boldsymbol{g}^{-11} = \{\boldsymbol{g}^{00}, \boldsymbol{g}^{10}, \boldsymbol{g}^{01}\}$ denote the union of the other three groups, and denote its corresponding population proportion parameters as

$$\boldsymbol{p}_{\boldsymbol{g}^{-11}} = \begin{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^{00}} \\ \boldsymbol{p}_{\boldsymbol{g}^{10}} \\ \boldsymbol{p}_{\boldsymbol{g}^{01}} \end{pmatrix},$$

then using a similar strategy we can show that if $\boldsymbol{\mathcal{T}} \boldsymbol{p}_{\boldsymbol{g}^{-11}} = \bar{\boldsymbol{\mathcal{T}}} \bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}}$ holds, we have $\mathbf{T}\boldsymbol{p} = \bar{\mathbf{T}}\bar{\boldsymbol{p}}$, where $\boldsymbol{\mathcal{T}}$ is given as follows:

$$\boldsymbol{\mathcal{T}} = \begin{pmatrix} 1 & 1 & 1 \\ \boldsymbol{\theta}_1^- & \boldsymbol{\theta}_1^+ & \boldsymbol{\theta}_1^- \\ \boldsymbol{\theta}_2^- & \boldsymbol{\theta}_2^- & \boldsymbol{\theta}_2^+ \\ \boldsymbol{\theta}_1^- \otimes \boldsymbol{\theta}_2^- & \boldsymbol{\theta}_1^+ \otimes \boldsymbol{\theta}_2^- & \boldsymbol{\theta}_1^- \otimes \boldsymbol{\theta}_2^+ \end{pmatrix} \otimes \boldsymbol{\mathcal{I}}, \tag{32}$$

with $\boldsymbol{\mathcal{I}} = \boldsymbol{\mathcal{I}}_{2^{K-2}}$ being the identity matrix of dimension $(K-2) \times (K-2)$. We now prove the claim.

- For any $\boldsymbol{r}$ s.t. $r_1 = r_2 = 0$, $\mathbf{T}_{\boldsymbol{r}}$ does not involve $\boldsymbol{\theta}_1^+$, $\boldsymbol{\theta}_1^-$, $\boldsymbol{\theta}_2^+$, $\boldsymbol{\theta}_2^-$. According to the construction of the $\mathbf{Q}$-matrix, item $2, \ldots, J$ either require both α_1 and α_2 or require neither, therefore the response distribution in groups $\boldsymbol{g}^{00}$, $\boldsymbol{g}^{10}$, $\boldsymbol{g}^{01}$ are the same, i.e., $t_{\boldsymbol{r}, \boldsymbol{g}_k^{00}} \equiv t_{\boldsymbol{r}, \boldsymbol{g}_k^{10}} \equiv t_{\boldsymbol{r}, \boldsymbol{g}_k^{01}}$ for $k \in [2^{K-2}]$. Since $\bar{\boldsymbol{\theta}}_j^+ = \boldsymbol{\theta}_j^+$ $\bar{\boldsymbol{\theta}}_j^- = \boldsymbol{\theta}_j^-$ for $j > 2$, we further have

$$t_{\boldsymbol{r}, \boldsymbol{g}_k^{00}} \equiv t_{\boldsymbol{r}, \boldsymbol{g}_k^{10}} \equiv t_{\boldsymbol{r}, \boldsymbol{g}_k^{01}} \equiv \bar{t}_{\boldsymbol{r}, \boldsymbol{g}_k^{00}} \equiv \bar{t}_{\boldsymbol{r}, \boldsymbol{g}_k^{10}} \equiv \bar{t}_{\boldsymbol{r}, \boldsymbol{g}_k^{01}}, \text{ and } t_{\boldsymbol{r}, \boldsymbol{g}_k^{11}} \equiv \bar{t}_{\boldsymbol{r}, \boldsymbol{g}_k^{11}} \text{ for } k \in [2^{K-2}]. \tag{33}$$

  Let $\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}$ be the first quartile of the vector $\mathbf{T}_{\boldsymbol{r}}$, and $\mathbf{T}_{\boldsymbol{r}}^{+}$ be the last quartile of the vector $\mathbf{T}_{\boldsymbol{r}}$, then $\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}$ is the response of group $\boldsymbol{g}^{00}$ and $\mathbf{T}_{\boldsymbol{r}}^{+}$ is the response of group $\boldsymbol{g}^{11}$. Then equation (33) indicates that $\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}$ is also the response of group $\boldsymbol{g}^{10}$ and $\boldsymbol{g}^{01}$. I.e., we have

$$\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})} = \prod_{j:r_j\neq 0} P\left(R_j = r_j \mid \mathbf{Q}, \boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{\alpha} = \boldsymbol{g}^{00}\right) = \prod_{j:r_j\neq 0} P\left(R_j = r_j \mid \mathbf{Q}, \boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{\alpha} = \boldsymbol{g}^{10}\right)$$
$$= \prod_{j:r_j\neq 0} P\left(R_j = r_j \mid \mathbf{Q}, \boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{\alpha} = \boldsymbol{g}^{01}\right),$$
$$\mathbf{T}_{\boldsymbol{r}}^{+} = \prod_{j:r_j\neq 0} P\left(R_j = r_j \mid \mathbf{Q}, \boldsymbol{\theta}^+, \boldsymbol{\theta}^-, \boldsymbol{\alpha} = \boldsymbol{g}^{11}\right).$$

Then equation (33) also indicates that $\bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{4})} = \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}$ and $\mathbf{T}_{\boldsymbol{r}}^{+} = \bar{\mathbf{T}}_{\boldsymbol{r}}^{+}$. Thus,

$$\mathbf{T}_{\boldsymbol{r}} = \left(\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}\ \ \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}\ \ \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}\ \ \mathbf{T}_{\boldsymbol{r}}^{+}\right) = \left(\mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \mathbf{T}_{\boldsymbol{r}}^{+}\right) = \left(\bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{4})}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \bar{\mathbf{T}}_{\boldsymbol{r}}^{+}\right).$$

combining $\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} = \left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\boldsymbol{p}_{\boldsymbol{g}^{-11}}$ in (32) and $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{11}} = \boldsymbol{p}_{\boldsymbol{g}^{11}}$ gives

$$\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \left(\bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{4})}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \bar{\mathbf{T}}_{\boldsymbol{r}}^{+}\right)\begin{pmatrix}\bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}}\\ \bar{\boldsymbol{p}}_{\boldsymbol{g}^{11}}\end{pmatrix} = \bar{\mathbf{T}}_{\boldsymbol{r}}^{(\frac{1}{4})}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} + \bar{\mathbf{T}}_{\boldsymbol{r}}^{+}\,\bar{\boldsymbol{p}}_{\boldsymbol{g}^{11}}$$
$$= \mathbf{T}_{\boldsymbol{r}}^{(\frac{1}{4})}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\boldsymbol{p}_{\boldsymbol{g}^{-11}} + \mathbf{T}_{\boldsymbol{r}}^{+}\,\boldsymbol{p}_{\boldsymbol{g}^{11}}$$
$$= \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}.$$

- For any $\boldsymbol{r}$ s.t. $r_1 \neq 0, r_2 = 0$, write $\boldsymbol{r} = r_1\boldsymbol{e}_1 + (\boldsymbol{r} - r_1\boldsymbol{e}_1)$, then $\mathbf{T}_{\boldsymbol{r}} = \mathbf{T}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)} \circ \mathbf{T}_{r_1\boldsymbol{e}_1}$. If we split the vector $\mathbf{T}_{r_1\boldsymbol{e}_1}$ via the third quartile, i.e.,

$$\mathbf{T}_{r_1\boldsymbol{e}_1} = \left(\mathbf{T}_{r_1\boldsymbol{e}_1}^{(\frac{3}{4})}\ \ \mathbf{T}_{r_1\boldsymbol{e}_1}^{+}\right), \tag{34}$$

where $\mathbf{T}_{r_1\boldsymbol{e}_1}^{(\frac{3}{4})}$ is the first three-fourths of the vector and $\mathbf{T}_{r_1\boldsymbol{e}_1}^{+}$ is the last fourth part of the vector. Then we know that $\mathbf{T}^{(\frac{3}{4})}_{r_1\boldsymbol{e}_1}$ is the response of group $\boldsymbol{g}^{-11}$ and $\mathbf{T}^{+}_{r_1\boldsymbol{e}_1}$ is the response of group $\boldsymbol{g}^{11}$, i.e.,

$$\mathbf{T}^{(\frac{3}{4})}_{r_1\boldsymbol{e}_1} = P\left(R_1 = r_1 \mid \mathbf{Q}, \boldsymbol{\theta}^{+}, \boldsymbol{\theta}^{-}, \boldsymbol{\alpha} = \boldsymbol{g}^{-11}\right),$$

$$\mathbf{T}^{+}_{r_1\boldsymbol{e}_1} = P\left(R_1 = r_1 \mid \mathbf{Q}, \boldsymbol{\theta}^{+}, \boldsymbol{\theta}^{-}, \boldsymbol{\alpha} = \boldsymbol{g}^{11}\right).$$

Since item 1 only requires α_1, the entries of $\mathbf{T}^{+}_{r_1\boldsymbol{e}_1}$ are positive, and since $\boldsymbol{\theta}^{+}_1 = \bar{\boldsymbol{\theta}}^{+}_1$, we have $\bar{\mathbf{T}}^{+}_{r_1\boldsymbol{e}_1} = \mathbf{T}^{+}_{r_1\boldsymbol{e}_1}$. Furthermore, response $(\boldsymbol{r} - r_1\boldsymbol{e}_1)$ belongs to the case we analyzed previously, therefore, we have

$$\bar{\mathbf{T}}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)} = \left(\bar{\mathbf{T}}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \ \bar{\mathbf{T}}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\right) \tag{35}$$

$$= \left(\mathbf{T}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \ \mathbf{T}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\right). \tag{36}$$

Hence,

$$\begin{aligned}
\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} &= \left(\bar{\mathbf{T}}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)} \circ \bar{\mathbf{T}}_{r_1\boldsymbol{e}_1}\right)\bar{\boldsymbol{p}} \\
&= \bar{\mathbf{T}}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\bar{\mathbf{T}}_{r_1\boldsymbol{e}_1} \circ \bar{\boldsymbol{p}}\right) \\
&\overset{(35)}{=\!=} \left(\bar{\mathbf{T}}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \ \bar{\mathbf{T}}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\right)\left(\bar{\mathbf{T}}_{r_1\boldsymbol{e}_1} \circ \bar{\boldsymbol{p}}\right) \\
&\overset{(36)}{=\!=} \left(\left(\mathbf{T}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right)\ \ \mathbf{T}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\right) \circ \bar{\mathbf{T}}_{r_1\boldsymbol{e}_1}\right)\bar{\boldsymbol{p}} \\
&\overset{(34)}{=\!=} \left(\mathbf{T}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\ \boldsymbol{\mathcal{I}}\right) \circ \bar{\mathbf{T}}^{(\frac{3}{4})}_{r_1\boldsymbol{e}_1}\ \ \mathbf{T}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)} \circ \bar{\mathbf{T}}^{+}_{r_1\boldsymbol{e}_1}\right)\bar{\boldsymbol{p}} \\
&= \mathbf{T}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\bar{\theta}^{-}_{1,r_1}\boldsymbol{\mathcal{I}}\ \ \bar{\theta}^{+}_{1,r_1}\boldsymbol{\mathcal{I}}\ \ \bar{\theta}^{-}_{1,r_1}\boldsymbol{\mathcal{I}}\right)\bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} + \mathbf{T}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)} \circ \bar{\mathbf{T}}^{+}_{r_1\boldsymbol{e}_1}\bar{\boldsymbol{p}}_{\boldsymbol{g}^{11}} \\
&= \mathbf{T}^{(\frac{1}{4})}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)}\left(\bar{\theta}^{-}_{1,r_1}\boldsymbol{\mathcal{I}}\ \ \bar{\theta}^{+}_{1,r_1}\boldsymbol{\mathcal{I}}\ \ \bar{\theta}^{-}_{1,r_1}\boldsymbol{\mathcal{I}}\right)\bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} + \mathbf{T}^{+}_{(\boldsymbol{r}-r_1\boldsymbol{e}_1)} \circ \mathbf{T}^{+}_{r_1\boldsymbol{e}_1}\boldsymbol{p}_{\boldsymbol{g}^{11}}.
\end{aligned}$$

Then $\begin{pmatrix}\bar{\boldsymbol{\theta}}_1^- \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_1^+ \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_1^- \otimes \boldsymbol{\mathcal{I}}\end{pmatrix} \bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} = \begin{pmatrix}\boldsymbol{\theta}_1^- \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_1^+ \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_1^- \otimes \boldsymbol{\mathcal{I}}\end{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^{-11}}$ in equation (32) guarantees that for $\forall r_1 \in [H_1]$,

$$\begin{pmatrix}\bar{\theta}_{1,r_1}^- \boldsymbol{\mathcal{I}} & \bar{\theta}_{1,r_1}^+ \boldsymbol{\mathcal{I}} & \bar{\theta}_{1,r_1}^- \boldsymbol{\mathcal{I}}\end{pmatrix} \bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} = \begin{pmatrix}\theta_{1,r_1}^- \boldsymbol{\mathcal{I}} & \theta_{1,r_1}^+ \boldsymbol{\mathcal{I}} & \theta_{1,r_1}^- \boldsymbol{\mathcal{I}}\end{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^{-11}}$$

Therefore, $\bar{\mathbf{T}}_{\boldsymbol{r}} \bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}} \boldsymbol{p}$.

- Similarly, for any $\boldsymbol{r}$ s.t. $r_1 = 0, r_2 \neq 0$, $\begin{pmatrix}\bar{\boldsymbol{\theta}}_2^- \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_2^- \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_2^+ \otimes \boldsymbol{\mathcal{I}}\end{pmatrix} \bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} = \begin{pmatrix}\boldsymbol{\theta}_2^- \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_2^- \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_2^+ \otimes \boldsymbol{\mathcal{I}}\end{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^{-11}}$ gives $\bar{\mathbf{T}}_{\boldsymbol{r}} \bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}} \boldsymbol{p}$.

- Similarly, for any $\boldsymbol{r}$ s.t. $r_1 \neq 0$, $r_2 \neq 0$, to ensure $\bar{\mathbf{T}}_{\boldsymbol{r}} \bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}} \boldsymbol{p}$, it suffices to have

$$\begin{pmatrix}\bar{\boldsymbol{\theta}}_1^- \otimes \bar{\boldsymbol{\theta}}_2^- \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_1^+ \otimes \bar{\boldsymbol{\theta}}_2^- \otimes \boldsymbol{\mathcal{I}} & \bar{\boldsymbol{\theta}}_1^- \otimes \bar{\boldsymbol{\theta}}_2^+ \otimes \boldsymbol{\mathcal{I}}\end{pmatrix} \bar{\boldsymbol{p}}_{\boldsymbol{g}^{-11}} = \begin{pmatrix}\boldsymbol{\theta}_1^- \otimes \boldsymbol{\theta}_2^- \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_1^+ \otimes \boldsymbol{\theta}_2^- \otimes \boldsymbol{\mathcal{I}} & \boldsymbol{\theta}_1^- \otimes \boldsymbol{\theta}_2^+ \otimes \boldsymbol{\mathcal{I}}\end{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^{-11}}.$$

Now we construct $(\bar{\boldsymbol{\theta}}_1^-, \bar{\boldsymbol{\theta}}_2^-, \bar{\boldsymbol{p}}_{\boldsymbol{g}^{00}}, \bar{\boldsymbol{p}}_{\boldsymbol{g}^{10}}, \bar{\boldsymbol{p}}_{\boldsymbol{g}^{01}})$ that satisfies $\bar{\boldsymbol{\mathcal{T}}} \bar{\boldsymbol{p}} = \boldsymbol{\mathcal{T}} \boldsymbol{p}$ as follows: let $\boldsymbol{p}_{\boldsymbol{g}^{10}} = \rho_1 \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}}$, $\boldsymbol{p}_{\boldsymbol{g}^{01}} = \rho_2 \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}}$, $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{00}} = u \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}}$, $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{10}} = v \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}}$ and $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{01}} = w \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}}$. Then $\bar{\boldsymbol{\mathcal{T}}} \bar{\boldsymbol{p}} = \boldsymbol{\mathcal{T}} \boldsymbol{p}$ can be simplified to

$$\bar{\boldsymbol{\mathcal{T}}} \begin{pmatrix} u \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}} \\ v \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}} \\ w \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}} \end{pmatrix} = \boldsymbol{\mathcal{T}} \begin{pmatrix} 1 \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}} \\ \rho_1 \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}} \\ \rho_2 \cdot \boldsymbol{p}_{\boldsymbol{g}^{00}} \end{pmatrix},$$

i.e., for $l_1 \in [H_1], \ \ l_2 \in [H_2]$,

$$\begin{cases} u\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + v\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + w\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} = \boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_1\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_2\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}}; \\ u\bar{\theta}^-_{1,l_1}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + v\bar{\theta}^+_{1,l_1}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + w\bar{\theta}^-_{1,l_1}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} = \theta^-_{1,l_1}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_1\theta^+_{1,l_1}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_2\theta^-_{1,l_1}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}}; \\ u\bar{\theta}^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + v\bar{\theta}^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + w\bar{\theta}^+_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} = \theta^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_1\theta^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_2\theta^+_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}}; \\ u\bar{\theta}^-_{1,l_1}\bar{\theta}^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + v\bar{\theta}^+_{1,l_1}\bar{\theta}^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + w\bar{\theta}^-_{1,l_1}\bar{\theta}^+_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} = \theta^-_{1,l_1}\theta^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_1\theta^+_{1,l_1}\theta^-_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}} + \rho_2\theta^-_{1,l_1}\theta^+_{2,l_2}\boldsymbol{\mathcal{I}}\boldsymbol{p}_{\boldsymbol{g}^{00}}. \end{cases}$$

Then it suffices to have

$$\begin{cases} u + v + w = 1 + \rho_1 + \rho_2; \\ u\bar{\theta}^-_{1,l_1} + v\bar{\theta}^+_{1,l_1} + w\bar{\theta}^-_{1,l_1} = \theta^-_{1,l_1} + \rho_1\theta^+_{1,l_1} + \rho_2\theta^-_{1,l_1}; \\ u\bar{\theta}^-_{2,l_2} + v\bar{\theta}^-_{2,l_2} + w\bar{\theta}^+_{2,l_2} = \theta^-_{2,l_2} + \rho_1\theta^-_{2,l_2} + \rho_2\theta^+_{2,l_2}; \\ u\bar{\theta}^-_{1,l_1}\bar{\theta}^-_{2,l_2} + v\bar{\theta}^+_{1,l_1}\bar{\theta}^-_{2,l_2} + w\bar{\theta}^-_{1,l_1}\bar{\theta}^+_{2,l_2} = \theta^-_{1,l_1}\theta^-_{2,l_2} + \rho_1\theta^+_{1,l_1}\theta^-_{2,l_2} + \rho_2\theta^-_{1,l_1}\theta^+_{2,l_2}. \end{cases} \tag{37}$$

Let some $\kappa \in (0, 1)$ s.t.

$$\begin{pmatrix} \theta^-_{1,l_1} \\ \bar{\theta}^-_{1,l_1} \\ \theta^+_{1,l_1} \\ \bar{\theta}^+_{1,l_1} \end{pmatrix} = \kappa^{l_1-1} \begin{pmatrix} \theta^-_{1,1} \\ \bar{\theta}^-_{1,1} \\ \theta^+_{1,1} \\ \bar{\theta}^+_{1,1} \end{pmatrix}, \text{ and } \begin{pmatrix} \theta^-_{2,l_2} \\ \bar{\theta}^-_{2,l_2} \\ \theta^+_{2,l_2} \\ \bar{\theta}^+_{2,l_2} \end{pmatrix} = \kappa^{l_2-1} \begin{pmatrix} \theta^-_{2,1} \\ \bar{\theta}^-_{2,1} \\ \theta^+_{2,1} \\ \bar{\theta}^+_{2,1} \end{pmatrix} \quad \text{for } \ l_1 \in [H_1], \ l_2 \in [H_2], \tag{38}$$

then equations (37) are now reduced to

$$\begin{cases} u+v+w=1+\rho_1+\rho_2; \\ \bar{\theta}^-_{1,1}+v\bar{\theta}^+_{1,1}+w\bar{\theta}^-_{1,1}=\theta^-_{1,1}+\rho_1\theta^+_{1,1}+\rho_2\theta^-_{1,1}; \\ u\bar{\theta}^-_{2,1}+v\bar{\theta}^-_{2,1}+w\bar{\theta}^+_{2,1}=\theta^-_{2,1}+\rho_1\theta^-_{2,1}+\rho_2\theta^+_{2,1}; \\ u\bar{\theta}^-_{1,1}\bar{\theta}^-_{2,1}+v\bar{\theta}^+_{1,1}\bar{\theta}^-_{2,1}+w\bar{\theta}^-_{1,1}\bar{\theta}^+_{2,1}=\theta^-_{1,1}\theta^-_{2,1}+\rho_1\theta^+_{1,1}\theta^-_{2,1}+\rho_2\theta^-_{1,1}\theta^+_{2,1}. \end{cases}$$

The above equation system contains $(u, v, w, \bar{\theta}^-_{1,1}, \bar{\theta}^-_{2,1})$ five parameters with four constraints, which gives infinitely many solutions. Thus the model parameters are not identifiable and condition C3 is necessary. □

A.3 Identifiability of Sequential DINA model

Theorem 2. *The Sequential DINA model parameters are identifiable if the $\mathbf{Q}^1$ matrix satisfies the following conditions S1-S3.*

Condition S1. *$\mathbf{Q}^1$-matrix is complete, i.e., under some permutation, $\mathbf{Q}^1_{1:K} = \boldsymbol{\mathcal{I}}_K$.*

Condition S2. *Each of the K attributes is required by at least three items' first categories.*

Condition S3. *Suppose $\mathbf{Q}^1_{1:K} = \boldsymbol{\mathcal{I}}_K$, then any two different columns of $\mathbf{Q}^1_{K+1:J}$ are distinct.*

Proof. We begin by showing that when $\mathbf{Q}^1$ meets the conditions S1-S3, $\bar{\mathbf{T}}^s\bar{\boldsymbol{p}} = \mathbf{T}^s\boldsymbol{p}$ gives $\bar{\boldsymbol{p}} = \boldsymbol{p}$ and $\bar{\beta}^+_{j,1} = \beta^+_{j,1}$, $\bar{\beta}^-_{j,1} = \beta^-_{j,1}$, for $j \in [J]$. As discussed in Section 3.3, the parameters $\big(\boldsymbol{p},\ (\beta^+_{j,1})_{j\in[J]},\ (\beta^-_{j,1})_{j\in[J]}\big)$ can be interpreted as parameters in the reduced binary DINA model. In this model, the binary item takes the form $I(\text{item } j \geq 1)$, and the corresponding **q**-vectors are $(\mathbf{q}_{j,1})_{j\in[J]}$, which indicates the attributes required to complete the first categories.

Moreover, $\mathbf{Q}^1$ is the $\mathbf{Q}$-matrix for this reduced DINA model, so the item parameters only involve $(\beta^+_{j,1})_{j\in[J]}, (\beta^-_{j,1})_{j\in[J]}$. Since the $\mathbf{Q}$-matrix for this reduced binary DINA model satisfies conditions S1-S3, according to the sufficient condition for the binary DINA model in Gu and Xu (2019b), parameters $\boldsymbol{p}$, $(\beta^+_{j,1})_{j\in[J]}$, $(\beta^-_{j,1})_{j\in[J]}$ are identified, i.e., $\bar{\boldsymbol{p}} = \boldsymbol{p}$ and $\bar{\beta}^+_{j,1} = \beta^+_{j,1}, \bar{\beta}^-_{j,1} = \beta^-_{j,1}$ for $j \in [J]$.

Next we identify $\beta^+_{j,l}, \beta^-_{j,l}$, for $l > 1$, by induction. Suppose categories $h\,(h < l)$ of item j have been identified, i.e.,

$$\bar{\beta}^+_{j,h} = \beta^+_{j,h}, \;\; \bar{\beta}^-_{j,h} = \beta^-_{j,h} \text{ for } h < l. \tag{39}$$

For each item j, we will use two rows of the $\mathbf{T}^s$-matrix in $\bar{\mathbf{T}}^s\bar{\boldsymbol{p}} = \mathbf{T}^s\boldsymbol{p}$ to infer that $\bar{\beta}^+_{j,l} = \beta^+_{j,l}$ and $\bar{\beta}^-_{j,l} = \beta^-_{j,l}$. For item j, its category l requires some attribute, and WLOG, suppose this is α_1. There are two possible cases.

Case 1: category l of item j requires solely α_1, i.e., $\mathbf{q}_{j,l} = \boldsymbol{e}_1^\top$. According to condition S2, there exists some other item j' whose first category requires α_1. So the $\mathbf{Q}$-matrix takes the following form:

$$\mathbf{Q} = \left(\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \textit{item } j, \textit{category } l & 1 & 0 & \dots & 0 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \textit{item } j', \textit{category } 1 & 1 & & \boldsymbol{v}^\top & \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right). \tag{40}$$

In this case, $\xi_{j',1,\boldsymbol{\alpha}} \leq \xi_{j,l,\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha}$, consider vectors

$$\mathbf{T}^s_{l\boldsymbol{e}_j} = P(R_j \geq l \mid \mathbf{Q}, \boldsymbol{\beta}^+, \boldsymbol{\beta}^-),$$

$$\mathbf{T}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}} = P(R_j \geq l, R_{j'} \geq 1 \mid \mathbf{Q}, \boldsymbol{\beta}^+, \boldsymbol{\beta}^-).$$

According to the assumption in equation (39), we have $t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}} = \bar{t}^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}$. Therefore,

$$t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}, \quad \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \bar{\beta}^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \bar{\beta}^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}. \tag{41}$$

Since $\bar{\beta}^+_{j',1} = \beta^+_{j',1}$, $\bar{\beta}^-_{j',1} = \beta^-_{j',1}$, we have

$$t^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'},\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j',1}\, t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j',1,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j',1}\, t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j',1,\boldsymbol{\alpha}} = 0 \end{cases}, \quad \bar{t}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'},\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j',1}\, \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} & \xi_{j',1,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j',1}\, \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j',1,\boldsymbol{\alpha}} = 0 \end{cases}. \tag{42}$$

Since $\bar{\mathbf{T}}^s_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}^s_{\boldsymbol{r}}\boldsymbol{p}$ for $\boldsymbol{r} = l\boldsymbol{e}_j$ and $\boldsymbol{r} = l\boldsymbol{e}_j + \boldsymbol{e}_{j'}$, $\bar{\boldsymbol{p}} = \boldsymbol{p}$, we have

$$\begin{cases} \beta^-_{j',1}(\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j} - \mathbf{T}^s_{l\boldsymbol{e}_j})\boldsymbol{p} = 0 \\ (\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}} - \mathbf{T}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}})\boldsymbol{p} = 0 \end{cases}$$

$$\Rightarrow [(\beta^-_{j',1}\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j} - \bar{\mathbf{T}}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}}) - (\beta^-_{j',1}\mathbf{T}^s_{l\boldsymbol{e}_j} - \mathbf{T}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}})]\boldsymbol{p} = 0.$$

Using equations (41-42), we have $(\beta^-_{j',1} - \beta^+_{j',1}) \sum_{\boldsymbol{\alpha}:\, \xi_{j',1,\boldsymbol{\alpha}=1}} (\bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} - t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}})\, p_{\boldsymbol{\alpha}} = 0$. According to the constructed $\mathbf{Q}$-matrix (40), $\xi_{j',1,\boldsymbol{\alpha}} = 1$ must imply $\xi_{j,l,\boldsymbol{\alpha}} = 1$. Therefore,

$$(\beta^-_{j',1} - \beta^+_{j',1})(\bar{\beta}^+_{j,l} - \beta^+_{j,l}) \sum_{\boldsymbol{\alpha}:\, \xi_{j',1,\boldsymbol{\alpha}=1}} t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}\, p_{\boldsymbol{\alpha}} = 0,$$

from which we conclude that $\bar{\beta}^+_{j,l} = \beta^+_{j,l}$.

Next consider $\mathbf{T}^s_{l\boldsymbol{e}_j}$: with $\bar{\beta}^+_{j,l} = \beta^+_{j,l}$, equation (41) can be written as

$$t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}, \quad \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \bar{\beta}^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}. \tag{43}$$

With $\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j}\bar{\boldsymbol{p}} = \mathbf{T}^s_{l\boldsymbol{e}_j}\boldsymbol{p}$, $\bar{\boldsymbol{p}} = \boldsymbol{p}$ and equation (43), we have

$$\begin{aligned} &(\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j} - \mathbf{T}^s_{l\boldsymbol{e}_j})\boldsymbol{p} = 0 \\ \Rightarrow &(\bar{\beta}^-_{j,l} - \beta^-_{j,l}) \cdot \sum_{\boldsymbol{\alpha}:\, \xi_{j,l,\boldsymbol{\alpha}}=0} t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}\, p_{\boldsymbol{\alpha}} = 0 \\ \Rightarrow &\bar{\beta}^-_{j,l} = \beta^-_{j,l}. \end{aligned}$$

Case 2: category l of item j requires some other attribute. WLOG, assume that this is the second attribute α_2. Then according to condition S1, α_1 or α_2 is required solely by some other item's first category. WLOG, assume that it is α_1, and it is item j''s ($j' \neq j$) first category that requires α_1, i.e., $\mathbf{q}_{j',1} = \boldsymbol{e}_1^\top$. So the $\mathbf{Q}$-matrix can be written as follows:

$$\mathbf{Q} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \textit{item } j, \textit{category } l & 1 & 1 & \boldsymbol{v}^\top & \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline \textit{item } j', \textit{category } 1 & 1 & 0 & \dots & 0 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}. \tag{44}$$

In this case, $\xi_{j,l,\boldsymbol{\alpha}} \leq \xi_{j',1,\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha}$. Consider vectors

$$\mathbf{T}^s_{l\boldsymbol{e}_j} = P(R_j \geq l \mid \mathbf{Q}, \boldsymbol{\beta}^+, \boldsymbol{\beta}^-), \tag{45}$$

$$\mathbf{T}^s_{l\boldsymbol{e}_j + \boldsymbol{e}_{j'}} = P(R_j \geq l, R_{j'} \geq 1 \mid \mathbf{Q}, \boldsymbol{\beta}^+, \boldsymbol{\beta}^-). \tag{46}$$

According to assumption (39), we have $t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}} = \bar{t}^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}$. Therefore,

$$t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}, \quad \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \bar{\beta}^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \bar{\beta}^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}. \tag{47}$$

Since $\bar{\beta}^+_{j',1} = \beta^+_{j',1}$, $\bar{\beta}^-_{j',1} = \beta^-_{j',1}$, we have

$$t^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'},\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j',1}\, t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j',1,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j',1}\, t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j',1,\boldsymbol{\alpha}} = 0 \end{cases}, \quad \bar{t}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'},\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j',1}\, \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} & \xi_{j',1,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j',1}\, \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j',1,\boldsymbol{\alpha}} = 0 \end{cases}. \tag{48}$$

Since $\bar{\mathbf{T}}^s_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}^s_{\boldsymbol{r}}\boldsymbol{p}$ holds for $\boldsymbol{r} = l\boldsymbol{e}_j$ and $\boldsymbol{r} = l\boldsymbol{e}_j + \boldsymbol{e}_{j'}$, and $\bar{\boldsymbol{p}} = \boldsymbol{p}$,

$$\begin{cases} \beta^+_{j',1}(\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j} - \mathbf{T}^s_{l\boldsymbol{e}_j})\boldsymbol{p} = 0 \\ (\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}} - \mathbf{T}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}})\boldsymbol{p} = 0 \end{cases}$$

$$\Rightarrow [(\beta^+_{j',1}\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j} - \bar{\mathbf{T}}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}}) - (\beta^+_{j',1}\mathbf{T}^s_{l\boldsymbol{e}_j} - \mathbf{T}^s_{l\boldsymbol{e}_j+\boldsymbol{e}_{j'}})]\boldsymbol{p} = 0.$$

Using equations (47-48), we have $(\beta^+_{j',1} - \beta^-_{j',1}) \sum_{\boldsymbol{\alpha}\,:\,\xi_{j',1,\boldsymbol{\alpha}=0}} (\bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} - t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}})\, p_{\boldsymbol{\alpha}} = 0$. According to the constructed $\mathbf{Q}$-matrix (44), $\xi_{j',1,\boldsymbol{\alpha}} = 0$ must imply $\xi_{j,l,\boldsymbol{\alpha}} = 0$. Therefore, using equation (47), we have

$$(\beta^+_{j',1} - \beta^-_{j',1})(\bar{\beta}^-_{j,l} - \beta^-_{j,l}) \sum_{\boldsymbol{\alpha}:\,\xi_{j',1,\boldsymbol{\alpha}=0}} t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}\, p_{\boldsymbol{\alpha}} = 0,$$

which we conclude that $\bar{\beta}^-_{j,l} = \beta^-_{j,l}$. Next consider $\mathbf{T}^s_{l\boldsymbol{e}_j}$: with $\bar{\beta}^-_{j,l} = \beta^-_{j,l}$, equation (47) can be written as

$$t^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \beta^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}, \quad \bar{t}^s_{l\boldsymbol{e}_j,\boldsymbol{\alpha}} = \begin{cases} \bar{\beta}^+_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 1 \\ \beta^-_{j,l}\, t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}, & \xi_{j,l,\boldsymbol{\alpha}} = 0 \end{cases}. \tag{49}$$

Using $\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j}\bar{\boldsymbol{p}} = \mathbf{T}^s_{l\boldsymbol{e}_j}\boldsymbol{p}$ and $\bar{\boldsymbol{p}} = \boldsymbol{p}$, and equation (49), we have

$$\begin{aligned} &(\bar{\mathbf{T}}^s_{l\boldsymbol{e}_j} - \mathbf{T}^s_{l\boldsymbol{e}_j})\boldsymbol{p} = 0 \\ \Rightarrow &(\bar{\beta}^+_{j,l} - \beta^+_{j,l}) \cdot \sum_{\boldsymbol{\alpha}\,:\,\xi_{j,l,\boldsymbol{\alpha}}=1} t^s_{(l-1)\boldsymbol{e}_j,\boldsymbol{\alpha}}\, p_{\boldsymbol{\alpha}} = 0 \\ \Rightarrow &\bar{\beta}^+_{j,l} = \beta^+_{j,l}. \end{aligned}$$

Therefore, for both cases, $\beta^+_{j,l}$ and $\beta^-_{j,l}$ are identified. By induction, we conclude that all parameters $(\boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{p})$ are identifiable and conditions S1-S3 are sufficient. □

Proposition 4 (Necessity of Conditions S2* and S3*)**.** *The Sequential DINA model parameters are identifiable only if the* $\mathbf{Q}$*-matrix satisfies the following conditions S2* and S3*.*

Condition S2* *Each of the K attributes is required by at least three categories (not necessarily the first categories), and the three categories must come from at least two different items.*

Condition S3* *Suppose* $\mathbf{Q}$*-matrix satisfies S1, i.e.,* $\mathbf{Q}^1_{1:K} = \boldsymbol{\mathcal{I}}_K$*, and any two different columns of the following matrix (which removes the identity matrix of* $\mathbf{Q}^1_{1:K}$ *from* $\mathbf{Q}$*)*

$$\begin{pmatrix} \mathbf{Q}^{-1}_{1:K} \\ \mathbf{Q}_{K+1:J} \end{pmatrix}$$

are distinct, where $\mathbf{Q}^{-1}_{1:K}$ *denotes the remaining submatrix of* $\mathbf{Q}_{1:K}$ *after removing* $\mathbf{Q}^1_{1:K}$*.*

Proof. We have shown the necessity of condition S1 in Proposition 3, so we may assume that the $\mathbf{Q}$-matrix satisfies condition S1.

Necessity of Condition S2*. Suppose the $\mathbf{Q}^1$ matrix satisfies condition S1, but does not satisfy condition S2*. We first show that each attribute must be required by more than one item. Suppose there exists some attribute that is only required by one item. WLOG, assume this is attribute one α_1, and is only required by the first item. So the $\mathbf{Q}$-matrix can be written as

$$\mathbf{Q} = \begin{pmatrix} \text{item } 1 & \begin{pmatrix} 1 & \mathbf{0}^\top \\ * & * \end{pmatrix} \\ \hdashline \text{item } (2:J) & \begin{pmatrix} \mathbf{0} & * \end{pmatrix} \end{pmatrix}. \tag{50}$$

We partition $\boldsymbol{\alpha}$ into two groups according to the first attribute:

$$\boldsymbol{g}^0 = \{\boldsymbol{\alpha} : \alpha_1 = 0\} = \{\boldsymbol{\alpha} = (0, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}\},$$

$$\boldsymbol{g}^1 = \{\boldsymbol{\alpha} : \alpha_1 = 1\} = \{\boldsymbol{\alpha} = (1, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-1}\}.$$

So each group has 2^{K-1} attribute profiles, and we index the entries in each group by

$$\boldsymbol{g}_1^0 = (0, \mathbf{0}),\ \boldsymbol{g}_2^0 = (0, \boldsymbol{e}_1), \ldots, \boldsymbol{g}_K^0 = (0, \boldsymbol{e}_{K-1}),\ \boldsymbol{g}_{K+1}^0 = (0, \boldsymbol{e}_1 + \boldsymbol{e}_2), \ldots, \boldsymbol{g}_{2^{K-1}}^0 = \left(0, \sum_{k=1}^{K-1} \boldsymbol{e}_k\right),$$

$$\boldsymbol{g}_1^1 = (1, \mathbf{0}),\ \boldsymbol{g}_2^1 = (1, \boldsymbol{e}_1), \ldots, \boldsymbol{g}_K^1 = (1, \boldsymbol{e}_{K-1}),\ \boldsymbol{g}_{K+1}^1 = (1, \boldsymbol{e}_1 + \boldsymbol{e}_2), \ldots, \boldsymbol{g}_{2^{K-1}}^1 = \left(1, \sum_{k=1}^{K-1} \boldsymbol{e}_k\right),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{K-1} \in \{0,1\}^{K-1}$ have $K-1$ elements. Therefore, the k-th ($k \in [2^{K-1}]$) entry of $\boldsymbol{g}^0$ and $\boldsymbol{g}^1$: $\boldsymbol{g}_k^0$ and $\boldsymbol{g}_k^1$, share the same attributes except for the first one α_1. Index the population proportion parameters $\boldsymbol{p}$ in the following way:

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_{\boldsymbol{g}^0} \\ \boldsymbol{p}_{\boldsymbol{g}^1} \end{pmatrix}, \quad \text{where } \boldsymbol{p}_{\boldsymbol{g}^0} = \begin{pmatrix} p_{\boldsymbol{g}_1^0} \\ p_{\boldsymbol{g}_2^0} \\ \vdots \\ p_{\boldsymbol{g}_{2^{K-1}}^0} \end{pmatrix} \quad \text{and } \boldsymbol{p}_{\boldsymbol{g}^1} = \begin{pmatrix} p_{\boldsymbol{g}_1^1} \\ p_{\boldsymbol{g}_2^1} \\ \vdots \\ p_{\boldsymbol{g}_{2^{K-1}}^1} \end{pmatrix}.$$

Recall that $\boldsymbol{\beta}_j^+ = \left(\beta_{j,1}^+,\ \beta_{j,1}^+\beta_{j,2}^+,\ \dots\ \prod_{l=1}^{H_j}\beta_{j,l}^+\right)$, $\boldsymbol{\beta}_j^- = \left(\beta_{j,1}^-,\ \beta_{j,1}^-\beta_{j,2}^-,\ \dots\ \prod_{l=1}^{H_j}\beta_{j,l}^-\right)$, for $j \in [J]$. Item parameters $\boldsymbol{\beta}^+ = (\boldsymbol{\beta}_1^+, \boldsymbol{\beta}_2^+, \dots, \boldsymbol{\beta}_J^+)$ and $\boldsymbol{\beta}^- = (\boldsymbol{\beta}_1^-, \boldsymbol{\beta}_2^-, \dots, \boldsymbol{\beta}_J^-)$. We now seek to construct $(\boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{p}) \neq (\bar{\boldsymbol{\beta}}^+, \bar{\boldsymbol{\beta}}^-, \bar{\boldsymbol{p}})$ such that (12) holds:

1. Take $\boldsymbol{\beta}_j^+ = \bar{\boldsymbol{\beta}}_j^+$, $\boldsymbol{\beta}_j^- = \bar{\boldsymbol{\beta}}_j^-$ for $j > 1$.
2. $\beta_{1,1}^- = \bar{\beta}_{1,1}^- = 0$, $\bar{\beta}_{1,l_1}^+ = \beta_{1,l_1}^+$, $\bar{\beta}_{1,l_1}^- = \beta_{1,l_1}^-$ for $l_1 > 1$.
3. $\bar{\boldsymbol{p}}_{\boldsymbol{g}^1} = \rho \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$, $\boldsymbol{p}_{\boldsymbol{g}^0} = u \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$, and $\boldsymbol{p}_{\boldsymbol{g}^1} = v \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$.

According to Lemma 2, in order for equation (12) to hold, it suffices to show that $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ holds for $\forall \boldsymbol{r}$.

- For any $\boldsymbol{r}$ s.t. $r_1 = 0$, $t_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv t_{\boldsymbol{r},\boldsymbol{g}_k^1}$ for $k \in [2^{K-1}]$. Since $\boldsymbol{\beta}_j^+ = \bar{\boldsymbol{\beta}}_j^+$, $\boldsymbol{\beta}_j^- = \bar{\boldsymbol{\beta}}_j^-$ for $j > 1$, $t_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv t_{\boldsymbol{r},\boldsymbol{g}_k^1} \equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^1}$ for $k \in [2^{K-1}]$, to ensure $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$, it suffices to have $1 + \rho = u + v$.
- For any $\boldsymbol{r}$ s.t. $r_1 = 1$, since $\beta_{1,1}^- = 0$, $t_{\boldsymbol{r},\boldsymbol{g}_k^0} = 0$, it suffices to have $\rho\bar{\beta}_{1,1}^+ = v\beta_{1,1}^+$ to guarantee $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$.
- For any $\boldsymbol{r}$ s.t. $r_1 > 1$, since $\beta_{1,1}^- = 0$ and $\bar{\beta}_{1,l_1}^+ = \beta_{1,l_1}^+$, $\bar{\beta}_{1,l_1}^- = \beta_{1,l_1}^-$ for $l_1 > 1$, $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ holds without additional conditions.

With three parameters $(u, v, \beta_{1,1}^+)$ and two constraints, the equation system has infinitely many solutions. Thus the construction exists. Therefore, each attribute must be required by more than one item.

Now suppose the $\mathbf{Q}$-matrix satisfies condition S1 and each attribute is required by at least two items, and suppose that there exists some attribute which is required by at most two categories. WLOG, assume this is α_1, and it is the first and second items that require

α_1. Assume that category $l_2^* \in [H_2]$ of item 2 requires α_1 and other categories of item 2 do not require. So the $\mathbf{Q}$ can be written as

$$\mathbf{Q} = \begin{pmatrix} \text{item 1} & \begin{cases} 1 & \mathbf{0}^\top \\ \mathbf{0} & * \end{cases} \\ \hdashline \text{item 2} & \begin{cases} \mathbf{0} & * \\ 1 & \boldsymbol{v}^\top \\ \mathbf{0} & * \end{cases} \\ \hdashline \text{item 3:J} & \begin{cases} \mathbf{0} & * \end{cases} \end{pmatrix}. \tag{51}$$

We now seek to construct $(\bar{\boldsymbol{\beta}}^+, \bar{\boldsymbol{\beta}}^-, \bar{\boldsymbol{p}}) \neq (\boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{p})$ such that $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ holds for $\forall \boldsymbol{r}$.

1. Take $\boldsymbol{\beta}_j^+ = \bar{\boldsymbol{\beta}}_j^+$, $\boldsymbol{\beta}_j^- = \bar{\boldsymbol{\beta}}_j^-$ for $j > 2$.

2. For $l_1 > 1$, $\bar{\beta}_{1,l_1}^- = \beta_{1,l_1}^-$ and $\bar{\beta}_{1,l_1}^+ = \beta_{1,l_1}^+$.

3. For $l_2 \neq l_2^*$, $\bar{\beta}_{2,l_2}^+ = \beta_{2,l_2}^+$, for $l_2 \in [H_2]$, $\bar{\beta}_{2,l_2}^- = \beta_{2,l_2}^-$.

4. $\bar{\boldsymbol{p}}_{\boldsymbol{g}^1} = \rho \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$, $\boldsymbol{p}_{\boldsymbol{g}^0} = u \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$, and $\boldsymbol{p}_{\boldsymbol{g}^1} = v \cdot \bar{\boldsymbol{p}}_{\boldsymbol{g}^0}$.

So the remaining parameters are $(\beta_{1,1}^+, \beta_{1,1}^-, \beta_{2,l_2^*}^+, \rho, u, v)$. We partition $\boldsymbol{\alpha}$ into two groups according to the first attribute as we did before.

- For any $\boldsymbol{r}$ s.t. $r_1 = 0, r_2 < l_2^*$, $t_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv t_{\boldsymbol{r},\boldsymbol{g}_k^1}$ for $k \in [2^{K-1}]$. Since $\boldsymbol{\beta}_j^+ = \bar{\boldsymbol{\beta}}_j^+$, $\boldsymbol{\beta}_j^- = \bar{\boldsymbol{\beta}}_j^-$ for $j > 1$, $t_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv t_{\boldsymbol{r},\boldsymbol{g}_k^1} \equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^0} \equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^1}$ for $k \in [2^{K-1}]$. To ensure $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$, it suffices to have $1 + \rho = u + v$.

- For any $\boldsymbol{r}$ s.t. $r_1 = 0, r_2 = l_2^*$, it suffices to have $\bar{\beta}_{2,l_2^*}^- + \rho\bar{\beta}_{2,l_2^*}^+ = u\beta_{2,l_2^*}^- + v\beta_{2,l_2^*}^+$. When this is met, for any $\boldsymbol{r}$ s.t. $r_1 = 0$, $r_2 > l_2^*$, $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ also holds, since for $l_2 > l_2^*$, $\bar{\beta}_{2,l_2}^+ = \beta_{2,l_2}^+$, $\bar{\beta}_{2,l_2}^- = \beta_{2,l_2}^-$.

- For any $\boldsymbol{r}$ s.t. $r_1 = 1$, $r_2 = 0$, it suffices to have $\bar{\beta}_{1,1}^- + \rho\bar{\beta}_{1,1}^+ = u\beta_{1,1}^- + v\beta_{1,1}^+$ to guarantee $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$. When this is met, for any $\boldsymbol{r}$ s.t. $r_1 > 1$, $r_2 = 0$, $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ also holds. Since for $l_1 > 1$, $\bar{\beta}_{1,l_1}^- = \beta_{1,l_1}^-$ and $\bar{\beta}_{1,l_1}^+ = \beta_{1,l_1}^+$.

- For any $\boldsymbol{r}$ s.t. $r_1 \neq 0$ and $r_2 \neq 0$, we need $\bar{\beta}_{1,1}^-\bar{\beta}_{2,l_2^*}^- + \rho\bar{\beta}_{1,1}^+\bar{\beta}_{2,l_2^*}^+ = u\beta_{1,1}^-\beta_{2,l_2^*}^- + v\beta_{1,1}^+\beta_{2,l_2^*}^+$.

With six parameters and four constraints, the equation system has infinitely many solutions. Thus the construction exists. Thus the parameters are not identifiable and the condition S2* is necessary.

Necessity of Condition S3*. Suppose $\mathbf{Q}$-matrix satisfies condition S1 and S2*, but does not satisfy condition S3*, i.e., there exists two columns of the matrix

$$\begin{pmatrix} \mathbf{Q}_{1:K}^{-1} \\ \mathbf{Q}_{K+1:J} \end{pmatrix}$$

are the same, and WLOG, assume they are columns 1 and 2. Since $\mathbf{Q}$ satisfies S1, $\mathbf{Q}^1_{1:K} = \boldsymbol{\mathcal{I}}_K$, so the $\mathbf{Q}$-matrix can be written as

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_1 \\ \hdashline \mathbf{Q}_2 \\ \hdashline \mathbf{Q}_{3:J} \end{pmatrix} = \begin{pmatrix} \text{item } 1 & \begin{cases} 1 & 0 & \ldots \\ \boldsymbol{v}_1 & \boldsymbol{v}_1 & \ldots \end{cases} \\ \hdashline \text{item } 2 & \begin{cases} 0 & 1 & \ldots \\ \boldsymbol{v}_2 & \boldsymbol{v}_2 & \ldots \end{cases} \\ \hdashline \text{item } (3:J) & \begin{cases} \boldsymbol{v} & \boldsymbol{v} & \ldots \end{cases} \end{pmatrix}.$$

We partition $\boldsymbol{\alpha}$ into four groups according to the first and the second attribute:

$$\boldsymbol{g}^{00} = \{\boldsymbol{\alpha} : \alpha_1 = 0, \alpha_2 = 0\} = \{\boldsymbol{\alpha} = (0, 0, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\},$$

$$\boldsymbol{g}^{10} = \{\boldsymbol{\alpha} : \alpha_1 = 1, \alpha_2 = 0\} = \{\boldsymbol{\alpha} = (1, 0, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\},$$

$$\boldsymbol{g}^{01} = \{\boldsymbol{\alpha} : \alpha_1 = 0, \alpha_2 = 1\} = \{\boldsymbol{\alpha} = (0, 1, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\},$$

$$\boldsymbol{g}^{11} = \{\boldsymbol{\alpha} : \alpha_1 = 1, \alpha_2 = 1\} = \{\boldsymbol{\alpha} = (1, 1, \boldsymbol{\alpha}^*),\ \boldsymbol{\alpha}^* \in \{0,1\}^{K-2}\}.$$

So each group has 2^{K-2} attribute profiles, and we index the entries in each group by

$$\boldsymbol{g}^{00}_1 = (0, 0, \mathbf{0}),\ \boldsymbol{g}^{00}_2 = (0, 0, \boldsymbol{e}_1), \ldots, \boldsymbol{g}^{00}_K = (0, 0, \boldsymbol{e}_{K-2}),\ \boldsymbol{g}^{00}_{K+1} = (0, 0, \boldsymbol{e}_1 + \boldsymbol{e}_2), \ldots, \boldsymbol{g}^{00}_{2^{K-2}} = \left(0, 0, \textstyle\sum_{k=1}^{K-2} \boldsymbol{e}_k\right),$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{K-2} \in \{0,1\}^{K-2}$ have $K-2$ elements. Similarly we index the elements of $\boldsymbol{g}^{10}$, $\boldsymbol{g}^{01}$, $\boldsymbol{g}^{11}$. Therefore, $\boldsymbol{g}^{00}_k$, $\boldsymbol{g}^{10}_k$, $\boldsymbol{g}^{01}_k$ and $\boldsymbol{g}^{11}_k$ for $k \in [2^{K-2}]$ share the same attributes except for the first and second attributes. Index the population proportion parameters $\boldsymbol{p}$ in the following way:

$$\boldsymbol{p}=\begin{pmatrix}\boldsymbol{p}_{\boldsymbol{g}^{00}}\\ \boldsymbol{p}_{\boldsymbol{g}^{10}}\\ \boldsymbol{p}_{\boldsymbol{g}^{01}}\\ \boldsymbol{p}_{\boldsymbol{g}^{11}}\end{pmatrix},\quad \text{where } \boldsymbol{p}_{\boldsymbol{g}^{00}}=\begin{pmatrix}p_{\boldsymbol{g}_1^{00}}\\ p_{\boldsymbol{g}_2^{00}}\\ \vdots\\ p_{\boldsymbol{g}_{2^{K-2}}^{00}}\end{pmatrix},\ \boldsymbol{p}_{\boldsymbol{g}^{10}}=\begin{pmatrix}p_{\boldsymbol{g}_1^{10}}\\ p_{\boldsymbol{g}_2^{10}}\\ \vdots\\ p_{\boldsymbol{g}_{2^{K-2}}^{10}}\end{pmatrix},\ \boldsymbol{p}_{\boldsymbol{g}^{01}}=\begin{pmatrix}p_{\boldsymbol{g}_1^{01}}\\ p_{\boldsymbol{g}_2^{01}}\\ \vdots\\ p_{\boldsymbol{g}_{2^{K-2}}^{01}}\end{pmatrix} \text{ and } \boldsymbol{p}_{\boldsymbol{g}^{11}}=\begin{pmatrix}p_{\boldsymbol{g}_1^{11}}\\ p_{\boldsymbol{g}_2^{11}}\\ \vdots\\ p_{\boldsymbol{g}_{2^{K-2}}^{11}}\end{pmatrix}.$$

We now seek to construct $(\bar{\boldsymbol{\beta}}^+,\bar{\boldsymbol{\beta}}^-,\bar{\boldsymbol{p}})\neq(\boldsymbol{\beta}^+,\boldsymbol{\beta}^-,\boldsymbol{p})$ such that $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}}=\mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ holds for $\forall\boldsymbol{r}$.

1. Take $\boldsymbol{\beta}_j^+=\bar{\boldsymbol{\beta}}_j^+,\ \boldsymbol{\beta}_j^-=\bar{\boldsymbol{\beta}}_j^-$ for $j>2$.

2. For $l_1>1$, $\bar{\beta}_{1,l_1}^-=\beta_{1,l_1}^-$ and $\bar{\beta}_{1,l_1}^+=\beta_{1,l_1}^+$; $\bar{\beta}_{1,1}^+=\beta_{1,1}^+$.

3. For $l_2>1$, $\bar{\beta}_{2,l_2}^+=\beta_{2,l_2}^+$ and $\bar{\beta}_{2,l_2}^-=\beta_{2,l_2}^-$; $\bar{\beta}_{2,1}^+=\beta_{2,1}^+$.

4. $\boldsymbol{p}_{\boldsymbol{g}^{11}}=\bar{\boldsymbol{p}}_{\boldsymbol{g}^{11}},\ \boldsymbol{p}_{\boldsymbol{g}^{00}}=\rho_1\boldsymbol{p}_{\boldsymbol{g}^{11}},\ \boldsymbol{p}_{\boldsymbol{g}^{10}}=\rho_2\boldsymbol{p}_{\boldsymbol{g}^{11}},\ \ \boldsymbol{p}_{\boldsymbol{g}^{01}}=\rho_3\boldsymbol{p}_{\boldsymbol{g}^{11}},$

   $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{00}}=u_1\boldsymbol{p}_{\boldsymbol{g}^{11}},\ \bar{\boldsymbol{p}}_{\boldsymbol{g}^{10}}=u_2\boldsymbol{p}_{\boldsymbol{g}^{11}}$ and $\bar{\boldsymbol{p}}_{\boldsymbol{g}^{01}}=u_3\boldsymbol{p}_{\boldsymbol{g}^{11}}$.

So the remaining parameters are $(\bar{\beta}_{1,1}^-,\bar{\beta}_{2,1}^-,u_1,u_2,u_3)$.

- For any $\boldsymbol{r}$ s.t. $r_1=0,r_2=0$, the response does not require α_1 and α_2, so $t_{\boldsymbol{r},\boldsymbol{g}_k^{00}}\equiv t_{\boldsymbol{r},\boldsymbol{g}_k^{10}}\equiv t_{\boldsymbol{r},\boldsymbol{g}_k^{01}}\equiv t_{\boldsymbol{r},\boldsymbol{g}_k^{11}}$ for $k\in[2^{K-2}]$. Since $\boldsymbol{\beta}_j^+=\bar{\boldsymbol{\beta}}_j^+,\ \boldsymbol{\beta}_j^-=\bar{\boldsymbol{\beta}}_j^-$ for $j>1$,

  $$t_{\boldsymbol{r},\boldsymbol{g}_k^{00}}\equiv t_{\boldsymbol{r},\boldsymbol{g}_k^{10}}\equiv t_{\boldsymbol{r},\boldsymbol{g}_k^{01}}\equiv t_{\boldsymbol{r},\boldsymbol{g}_k^{11}}\equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^{00}}\equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^{10}}\equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^{01}}\equiv \bar{t}_{\boldsymbol{r},\boldsymbol{g}_k^{11}},\ \ \text{for } k\in[2^{K-2}].$$

  To ensure $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}}=\mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$, it suffices to have $\rho_1+\rho_2+\rho_3=u_1+u_2+u_3$.

- For any $\boldsymbol{r}$ s.t. $r_1=1,r_2=0$, it suffices to have $\rho_1\beta_{1,1}^-+\rho_2\beta_{1,1}^++\rho_3\beta_{1,1}^-+\beta_{1,1}^+=u_1\bar{\beta}_{1,1}^-+u_2\bar{\beta}_{1,1}^++u_3\bar{\beta}_{1,1}^-+\bar{\beta}_{1,1}^+$ to ensure $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}}=\mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$. Since $\beta_{1,1}^+=\bar{\beta}_{1,1}^+$, it suffices to have $\rho_1\beta_{1,1}^- + \rho_2\beta_{1,1}^+ + \rho_3\beta_{1,1}^- = u_1\bar{\beta}_{1,1}^- + u_2\bar{\beta}_{1,1}^+ + u_3\bar{\beta}_{1,1}^-$. When this is met, for any $\boldsymbol{r}$ s.t. $r_1 > 1$, $r_2 = 0$, $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ also holds, since for $l_1 > 1$, $\bar{\beta}_{1,l_1}^- = \beta_{1,l_1}^-$ and $\bar{\beta}_{1,l_1}^+ = \beta_{1,l_1}^+$.

- For any $\boldsymbol{r}$ s.t. $r_1 = 0$, $r_2 = 1$, it suffices to have $\rho_1\beta_{2,1}^- + \rho_2\beta_{2,1}^- + \rho_3\beta_{2,1}^+ + \beta_{2,1}^+ = u_1\bar{\beta}_{2,1}^- + u_2\bar{\beta}_{2,1}^- + u_3\bar{\beta}_{2,1}^+ + \bar{\beta}_{2,1}^+$ to ensure $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$. Since $\beta_{2,1}^+ = \bar{\beta}_{2,1}^+$, it suffices to have $\rho_1\beta_{2,1}^- + \rho_2\beta_{2,1}^- + \rho_3\beta_{2,1}^+ = u_1\bar{\beta}_{2,1}^- + u_2\bar{\beta}_{2,1}^- + u_3\bar{\beta}_{2,1}^+$. When this is met, for any $\boldsymbol{r}$ s.t. $r_1 = 0$, $r_2 > 1$, $\bar{\mathbf{T}}_{\boldsymbol{r}}\bar{\boldsymbol{p}} = \mathbf{T}_{\boldsymbol{r}}\boldsymbol{p}$ also holds, since for $l_2 > 1$, $\bar{\beta}_{2,l_2}^- = \beta_{2,l_2}^-$ and $\bar{\beta}_{2,l_2}^+ = \beta_{2,l_2}^+$.

- For any $\boldsymbol{r}$ s.t. $r_1 \neq 0$ and $r_2 \neq 0$, we also need $\rho_1\beta_{1,1}^-\beta_{2,1}^- + \rho_2\beta_{1,1}^+\beta_{2,1}^- + \rho_3\beta_{1,1}^-\beta_{2,1}^+ = u_1\bar{\beta}_{1,1}^-\bar{\beta}_{2,1}^- + u_2\bar{\beta}_{1,1}^+\bar{\beta}_{2,1}^- + u_3\bar{\beta}_{1,1}^-\bar{\beta}_{2,1}^+$.

With five parameters and four constraints, the equation system has infinitely many solutions, thus the construction exists. So the model parameters are not identifiable and condition S3* is necessary.

□

A.4 Proofs of Examples

Our proofs utilize certain results from existing literature, which we have summarized as lemmas below.

Lemma 3. *When $K = 1$, the parameters of the binary DINA model with the following* **Q***-matrix are identifiable.*

$$\mathbf{Q} = \begin{pmatrix} 1 \\ \hdashline 1 \\ \hdashline 1 \end{pmatrix} \tag{52}$$

This lemma is a direct result from Theorem 1 in Gu and Xu (2019b).

Lemma 4. *Given that the item parameters are known(identified), i.e., only the population proportion parameters* $\boldsymbol{p}$ *need to be identified, and if the* **Q**-*matrix contains an identity matrix, then the binary DINA model is identifiable.*

This lemma is a result from Theorem 1 in Xu and Zhang (2016).

A.4.1 Proof of Example 7

Assuming that $0 < \beta_{j,l}^{-} < \beta_{j,l}^{+} \leq 1$, the Sequential DINA model parameters with the following **Q**-matrix are identifiable:

$$\mathbf{Q} = \begin{pmatrix} item1 \begin{cases} 1 & 1 \\ 0 & 1 \end{cases} \\ \hdashline item2 \begin{cases} 1 & 1 \end{cases} \\ \hdashline item3 \begin{cases} 1 & 1 \end{cases} \\ \hdashline item4 \begin{cases} 1 & 0 \end{cases} \\ \hdashline item5 \begin{cases} 1 & 0 \end{cases} \\ \hdashline item6 \begin{cases} 1 & 0 \end{cases} \end{pmatrix} \text{ and } \mathbf{Q}^1 = \begin{pmatrix} 1 & 1 \\ \hdashline 1 & 1 \\ \hdashline 1 & 1 \\ \hdashline 1 & 0 \\ \hdashline 1 & 0 \\ \hdashline 1 & 0 \end{pmatrix}. \tag{53}$$

Proof. If we consider the first categories of the first three items, whose **Q**-matrix can be written as

$$
\mathbf{Q}^1_{1:3} = \begin{pmatrix} 1 & 1 \\ \hdashline 1 & 1 \\ \hdashline 1 & 1 \end{pmatrix}.
$$

and we regard $(1,1)$ together as "one attribute", then the above $\mathbf{Q}$-matrix can be viewed as the $\mathbf{Q}$-matrix for the binary DINA model with only one attribute $(1,1)$. Using Lemma 3 and the equations $\mathbf{T}^s_{\boldsymbol{r}}\boldsymbol{p} = \bar{\mathbf{T}}^s_{\boldsymbol{r}}\bar{\boldsymbol{p}}$ with $\boldsymbol{r} = \boldsymbol{e}_j,\ j \in [3]$ and their sums, we have $p_{(11)} = \bar{p}_{(11)}$, $\beta^+_{j,1} = \bar{\beta}^+_{j,1}$ and $\beta^-_{j,1} = \bar{\beta}^-_{j,1}$ for $j \in [3]$. Similarly if we consider the first categories of the last three items, we obtain $p_{(10)} + p_{(11)} = \bar{p}_{(10)} + \bar{p}_{(11)}$ and $\beta^+_{j,1} = \bar{\beta}^+_{j,1}$, $\beta^-_{j,1} = \bar{\beta}^-_{j,1}$ for $j \in \{4,5,6\}$. Therefore, $p_{(10)} = \bar{p}_{(10)}$. Next we identify $\beta^+_{1,2}$, $\beta^-_{1,2}$, $p_{(00)}$, $p_{(01)}$: consider equations $\mathbf{T}^s_{\boldsymbol{r}}\boldsymbol{p} = \bar{\mathbf{T}}^s_{\boldsymbol{r}}\bar{\boldsymbol{p}}$ with $\boldsymbol{r} \in \{2\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_4\}$ and their sums. With $p_{(11)} = \bar{p}_{(11)}$, $p_{(10)} = \bar{p}_{(10)}$ and $\beta^+_{j,1} = \bar{\beta}^+_{j,1}$, $\beta^-_{j,1} = \bar{\beta}^-_{j,1}$ for $j \in \{1,2,4\}$, these equations give

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\beta^-_{1,1} & \beta^-_{1,1} & \beta^-_{1,1} & \beta^+_{1,1} & \beta^-_{1,1} & \beta^-_{1,1} & \beta^-_{1,1} & \beta^+_{1,1} \\
\beta^-_{1,1}\beta^-_{1,2} & \beta^-_{1,1}\beta^-_{1,2} & \beta^-_{1,1}\beta^+_{1,2} & \beta^+_{1,1}\beta^+_{1,2} & \beta^-_{1,1}\bar{\beta}^-_{1,2} & \beta^-_{1,1}\bar{\beta}^-_{1,2} & \beta^-_{1,1}\bar{\beta}^+_{1,2} & \beta^+_{1,1}\bar{\beta}^+_{1,2} \\
\beta^-_{2,1}\beta^-_{1,1} & \beta^-_{2,1}\beta^-_{1,1} & \beta^-_{2,1}\beta^-_{1,1} & \beta^+_{2,1}\beta^+_{1,1} & \beta^-_{2,1}\beta^-_{1,1} & \beta^-_{2,1}\beta^-_{1,1} & \beta^-_{2,1}\beta^-_{1,1} & \beta^+_{2,1}\beta^+_{1,1} \\
\beta^-_{4,1}\beta^-_{1,1} & \beta^+_{4,1}\beta^-_{1,1} & \beta^-_{4,1}\beta^-_{1,1} & \beta^+_{4,1}\beta^+_{1,1} & \beta^-_{4,1}\beta^-_{1,1} & \beta^+_{4,1}\beta^-_{1,1} & \beta^-_{4,1}\beta^-_{1,1} & \beta^+_{4,1}\beta^+_{1,1} \\
\beta^-_{2,1}\beta^-_{1,1}\beta^-_{1,2} & \beta^-_{2,1}\beta^-_{1,1}\beta^-_{1,2} & \beta^-_{2,1}\beta^-_{1,1}\beta^+_{1,2} & \beta^+_{2,1}\beta^+_{1,1}\beta^+_{1,2} & \beta^-_{2,1}\beta^-_{1,1}\bar{\beta}^-_{1,2} & \beta^-_{2,1}\beta^-_{1,1}\bar{\beta}^-_{1,2} & \beta^-_{2,1}\beta^-_{1,1}\bar{\beta}^+_{1,2} & \beta^+_{2,1}\beta^+_{1,1}\bar{\beta}^+_{1,2} \\
\beta^-_{4,1}\beta^-_{1,1}\beta^-_{1,2} & \beta^+_{4,1}\beta^-_{1,1}\beta^-_{1,2} & \beta^-_{4,1}\beta^-_{1,1}\beta^+_{1,2} & \beta^+_{4,1}\beta^+_{1,1}\beta^+_{1,2} & \beta^-_{4,1}\beta^-_{1,1}\bar{\beta}^-_{1,2} & \beta^+_{4,1}\beta^-_{1,1}\bar{\beta}^-_{1,2} & \beta^-_{4,1}\beta^-_{1,1}\bar{\beta}^+_{1,2} & \beta^+_{4,1}\beta^+_{1,1}\bar{\beta}^+_{1,2}
\end{pmatrix}
\begin{pmatrix}
p_{(00)} \\ p_{(10)} \\ p_{(01)} \\ p_{(11)} \\ -\bar{p}_{(00)} \\ -\bar{p}_{(10)} \\ -\bar{p}_{(01)} \\ -\bar{p}_{(11)}
\end{pmatrix}
= \mathbf{0},
$$

if we abbreviate

$$
\widetilde{\boldsymbol{p}} = (p_{(00)}\ \ p_{(10)}\ \ p_{(01)}\ \ p_{(11)}\ \ -\bar{p}_{(00)}\ \ -\bar{p}_{(10)}\ \ -\bar{p}_{(01)}\ \ -\bar{p}_{(11)}\ \ -\bar{p}_{(11)})^\top, \tag{54}
$$

then

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & \beta^-_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & \beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) \\ 0 & 0 & \beta^-_{2,1}\beta^-_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{2,1}\beta^+_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & \beta^-_{2,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{2,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{2,1}\beta^-_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{2,1}\beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) \\ 0 & 0 & \beta^-_{4,1}\beta^-_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{4,1}\beta^+_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & \beta^-_{4,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^+_{4,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{4,1}\beta^-_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{4,1}\beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) \end{pmatrix} \widetilde{\boldsymbol{p}} = \mathbf{0},$$

thus

$$\begin{pmatrix} 0 & 0 & \beta^-_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & 0 & \beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^+_{1,2}) \\ 0 & 0 & \beta^-_{2,1}\beta^-_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & 0 & \beta^-_{2,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{2,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{2,1}\beta^-_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{2,1}\beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^+_{1,2}) \\ 0 & 0 & \beta^-_{4,1}\beta^-_{1,1}(\beta^+_{1,2}-\beta^-_{1,2}) & 0 & \beta^-_{4,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^+_{4,1}\beta^-_{1,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{4,1}\beta^-_{1,1}(\bar\beta^+_{1,2}-\beta^-_{1,2}) & \beta^+_{4,1}\beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^+_{1,2}) \end{pmatrix} \widetilde{\boldsymbol{p}} = \mathbf{0}, \tag{55}$$

which gives $(\beta^+_{2,1}-\beta^-_{2,1})\beta^+_{1,1}(\bar\beta^+_{1,2}-\beta^+_{1,2}) = 0$, so we have $\bar\beta^+_{1,2} = \beta^+_{1,2}$. Taking this back to (55), we have

$$\beta^-_{1,1} \begin{pmatrix} 0 & 0 & (\beta^+_{1,2}-\beta^-_{1,2}) & 0 & (\bar\beta^-_{1,2}-\beta^-_{1,2}) & (\bar\beta^-_{1,2}-\beta^-_{1,2}) & (\beta^+_{1,2}-\beta^-_{1,2}) & 0 \\ 0 & 0 & \beta^-_{4,1}(\beta^+_{1,2}-\beta^-_{1,2}) & 0 & \beta^-_{4,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^+_{4,1}(\bar\beta^-_{1,2}-\beta^-_{1,2}) & \beta^-_{4,1}(\beta^+_{1,2}-\beta^-_{1,2}) & 0 \end{pmatrix} \widetilde{\boldsymbol{p}} = 0, \tag{56}$$

which gives

$$\beta^-_{1,1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & (\beta^+_{4,1}-\beta^-_{4,1})(\bar\beta^-_{1,2}-\beta^-_{1,2}) & 0 & 0 \end{pmatrix} \widetilde{\boldsymbol{p}} = 0.$$

Assuming $\beta^-_{1,1} > 0$, we have $\bar\beta^-_{1,2} = \beta^-_{1,2}$. Taking this back to (56), we have

$$\begin{pmatrix} 0 & 0 & (\beta^+_{1,2}-\beta^-_{1,2}) & 0 & 0 & 0 & (\beta^+_{1,2}-\beta^-_{1,2}) & 0 \end{pmatrix} \widetilde{\boldsymbol{p}} = 0.$$

Thus $p_{(01)} = \bar p_{(01)}$, and also $p_{(00)} = \bar p_{(00)}$. Therefore, the parameters of the Sequential DINA model with the above **Q**-matrix (53) are identifiable. □

A.4.2 Proof of Example 8

The Sequential DINA model parameters with the following $\mathbf{Q}$-matrix are not identifiable:

$$\mathbf{Q} = \begin{pmatrix} \textit{item 1} \left\{ \begin{matrix} 1 & 0 \end{matrix} \right. \\ \hdashline \textit{item 2} \left\{ \begin{matrix} 0 & 1 \end{matrix} \right. \\ \hdashline \textit{item 3} \left\{ \begin{matrix} 0 & 1 \end{matrix} \right. \\ \hdashline \textit{item 4} \left\{ \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \right. \end{pmatrix} \text{ and } \mathbf{Q}^1 = \begin{pmatrix} 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 0 & 1 \\ \hdashline 1 & 1 \end{pmatrix}.$$

Proof. Let $\bar{\beta}^+_{1,1} = \beta^+_{1,1}$, $\bar{\beta}^+_{2,1} = \beta^+_{2,1}$ $\bar{\beta}^+_{3,1} = \beta^+_{3,1}$, $\bar{\beta}^+_{4,2} = \beta^+_{4,2}$, $\bar{\beta}^-_{2,1} = \beta^-_{2,1}$, $\bar{\beta}^-_{3,1} = \beta^-_{3,1}$, $\bar{\beta}^-_{4,1} = \beta^-_{4,1} = 0$. Since $\beta^-_{4,1} = 0$, $\beta^-_{4,2}$ is not defined (or equals 0). Then $\bar{\mathbf{T}}\bar{\boldsymbol{p}} = \mathbf{T}\boldsymbol{p}$ holds if and only if the following equations hold:

$$\begin{cases} \bar{p}_{(00)} + \bar{p}_{(10)} + \bar{p}_{(01)} + \bar{p}_{(11)} = p_{(00)} + p_{(10)} + p_{(01)} + p_{(11)}; \\ \bar{p}_{(01)} + \bar{p}_{(11)} = p_{(01)} + p_{(11)}; \\ \bar{\beta}^-_{1,1}[\bar{p}_{(00)} + \bar{p}_{(01)}] + \beta^+_{1,1}[\bar{p}_{(10)} + \bar{p}_{(11)}] = \beta^-_{1,1}[p_{(00)} + p_{(01)}] + \beta^+_{1,1}[p_{(10)} + p_{(11)}]; \\ \bar{\beta}^-_{1,1}\beta^-_{2,1}\bar{p}_{(00)} + \bar{\beta}^-_{1,1}\beta^+_{2,1}\bar{p}_{(01)} + \beta^+_{1,1}\beta^-_{2,1}\bar{p}_{(10)} + \beta^+_{1,1}\beta^+_{2,1}\bar{p}_{(11)} = \beta^-_{1,1}\beta^-_{2,1}p_{(00)} + \beta^-_{1,1}\beta^+_{2,1}p_{(01)} + \beta^+_{1,1}\beta^-_{2,1}p_{(10)} + \beta^+_{1,1}\beta^+_{2,1}p_{(11)}; \\ \bar{\beta}^+_{4,1}\bar{p}_{(11)} = \beta^+_{4,1}p_{(11)}. \end{cases}$$

There are five equations with six parameters $(p_{(00)}, p_{(10)}, p_{(01)}, p_{(11)}, \beta^-_{1,1}, \beta^+_{4,1})$, thus there are infinitely many solutions and the parameters are not identifiable. □

A.4.3 Proof of Example 9

The Sequential DINA model parameters with the following **Q**-matrix are not identifiable:

$$\mathbf{Q} = \begin{pmatrix} \textit{item } 1 \{ & 1 & 0 \\ \hdashline \textit{item } 2 \{ & 0 & 1 \\ \hdashline \multirow{2}{*}{\textit{item } 3} \{ & 1 & 1 \\ & 1 & 0 \\ \hdashline \textit{item } 4 \{ & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q}^1 = \begin{pmatrix} 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 1 & 1 \\ \hdashline 1 & 1 \end{pmatrix}.$$

Proof. Let $\bar{\beta}^+_{j,l} = \beta^+_{j,l}$ for all j and $l \in [H_j]$, $\bar{\beta}^-_{4,1} = \beta^-_{4,1}$, $\bar{\beta}^-_{3,1} = \beta^-_{3,1} = 0$, and $\bar{p}_{(11)} = p_{(11)}$. Since $\beta^-_{3,1} = 0$, $\beta^-_{3,2}$ is not defined (or equals 0). Then $\bar{\mathbf{T}}\bar{\boldsymbol{p}} = \mathbf{T}\boldsymbol{p}$ holds if and only if the following equations hold:

$$\begin{cases} \bar{p}_{(00)} + \bar{p}_{(10)} + \bar{p}_{(01)} = p_{(00)} + p_{(10)} + p_{(01)}; \\ \bar{\beta}^-_{1,1}[\bar{p}_{(00)} + \bar{p}_{(01)}] + \beta^+_{1,1}[\bar{p}_{(10)} + p_{(11)}] = \beta^-_{1,1}[p_{(00)} + p_{(01)}] + \beta^+_{1,1}[p_{(10)} + p_{(11)}]; \\ \bar{\beta}^-_{2,1}[\bar{p}_{(00)} + \bar{p}_{(10)}] + \beta^+_{2,1}[\bar{p}_{(01)} + p_{(11)}] = \beta^-_{2,1}[p_{(00)} + p_{(10)}] + \beta^+_{2,1}[p_{(01)} + p_{(11)}]; \\ \bar{\beta}^-_{1,1}\bar{\beta}^-_{2,1}\bar{p}_{(00)} + \bar{\beta}^-_{1,1}\bar{\beta}^+_{2,1}\bar{p}_{(01)} + \beta^+_{1,1}\bar{\beta}^-_{2,1}\bar{p}_{(10)} + \beta^+_{1,1}\beta^+_{2,1}p_{(11)} = \beta^-_{1,1}\beta^-_{2,1}p_{(00)} + \beta^-_{1,1}\beta^+_{2,1}p_{(01)} + \beta^+_{1,1}\beta^-_{2,1}p_{(10)} + \beta^+_{1,1}\beta^+_{2,1}p_{(11)}. \end{cases}$$

There are four equations with five parameters $(p_{(00)}, p_{(10)}, p_{(01)}, \beta^-_{1,1}, \beta^-_{2,1})$, thus there are infinitely many solutions and the parameters are not identifiable. $\square$

A.4.4 Proof of Example 10

The Sequential DINA model parameters with the following $\mathbf{Q}$-matrix are identifiable:

$$\mathbf{Q} = \begin{pmatrix} \textit{item 1} \begin{cases} 1 & 0 \\ 0 & 1 \end{cases} \\ \hdashline \textit{item 2} \begin{cases} 1 & 0 \\ 0 & 1 \end{cases} \\ \hdashline \textit{item 3} \begin{cases} 0 & 1 \\ 1 & 0 \end{cases} \\ \hdashline \textit{item 4} \begin{cases} 0 & 1 \\ 1 & 0 \end{cases} \end{pmatrix} \quad \text{and} \quad \mathbf{Q}^1 = \begin{pmatrix} 1 & 0 \\ \hdashline 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 0 & 1 \end{pmatrix}. \tag{57}$$

Proof. Consider equations $\mathbf{T}^s_{\boldsymbol{r}}\boldsymbol{p} = \bar{\mathbf{T}}^s_{\boldsymbol{r}}\bar{\boldsymbol{p}}$ with $\boldsymbol{r} \in \{\boldsymbol{e}_1, 2\boldsymbol{e}_1, \ \boldsymbol{e}_3, \ \boldsymbol{e}_4\}$ and their sums. Using a similar calculation, these equations give

$$\begin{pmatrix} \mathbf{0}_4^\top & \bar\beta^-_{1,1}(\bar\beta^-_{1,2} - \beta^-_{1,2})(\bar\beta^-_{3,1} - \beta^+_{3,1}) & \bar\beta^+_{1,1}(\bar\beta^-_{1,2} - \beta^-_{1,2})(\bar\beta^-_{3,1} - \beta^+_{3,1}) & \bar\beta^-_{1,1}(\bar\beta^+_{1,2} - \beta^-_{1,2})(\bar\beta^+_{3,1} - \beta^+_{3,1}) & \bar\beta^+_{1,1}(\bar\beta^+_{1,2} - \beta^-_{1,2})(\bar\beta^+_{3,1} - \beta^+_{3,1}) \\ \mathbf{0}_4^\top & \bar\beta^-_{1,1}(\bar\beta^-_{1,2} - \beta^-_{1,2})(\bar\beta^-_{3,1} - \beta^+_{3,1})\bar\beta^-_{4,1} & \bar\beta^+_{1,1}(\bar\beta^-_{1,2} - \beta^-_{1,2})(\bar\beta^-_{3,1} - \beta^+_{3,1})\bar\beta^-_{4,1} & \bar\beta^-_{1,1}(\bar\beta^+_{1,2} - \beta^-_{1,2})(\bar\beta^+_{3,1} - \beta^+_{3,1})\bar\beta^+_{4,1} & \bar\beta^+_{1,1}(\bar\beta^+_{1,2} - \beta^-_{1,2})(\bar\beta^+_{3,1} - \beta^+_{3,1})\bar\beta^+_{4,1} \end{pmatrix} \widetilde{\boldsymbol{p}} = \mathbf{0}, \tag{58}$$

where $\mathbf{0}_4$ denotes all-zero vectors of dimension four and $\widetilde{\boldsymbol{p}}$ is (54). Thus,

$$\begin{pmatrix} \mathbf{0}_5^\top & \bar\beta^-_{1,1}(\bar\beta^+_{1,2} - \beta^-_{1,2})(\bar\beta^+_{3,1} - \beta^+_{3,1})(\bar\beta^+_{4,1} - \bar\beta^-_{4,1}) & \bar\beta^+_{1,1}(\bar\beta^+_{1,2} - \beta^-_{1,2})(\bar\beta^+_{3,1} - \beta^+_{3,1})(\bar\beta^+_{4,1} - \bar\beta^-_{4,1}) \end{pmatrix} \widetilde{\boldsymbol{p}} = \mathbf{0}.$$

Therefore,

$$(\bar{\beta}^-_{1,1}\bar{p}_{01} + \bar{\beta}^+_{1,1}\bar{p}_{11})(\bar{\beta}^+_{1,2} - \beta^-_{1,2})(\bar{\beta}^+_{3,1} - \beta^+_{3,1})(\bar{\beta}^+_{4,1} - \bar{\beta}^-_{4,1}) = 0.$$

Since $0 \le \beta^-_{j,l} < \beta^+_{j,l} \le 1$, $(\bar{\beta}^+_{1,2} - \beta^-_{1,2})(\bar{\beta}^+_{3,1} - \beta^+_{3,1}) = 0$. But if $\bar{\beta}^+_{3,1} \neq \beta^+_{3,1}$ and $\bar{\beta}^+_{1,2} = \beta^-_{1,2}$, we can swap the parameters $\boldsymbol{\beta}$ and $\bar{\boldsymbol{\beta}}$, and show that $\beta^+_{1,2} = \bar{\beta}^-_{1,2}$ by symmetry. Yet this will indicate that $\beta^+_{1,2} = \bar{\beta}^-_{1,2} < \bar{\beta}^+_{1,2} = \beta^-_{1,2}$, which leads to a contradiction, thus we must have $\bar{\beta}^+_{3,1} = \beta^+_{3,1}$. By symmetry we can also show that $\bar{\beta}^+_{4,1} = \beta^+_{4,1}$, and similarly for $\bar{\beta}^+_{1,1} = \beta^+_{1,1}$ and $\bar{\beta}^+_{2,1} = \beta^+_{2,1}$. Taking these back to equation (58), we have $\bar{\beta}^-_{1,2} = \beta^-_{1,2}$. Samely we obtain $\bar{\beta}^-_{j,2} = \beta^-_{j,2}$ for $j \in [4]$. Thus all the item parameters are identified, and according to Lemma 4, we know that when these parameters are known, the completeness of the **Q**-matrix will suffice to identify parameters $\boldsymbol{p}$. Therefore, we must have $(\boldsymbol{\beta}^+, \boldsymbol{\beta}^-, \boldsymbol{p}) = (\bar{\boldsymbol{\beta}}^+, \bar{\boldsymbol{\beta}}^-, \bar{\boldsymbol{p}})$, i.e., the model parameters are identifiable. □

A.4.5 Proof of Example 11

The Sequential DINA model parameters with the following **Q**-matrix are identifiable:

$$
\mathbf{Q} = \begin{pmatrix}
\textit{item 1} \begin{cases} 1 & 0 \\ 0 & 1 \end{cases} \\
\hdashline
\textit{item 2} \begin{cases} 0 & 1 \\ 1 & 0 \end{cases} \\
\hdashline
\textit{item 3} \begin{cases} 1 & 1 \end{cases} \\
\hdashline
\textit{item 4} \begin{cases} 1 & 1 \end{cases} \\
\hdashline
\textit{item 5} \begin{cases} 1 & 1 \end{cases}
\end{pmatrix} \quad \text{and} \quad \mathbf{Q}^1 = \begin{pmatrix} 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 1 & 1 \\ \hdashline 1 & 1 \\ \hdashline 1 & 1 \end{pmatrix}.
$$

Proof. Consider the last three items, similar to the proof in Example 7, according to Lemma 3, we have $p_{(11)} = \bar{p}_{(11)}$, and $\beta^+_{j,1} = \bar{\beta}^+_{j,1}$, $\beta^-_{j,1} = \bar{\beta}^-_{j,1}$ for $j \in \{3, 4, 5\}$. Next using equations $\mathbf{T}^s_{\boldsymbol{r}}\boldsymbol{p} = \bar{\mathbf{T}}^s_{\boldsymbol{r}}\bar{\boldsymbol{p}}$ with $\boldsymbol{r} = \boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_3$, we have

$$
\begin{pmatrix}
\beta^-_{1,1} & \beta^+_{1,1} & \beta^-_{1,1} & \beta^+_{1,1} & \bar{\beta}^-_{1,1} & \bar{\beta}^+_{1,1} & \bar{\beta}^-_{1,1} & \bar{\beta}^+_{1,1} \\
\beta^-_{1,1}\beta^-_{3,1} & \beta^+_{1,1}\beta^-_{3,1} & \beta^-_{1,1}\beta^-_{3,1} & \beta^+_{1,1}\beta^+_{3,1} & \bar{\beta}^-_{1,1}\beta^-_{3,1} & \bar{\beta}^+_{1,1}\beta^-_{3,1} & \bar{\beta}^-_{1,1}\beta^-_{3,1} & \bar{\beta}^+_{1,1}\beta^+_{3,1}
\end{pmatrix} \widetilde{\boldsymbol{p}} = \mathbf{0},
$$

which gives

$$
\begin{pmatrix} 0 & 0 & 0 & \beta^+_{1,1}(\beta^+_{3,1} - \beta^-_{3,1}) & 0 & 0 & 0 & \bar{\beta}^+_{1,1}(\beta^+_{3,1} - \beta^-_{3,1}) \end{pmatrix} \widetilde{\boldsymbol{p}} = 0,
$$

where $\widetilde{\boldsymbol{p}}$ is given by (54). Since $p_{(11)} = \bar{p}_{(11)}$, we must have $\beta^+_{1,1} = \bar{\beta}^+_{1,1}$. Next combining

$\boldsymbol{r} = 2\boldsymbol{e}_1$ and $\boldsymbol{r} = 2\boldsymbol{e}_1 + \boldsymbol{e}_3$,

$$\begin{pmatrix} \beta^-_{1,1}\beta^-_{1,2} & \beta^+_{1,1}\beta^-_{1,2} & \beta^-_{1,1}\beta^+_{1,2} & \beta^+_{1,1}\beta^+_{1,2} & \bar{\beta}^-_{1,1}\bar{\beta}^-_{1,2} & \bar{\beta}^+_{1,1}\bar{\beta}^-_{1,2} & \bar{\beta}^-_{1,1}\bar{\beta}^+_{1,2} & \bar{\beta}^+_{1,1}\bar{\beta}^+_{1,2} \\ \beta^-_{1,1}\beta^-_{1,2}\beta^-_{3,1} & \beta^+_{1,1}\beta^-_{1,2}\beta^-_{3,1} & \beta^-_{1,1}\beta^+_{1,2}\beta^-_{3,1} & \beta^+_{1,1}\beta^+_{1,2}\beta^+_{3,1} & \bar{\beta}^-_{1,1}\bar{\beta}^-_{1,2}\beta^-_{3,1} & \bar{\beta}^+_{1,1}\bar{\beta}^-_{1,2}\beta^-_{3,1} & \bar{\beta}^-_{1,1}\bar{\beta}^+_{1,2}\beta^-_{3,1} & \bar{\beta}^+_{1,1}\bar{\beta}^+_{1,2}\beta^+_{3,1} \end{pmatrix} \widetilde{\boldsymbol{p}} = \boldsymbol{0},$$

which gives $\beta^+_{1,2} = \bar{\beta}^+_{1,2}$. Similarly we can obtain $\beta^+_{2,1} = \bar{\beta}^+_{2,1}$ and $\beta^+_{2,2} = \bar{\beta}^+_{2,2}$. Next consider $\boldsymbol{r} \in \{\boldsymbol{e}_1, 2\boldsymbol{e}_1, \boldsymbol{e}_1 + \boldsymbol{e}_2, 2\boldsymbol{e}_1 + \boldsymbol{e}_2\}$, using similar calculations, we have,

$$\begin{pmatrix} 0 & 0 & \beta^-_{1,1}(\beta^+_{1,2} - \beta^-_{1,2}) & \beta^+_{1,1}(\beta^+_{1,2} - \beta^-_{1,2}) & \bar{\beta}^-_{1,1}(\bar{\beta}^-_{1,2} - \beta^-_{1,2}) & \beta^+_{1,1}(\bar{\beta}^-_{1,2} - \beta^-_{1,2}) & \bar{\beta}^-_{1,1}(\beta^+_{1,2} - \beta^-_{1,2}) & \beta^+_{1,1}(\beta^+_{1,2} - \beta^-_{1,2}) \\ 0 & 0 & \beta^-_{1,1}(\beta^+_{1,2} - \beta^-_{1,2})\beta^+_{2,1} & \beta^+_{1,1}(\beta^+_{1,2} - \beta^-_{1,2})\beta^+_{2,1} & \bar{\beta}^-_{1,1}(\bar{\beta}^-_{1,2} - \beta^-_{1,2})\bar{\beta}^-_{2,1} & \beta^+_{1,1}(\bar{\beta}^-_{1,2} - \beta^-_{1,2})\bar{\beta}^-_{2,1} & \bar{\beta}^-_{1,1}(\beta^+_{1,2} - \beta^-_{1,2})\beta^+_{2,1} & \beta^+_{1,1}(\beta^+_{1,2} - \beta^-_{1,2})\beta^+_{2,1} \end{pmatrix} \widetilde{\boldsymbol{p}} = \boldsymbol{0},$$

which gives

$$(\bar{\beta}^-_{1,1}\bar{p}_{(00)} + \bar{\beta}^+_{1,1}\bar{p}_{(10)})(\bar{\beta}^-_{1,2} - \beta^-_{1,2})(\bar{\beta}^-_{2,1} - \beta^+_{2,1}) = 0,$$

thus we have $\bar{\beta}^-_{1,2} = \beta^-_{1,2}$. Finally we identify $\beta^-_{2,1}$,

$$\begin{pmatrix} \beta^-_{1,1} & \beta^+_{1,1} & \beta^-_{1,1} & \beta^+_{1,1} & \bar{\beta}^-_{1,1} & \beta^+_{1,1} & \bar{\beta}^-_{1,1} & \beta^+_{1,1} \\ \beta^-_{1,1}\beta^-_{1,2} & \beta^+_{1,1}\beta^-_{1,2} & \beta^-_{1,1}\beta^+_{1,2} & \beta^+_{1,1}\beta^+_{1,2} & \bar{\beta}^-_{1,1}\bar{\beta}^-_{1,2} & \beta^+_{1,1}\bar{\beta}^-_{1,2} & \bar{\beta}^-_{1,1}\beta^+_{1,2} & \beta^+_{1,1}\beta^+_{1,2} \\ \beta^-_{1,1}\beta^-_{2,1} & \beta^+_{1,1}\beta^-_{2,1} & \beta^-_{1,1}\beta^+_{2,1} & \beta^+_{1,1}\beta^+_{2,1} & \bar{\beta}^-_{1,1}\bar{\beta}^-_{2,1} & \beta^+_{1,1}\bar{\beta}^-_{2,1} & \bar{\beta}^-_{1,1}\beta^+_{2,1} & \beta^+_{1,1}\beta^+_{2,1} \\ \beta^-_{1,1}\beta^-_{1,2}\beta^-_{2,1} & \beta^+_{1,1}\beta^-_{1,2}\beta^-_{2,1} & \beta^-_{1,1}\beta^+_{1,2}\beta^+_{2,1} & \beta^+_{1,1}\beta^+_{1,2}\beta^+_{2,1} & \bar{\beta}^-_{1,1}\bar{\beta}^-_{1,2}\bar{\beta}^-_{2,1} & \beta^+_{1,1}\bar{\beta}^-_{1,2}\bar{\beta}^-_{2,1} & \bar{\beta}^-_{1,1}\beta^+_{1,2}\beta^+_{2,1} & \beta^+_{1,1}\beta^+_{1,2}\beta^+_{2,1} \end{pmatrix} \widetilde{\boldsymbol{p}} = \boldsymbol{0}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & \beta^-_{1,1}(\beta^+_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}(\beta^+_{2,1} - \beta^-_{2,1}) & \bar{\beta}^-_{1,1}(\bar{\beta}^-_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}(\bar{\beta}^-_{2,1} - \beta^-_{2,1}) & \bar{\beta}^-_{1,1}(\beta^+_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}(\beta^+_{2,1} - \beta^-_{2,1}) \\ 0 & 0 & \beta^-_{1,1}\beta^+_{1,2}(\beta^+_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}\beta^+_{1,2}(\beta^+_{2,1} - \beta^-_{2,1}) & \bar{\beta}^-_{1,1}\bar{\beta}^-_{1,2}(\bar{\beta}^-_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}\bar{\beta}^-_{1,2}(\bar{\beta}^-_{2,1} - \beta^-_{2,1}) & \bar{\beta}^-_{1,1}\beta^+_{1,2}(\beta^+_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}\beta^+_{1,2}(\beta^+_{2,1} - \beta^-_{2,1}) \end{pmatrix} \widetilde{\boldsymbol{p}} = \boldsymbol{0}.$$

Hence

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \bar{\beta}^-_{1,1}(\bar{\beta}^-_{1,2} - \beta^+_{1,2})(\bar{\beta}^-_{2,1} - \beta^-_{2,1}) & \beta^+_{1,1}(\bar{\beta}^-_{1,2} - \beta^+_{1,2})(\bar{\beta}^-_{2,1} - \beta^-_{2,1}) & 0 & 0 \end{pmatrix} \widetilde{\boldsymbol{p}} = 0.$$

Therefore, we have

$$(\bar{\beta}^{-}_{1,1}\bar{p}_{(00)} + \bar{\beta}^{+}_{1,1}\bar{p}_{(10)})(\bar{\beta}^{-}_{1,2} - \beta^{+}_{1,2})(\bar{\beta}^{-}_{2,1} - \beta^{-}_{2,1}) = 0,$$

thus $\bar{\beta}^{-}_{2,1} = \beta^{-}_{2,1}$. Using the same strategy we can show that $\bar{\beta}^{-}_{2,2} = \beta^{-}_{2,2}$ and $\bar{\beta}^{-}_{1,1} = \beta^{-}_{1,1}$, which completes the proof. □