Supplementary Material for "Polytomous effectiveness indicators in complex problem-solving tasks and their applications in developing measurement model"

S1. Design and results of the simulation study.

The capital letter A represents the initial state, with G serving as the target state. BCDEF are intermediate states. Seven states are paired to form six step-by-step transitions (A→B, B→C, C→D, D→E, E→F, and F→G). Furthermore, three forward transitions (A→C, B→E, and D→F) are added, resulting in two optimal paths (A→B→E→F→G and A→C→D→F→G). Counter-intuitively, states C and D are equidistant from the target state G, as are E and F. There are also eight backward transitions between the intermediate states (B→A, C→A, C→B, D→B, D→C, E→C, F→C, and F→D). Additionally, the symbol # represents the incorrect end state, and the transitions A→#, B→#, D→#, and E→# represent four incorrect end transitions in this task. When these transitions occur, the problem-solving process concludes prematurely. The effectiveness indicators of these incorrect end transitions are assigned a minimum value of -3, which is lower than the effectiveness of all backward transitions. The effectiveness of the states and transitions is illustrated in Table B1.

Table S1. The effectiveness of all states and transitions in the simulated task with multiple optimal paths.

S	*S*'							
	A (4)	B (3)	C (3)	D (2)	E (2)	F (1)	G (0)	End
A (4)		AB (1)	AC (1)					A# (-3)
B (3)	BA (-1)		BC (0)		BE (1)			B# (-3)
C (3)	CA (-2)	CB (0)		CD (1)				
D (2)		DB (-1)	DC (-1)		DE (0)	DF (1)		D# (-3)
E (2)			EC (-1)			EF (1)		E# (-3)
F (1)			FC (-2)	FD (-1)			FG (1)	

Note: The effectiveness is in parentheses. Omit the arrow indicating the transition in two letters (same in the tables and figures below).

Table S2. The true values of transition tendency parameters for different sequence lengths in the simulated problem-solving task with multiple optimal paths.

Transition parameters	Sequence length	
	Short	Long
$\lambda_{A\#}$	-3.482	-3.482
λ_{AB}	1.241	2.241
$\lambda_{A\mathrm{C}}$	2.241	1.241
λ_{BA}	0.140	1.140
$\lambda_{B\#}$	-1.328	-1.328
λ_{BC}	1.097	1.097
$\lambda_{B\mathrm{E}}$	0.091	-0.909
λ_{CA}	0.684	1.684
λ_{CB}	0.012	-0.988
λ_{CD}	-0.696	-0.696
λ_{DB}	-0.116	0.884
λ_{DC}	0.384	0.884
$\lambda_{D\#}$	-0.673	-0.673
λ_{DE}	0.502	0.002
$\lambda_{D\mathrm{F}}$	-0.097	-1.097
λ_{EC}	0.435	0.935
$\lambda_{E\#}$	-0.776	-0.776
λ_{EF}	0.341	-0.159
λ_{FC}	0.704	0.704
λ_{FD}	0.229	1.229
λ_{FG}	-0.933	-1.933

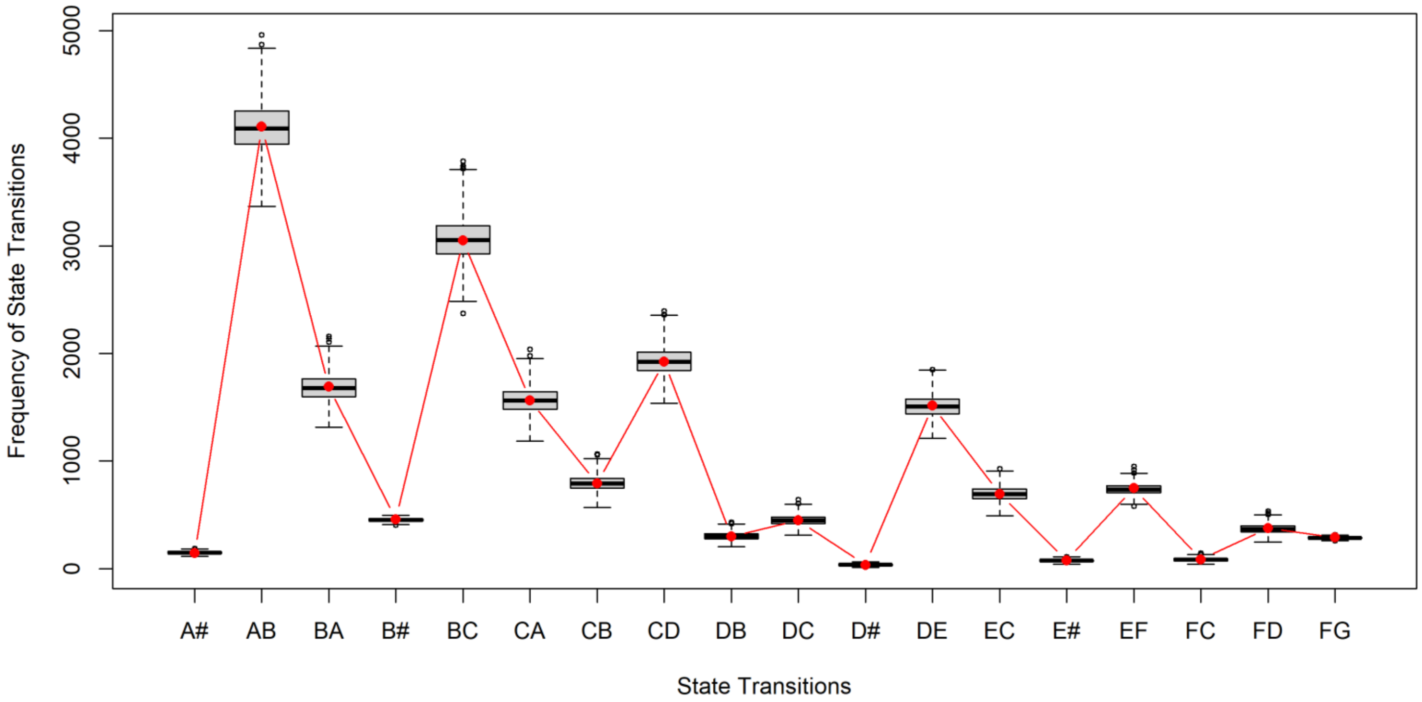

Figure S1. Frequency of transitions in observed data and posterior predictive data in the simulation study.
Note: The frequency of the observed data is indicated by the points. The box for each transition depicts the range of the posterior predictive distributions. The notch in the middle indicates the median. The whiskers represent the 2.5th and 97.5th percentiles.

Table S3. The frequency and estimation accuracy of all transition parameters in the simulated problem-solving tasks with multiple optimal paths.

Sample size		Short sequence				Long sequence			
		non-inf.		inf.		non-inf.		inf.	
		Frequency	RMSE	Frequency	RMSE	Frequency	RMSE	Frequency	RMSE
200	$\lambda_{A\#}$	37	0.320	37	0.202	35	0.219	35	0.230
	λ_{AB}	251	0.169	251	0.107	1484	0.109	1484	0.118
	$\lambda_{A\mathrm{C}}$	691	0.157	691	0.106	548	0.113	548	0.114
	λ_{BA}	112	0.065	112	0.065	748	0.037	748	0.039
	$\lambda_{B\#}$	50	0.168	50	0.152	81	0.144	81	0.111
	λ_{BC}	294	0.051	294	0.056	711	0.039	711	0.040
	$\lambda_{B\mathrm{E}}$	109	0.149	109	0.144	173	0.143	173	0.103
	λ_{CA}	667	0.103	667	0.092	1119	0.115	1119	0.087
	λ_{CB}	283	0.055	283	0.055	88	0.070	88	0.067
	λ_{CD}	205	0.141	205	0.130	364	0.151	364	0.110
	λ_{DB}	31	0.142	31	0.135	142	0.084	142	0.079
	λ_{DC}	51	0.121	51	0.111	142	0.067	142	0.078
	$\lambda_{D\#}$	14	0.238	14	0.208	13	0.226	13	0.203
	λ_{DE}	79	0.104	79	0.096	105	0.089	105	0.085
	$\lambda_{D\mathrm{F}}$	76	0.153	76	0.142	71	0.154	71	0.122
	λ_{EC}	65	0.091	65	0.102	131	0.093	131	0.091
	$\lambda_{E\#}$	18	0.165	18	0.197	14	0.205	14	0.182
	λ_{EF}	105	0.138	105	0.165	134	0.168	134	0.132
	λ_{FC}	55	0.128	55	0.137	37	0.146	37	0.129
	λ_{FD}	45	0.099	45	0.107	110	0.096	110	0.084
	λ_{FG}	81	0.155	81	0.165	58	0.183	58	0.149
	Average	158	0.174	158	0.161	300	0.162	300	0.144

500	$\lambda_{A\#}$	81	0.193	82	0.140	81	0.156	81	0.139
	λ_{AB}	638	0.099	637	0.073	4929	0.079	5008	0.069
	$\lambda_{A\mathrm{C}}$	1727	0.097	1734	0.072	1814	0.078	1843	0.072
	λ_{BA}	300	0.050	303	0.047	2194	0.025	2229	0.024
	$\lambda_{B\#}$	128	0.126	129	0.103	216	0.099	216	0.093
	λ_{BC}	786	0.050	784	0.038	2745	0.035	2798	0.034
	$\lambda_{B\mathrm{E}}$	351	0.089	352	0.075	584	0.070	594	0.070
	λ_{CA}	1645	0.045	1650	0.043	4130	0.040	4202	0.043
	λ_{CB}	845	0.022	846	0.020	413	0.027	419	0.030
	λ_{CD}	527	0.047	528	0.045	946	0.047	967	0.046
	λ_{DB}	84	0.075	85	0.077	397	0.048	410	0.051
	λ_{DC}	140	0.057	142	0.068	398	0.046	405	0.044
	$\lambda_{D\#}$	43	0.166	42	0.114	34	0.138	35	0.141
	λ_{DE}	219	0.063	216	0.065	321	0.051	324	0.059
	$\lambda_{D\mathrm{F}}$	204	0.109	203	0.092	243	0.080	245	0.081
	λ_{EC}	189	0.051	186	0.053	401	0.054	409	0.055
	$\lambda_{E\#}$	52	0.136	51	0.116	38	0.131	38	0.132
	λ_{EF}	330	0.119	331	0.100	466	0.095	471	0.100
	λ_{FC}	175	0.077	178	0.079	130	0.066	133	0.075
	λ_{FD}	162	0.048	161	0.057	448	0.043	453	0.052
	λ_{FG}	196	0.082	195	0.089	131	0.072	131	0.088
	Average	420	0.111	420	0.096	1003	0.092	1019	0.091
1000	$\lambda_{A\#}$	158	0.106	161	0.108	162	0.097	161	0.092
	λ_{AB}	1281	0.053	1274	0.058	9935	0.049	9943	0.046
	$\lambda_{A\mathrm{C}}$	3474	0.055	3471	0.054	3664	0.049	3667	0.047
	λ_{BA}	610	0.033	603	0.034	4426	0.019	4426	0.015

	$\lambda_{B\#}$	258	0.074	258	0.078	428	0.072	433	0.074
	λ_{BC}	1578	0.029	1562	0.028	5550	0.026	5543	0.025
	$\lambda_{B\mathrm{E}}$	706	0.059	707	0.057	1172	0.052	1176	0.055
	λ_{CA}	3303	0.034	3304	0.030	8335	0.030	8346	0.030
	λ_{CB}	1698	0.016	1684	0.015	832	0.018	835	0.020
	λ_{CD}	1065	0.037	1054	0.034	1928	0.030	1919	0.030
	λ_{DB}	173	0.052	172	0.058	810	0.032	800	0.032
	λ_{DC}	286	0.049	281	0.047	812	0.034	814	0.032
	$\lambda_{D\#}$	87	0.099	85	0.099	69	0.103	67	0.099
	λ_{DE}	439	0.044	437	0.046	651	0.039	648	0.033
	$\lambda_{D\mathrm{F}}$	413	0.069	408	0.065	494	0.059	493	0.065
	λ_{EC}	373	0.038	373	0.040	806	0.043	813	0.036
	$\lambda_{E\#}$	102	0.091	103	0.076	76	0.090	75	0.087
	λ_{EF}	670	0.079	667	0.065	941	0.070	937	0.069
	λ_{FC}	355	0.055	355	0.054	263	0.051	263	0.054
	λ_{FD}	332	0.035	328	0.042	907	0.032	903	0.034
	λ_{FG}	395	0.066	392	0.062	265	0.064	263	0.063
	Average	846	0.072	842	0.070	2025	0.066	2025	0.065
2000	$\lambda_{A\#}$	318	0.073	320	0.075	322	0.074	324	0.065
	λ_{AB}	2552	0.038	2559	0.039	19803	0.035	19824	0.032
	$\lambda_{A\mathrm{C}}$	6927	0.036	6963	0.037	7278	0.039	7284	0.034
	λ_{BA}	1207	0.020	1214	0.021	8824	0.013	8825	0.012
	$\lambda_{B\#}$	519	0.060	519	0.059	862	0.050	864	0.053
	λ_{BC}	3133	0.025	3143	0.024	11051	0.017	11052	0.020
	$\lambda_{B\mathrm{E}}$	1405	0.043	1405	0.045	2346	0.039	2354	0.038
	λ_{CA}	6591	0.023	6628	0.024	16578	0.024	16607	0.024

λ_{CB}	3370	0.011	3380	0.010	1667	0.013	1665	0.013
λ_{CD}	2108	0.025	2114	0.025	3836	0.024	3826	0.024
λ_{DB}	341	0.036	343	0.040	1613	0.022	1606	0.024
λ_{DC}	562	0.032	565	0.034	1610	0.022	1610	0.023
$\lambda_{D\#}$	174	0.075	169	0.076	139	0.073	137	0.078
λ_{DE}	863	0.036	868	0.032	1293	0.031	1293	0.036
$\lambda_{D\mathrm{F}}$	817	0.054	818	0.050	985	0.046	991	0.046
λ_{EC}	743	0.032	741	0.023	1620	0.025	1622	0.029
$\lambda_{E\#}$	206	0.057	206	0.056	152	0.057	150	0.064
λ_{EF}	1320	0.048	1326	0.054	1867	0.049	1874	0.051
λ_{FC}	705	0.037	710	0.038	524	0.038	530	0.042
λ_{FD}	649	0.029	649	0.029	1803	0.020	1810	0.022
λ_{FG}	783	0.045	785	0.043	525	0.042	525	0.049
Average	1680	0.050	1687	0.051	4033	0.046	4037	0.048

Note: Average is the average RMSE of all parameters in each condition, which is the same as the column of RMSE for λ in Table 2.

S2. Supplementary results of the empirical study on the Balance Beam task.

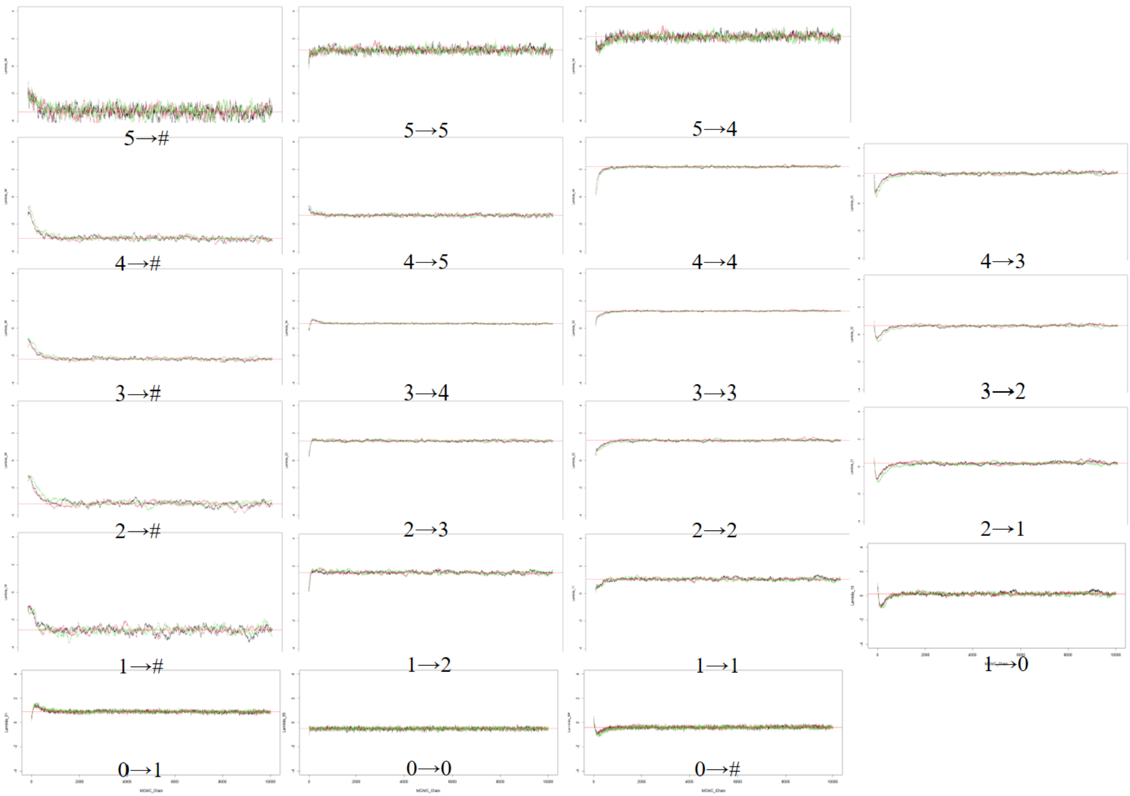

Figure S2. The trace plots of all the transition tendency parameters estimated by SRM-PEI on the Balance Beam task.[1]

[1] Thanks to the anonymous reviewers for their findings. When estimating SRM (Han et al., 2022) and our SRM-PEI using the same MCMC algorithm, low mixing of transition tendency parameters may arise. This issue, while not impacting posterior means, tends to underestimate posterior standard deviations. Therefore, investigating advanced sampling techniques and effective proposal distributions is recommended.

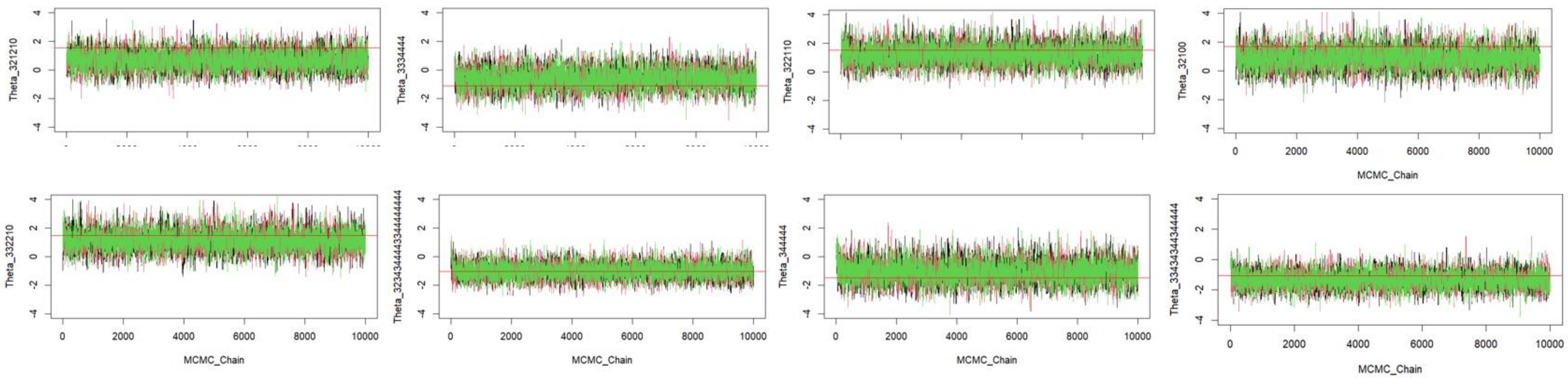

Figure S3. The trace plots of latent abilities of some typical sequences estimated by SRM-PEI on the Balance Beam task.

Table S4. The transition tendency parameters estimated by SRM and SRMM with two forms of simplified polytomous effectiveness on the Balance Beam task.

Transition	SRM-v1			SRM-v2			SRMM-v1			SRMM-v2		
	Effectiveness	Mean	SD	Effectiveness	Mean	SD	Effectiveness	Mean	SD	Effectiveness	Mean	SD
5→#	-1	-2.981	0.348	-1	-2.756	0.369	0	-4.921	0.484	0	-1.003	0.330
5→5	1	1.215	0.190	-1	0.705	0.201	1	2.167	0.256	0	-0.919	0.331
5→4	1	1.765	0.186	1	2.050	0.209	1	2.754	0.249	1	1.922	0.076
4→#	-1	-2.632	0.128	-1	-2.416	0.109	0	-3.733	0.508	0	-0.154	0.235
4→5	-1	-1.546	0.085	-1	-1.339	0.079	0	-3.799	0.501	0	-0.878	0.242
4→4	1	2.262	0.061	-1	1.729	0.053	1	3.940	0.090	0	-0.245	0.304
4→3	1	1.916	0.062	1	2.025	0.086	1	3.592	0.090	1	1.278	0.031
3→#	-1	-2.000	0.061	-1	-1.921	0.057	0	-1.521	0.334	0	-0.138	0.359
3→4	-1	0.219	0.046	-1	0.309	0.035	0	-0.770	0.342	0	0.227	0.381
3→3	1	1.228	0.043	-1	1.017	0.033	1	1.483	0.059	0	-0.326	0.568
3→2	1	0.553	0.045	1	0.594	0.080	1	0.808	0.061	1	0.237	0.029
2→#	-1	-3.083	0.149	-1	-3.103	0.170	0	-0.009	0.406	0	0.637	0.461
2→3	-1	1.367	0.065	-1	1.362	0.062	0	-0.248	0.420	0	-0.092	0.388
2→2	1	1.348	0.066	-1	1.435	0.062	1	0.618	0.062	0	0.081	0.378
2→1	1	0.368	0.068	1	0.307	0.100	1	-0.361	0.067	1	-0.625	0.033
1→#	-1	-2.881	0.264	-1	-2.948	0.247	0	-0.125	0.750	0	0.129	0.324
1→2	-1	1.536	0.103	-1	1.426	0.095	0	0.398	0.729	0	0.307	0.339
1→1	1	0.866	0.102	-1	1.191	0.094	1	0.058	0.082	0	0.014	0.397
1→0	1	0.479	0.104	1	0.331	0.120	1	-0.330	0.086	1	-0.450	0.043
0→1	-1	0.867	0.101	-1	0.648	0.086	0	2.136	0.258	0	0.163	0.391
0→0	1	-0.823	0.106	-1	-0.343	0.106	1	-1.443	0.167	0	0.224	0.393

0→#	1	0.087	0.043	1	-0.305	0.111	1	-0.693	0.137	1	-0.387	0.135