Appendix A

We here provide the general model in matrix formulation. Matrix formulation is more general than a scalar version and particularly useful for discussing model identification discussed in Section 3.8. We write the general model (Equation (2) in the manuscript) for person p, assuming $K = 2$ and $n = 1$ as follows (subscript p will be dropped for conciseness):

$$\mathbf{y} = \boldsymbol{\beta} + \Lambda\boldsymbol{\theta}' + \boldsymbol{\epsilon}, \tag{1}$$

where $\mathbf{y}$ is a $I = (S_1 \times S_2)$ dimensional vector of observed responses, $\boldsymbol{\beta}$ is a I dimensional vector of the means (or intercepts), Λ is a $I \times (1 + S_1 + S_2)$ matrix of factor loadings, $\boldsymbol{\theta}$ is a $(1 + S_1 + S_2)$ vector of all factors, $\boldsymbol{\theta} = (\theta^G, \theta_1^{S_1}, ..., \theta_{s_1}^{S_1}, ..., \theta_{S_1}^{S_1}, \theta_1^{S_2}, ..., \theta_{s_2}^{S_2}, ..., \theta_{S_2}^{S_2})'$, and $\boldsymbol{\epsilon}$ is a I dimensional vector of residuals. We set $E(\boldsymbol{\theta}) = \mathbf{0}$, $E(\boldsymbol{\epsilon}) = \mathbf{0}$, $\boldsymbol{\theta} \sim N(\mathbf{0}, \Phi)$ and $\boldsymbol{\epsilon} \sim N(\mathbf{0}, \Psi)$, where Φ is a $(1 + S_1 + S_2) \times (1 + S_1 + S_2)$ identity matrix (if all factor loadings are freely estimated) and Ψ is a $I \times I$ diagonal matrix with free diagonal elements, ψ_{ii} $(i = 1, ..., I)$. Hence, $\text{Cov}(\mathbf{y}) = \Sigma = \Lambda\Phi\Lambda' + \Psi$.

The factor loading matrix Λ can be partitioned as

$$\begin{aligned}\Lambda &= \left[\Lambda^G | \Lambda^{S_1} | \Lambda^{S_2}\right], \\ &= \left[\begin{array}{c|c|cccc} A_1^G & A_1^{S_1} & A_1^{S_2} & 0 & \cdots & 0 \\ A_2^G & A_2^{S_1} & 0 & A_2^{S_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_M^G & A_{S_2}^{S_1} & 0 & 0 & \cdots & A_{S_2}^{S_2} \end{array}\right],\end{aligned} \tag{2}$$

where Λ^G is $I \times 1$, Λ^{S_1} is $I \times S_1$, and Λ^{S_2} is $I \times M = S_2$. $A_{s_1}^G$ is a S_1 dimensional column vector, $A_{s_2}^{S_1}$ is a $S_1 \times S_1$ diagonal submatrix, and $A_{s_2}^{S_2}$ is a S_1 dimensional column vector.

The factor covariance matrix Φ can be partitioned as

$$\Phi = \begin{bmatrix} \Phi_{GG} & \Phi_{GS_1} & \Phi_{GS_2} \\ \Phi_{S_1G} & \Phi_{S_1S_1} & \Phi_{S_1S_2} \\ \Phi_{S_2G} & \Phi_{S_2S_1} & \Phi_{S_2S_2} \end{bmatrix}. \tag{3}$$

Along the diagonal, Φ_{GG} is a 1×1 submatrix of Φ that contains the general factor variance, $\Phi_{S_1S_1}$ is a $S_1 \times S_1$ submatrix of Φ that contains the covariance matrix of $\boldsymbol{\theta}^{S_2} = (\theta_t^{S_1}, \ldots, \theta_T^{S_1})'$, and $\Phi_{S_2S_2}$ is a $S_2 \times S_2$ submatrix of Φ that contains the covariance matrix of $\boldsymbol{\theta}^{S_2} = (\theta_1^{S_2}, \ldots, \theta_{S_2}^{S_2})'$. The diagonals of these matrices and therefore of Φ is constrained to 1 by Assumption 3. The three off-diagonal submatrices have all elements equal to zero by Assumption 4; specifically, Φ_{S_1G} and Φ_{S_2G} are $S_1 \times 1$ and $S_2 \times 1$ covariance matrices between $\boldsymbol{\theta}^{S_1}$ and θ^G and between $\boldsymbol{\theta}^{S_2}$ and θ^G, respectively, and $\Phi_{S_1S_2}$ is the $S_1 \times S_2$ matrix of covariances between $\boldsymbol{\theta}^{S_1}$ and $\boldsymbol{\theta}^{S_2}$.

A matrix form of the proportional model (Equation (3) in the manuscript) can be obtained in a straightforward way by imposing equality constraints on the factor loadings of the model and by freeing the diagonal elements of the covariance matrices $\Phi_{S_1S_1}$ and $\Phi_{S_2S_2}$ in (3) above.

Appendix B

We show that the proportional model (Equation (3) in the manuscript) is not equivalent to the second-order version where the general factor has direct effects on the two sets of specific factors. To derive a second-order factor version, we first write the model for two sets of specific (first-order) factors as follows:

$$y_{ip} = \beta_i + \alpha_{is_1}^{S_1}\theta_{s_1p}^{S_1} + \alpha_{is_2}^{S_2}\theta_{s_2p}^{S_2} + \epsilon_{ip}. \tag{4}$$

The first-order factors $\theta_{s_1p}^{S_1}$ and $\theta_{s_2p}^{S_2}$ have the the second-order (general) factor θ_p^G as a regressor:

$$\theta_{s_1p}^{S_1} = \alpha_{s_1}^{G}\theta_p^G + \xi_{s_1p}^{S_1}, \tag{5}$$

$$\theta_{s_2p}^{S_2} = \alpha_{s_2}^{G}\theta_p^G + \xi_{s_2p}^{S_2}, \tag{6}$$

where $\alpha_{s_1}^G$ and $\alpha_{s_2}^G$ are the factor loadings of the first-order factors ($\theta_{s_1p}^{S_1}$ and $\theta_{s_2p}^{S_2}$) on the second-order factor θ_p^G and $\xi_{s_1p}^{S_1}$ and $\xi_{s_2p}^{S_2}$ are the residuals. Combining (4) with (5) and (6), we obtain

$$y_{ip} = \beta_i + \alpha_{is_1}^{S_1}(\alpha_{s_1}^{G}\theta_p^G + \xi_{s_1p}^{S_1}) + \alpha_{s_2i}^{S_2}(\alpha_{s_1}^{G}\theta_p^G + \xi_{s_2p}^{S_2}) + \epsilon_{ip}. \tag{7}$$

We further parameterize (7):

$$
\begin{aligned}
y_{ip} &= \beta_i + \alpha^{S_1}_{is_1}\alpha^G_{s_1}\theta^G_p + \alpha^{S_1}_{is_1}\xi^{S_1}_{s_1p} + \alpha^{S_2}_{is_2}\alpha^G_{s_1}\theta^G_p + \alpha^{S_2}_{is_2}\xi^{S_2}_{s_2p} + \epsilon_{ip}, && (8)\\
&= \beta_i + (\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})\theta^G_p + \alpha^{S_1}_{is_1}\xi^{S_1}_{s_1p} + \alpha^{S_2}_{is_2}\xi^{S_2}_{s_2p} + \epsilon_{ip}, && (9)\\
&= \beta_i + (\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_1})\left\{\theta^G_p + \frac{\alpha^{S_1}_{is_1}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}\xi^{S_1}_{s_1p} + \frac{\alpha^{S_2}_{is_2}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}\xi^{S_2}_{s_2p}\right\} + \epsilon_{ip}, && (10)\\
&= \beta_i + \alpha^{*G}_i\left\{\theta^G_p + \frac{\alpha^{S_1}_{is_1}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}\xi^{S_1}_{s_1p} + \frac{\alpha^{S_2}_{is_2}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}\xi^{S_2}_{s_2p}\right\} + \epsilon_{ip}, && (11)
\end{aligned}
$$

where $\alpha^{*G}_i = (\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})$. Equation (11) contains the common factor loadings α^{*G}_i similar to the proportional model. However, the proportionality constants $\frac{\alpha^{S_1}_{is_1}}{(\alpha^{S_1}_{is_1}\alpha^G_t + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}$ and $\frac{\alpha^{S_2}_{is_2}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}$ cannot be simplified to $C^{*\ S_1}_{s_1}$ or $C^{*\ S_2}_{s_2}$, as in the proportional model (Equation (3) in the manuscript), because they depend on i, s_1, and s_2, rather than only s_1 and s_2. That is, $C^{*\ S_1}_{s_1} \neq \frac{\alpha^{S_1}_{is_1}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}$ and $C^{*\ S_2}_{s_2} \neq \frac{\alpha^{S_2}_{is_2}}{(\alpha^{S_1}_{is_1}\alpha^G_{s_1} + \alpha^{S_2}_{is_2}\alpha^G_{s_2})}$. Therefore, the proportional model is not equivalent to the second-order version (11). In fact, the second-order factor version is more complex than the proportional model as it requires $(2I + S_1 + S_2)$ loading parameters while the proportional model contains $(I + S_1 + S_2)$ loading parameters. In addition, the second-order factor version cannot be reduced to the proportional by imposing constraints on the parameters. That is, the proportional model is not nested within the second-order version.

Appendix C

We here show how the Wald rank rule can be applied to test local identification of the proposed models. To illustrate, we first consider the general model for the mean-centered data with $S_1 = S_2 = 3$ and $n = 1$ (i.e., $I = 9$). As indicated in the article, local identification evaluation is based on a matrix formulation of the models (provided in Appendix A).

First, we define the factor loading matrix Λ (of size 9×9) and construct the covariance matrix $\Sigma = \Lambda\Phi\Lambda' + \Psi$, where Φ is the identity matrix and Ψ is a 9×9 diagonal matrix. Second, define ϑ, a vector or nonredundant elements of the covariance matrix (Σ) and φ, a vector of unknown model parameters. Third, compute the Jacobian matrix $J(\boldsymbol{\theta})$ as $\frac{\partial\boldsymbol{\vartheta}}{\partial\boldsymbol{\varphi}}$, which is the first derivatives of $\boldsymbol{\vartheta}$ with respect to the unknown parameter vector $\boldsymbol{\varphi}$. Fourth, check whether the Jacobian matrix $J(\boldsymbol{\theta})$ is full rank (or its rank is equal to the dimension of $\boldsymbol{\varphi}$).

Below is the Mathematica (Wolfram Research, Inc, 2010) code to implement the procedure for testing local identification of the general model. For the proportional model, the same procedure can be applied by simply altering the factor loading matrix (from Equation (1) to Equation (4) in the manuscript) (i.e., by modifying `lam` in the code).

```
* Factor loading matrix Lambda
lam = {{a1m1, 0, 0, a1t1, 0, 0, a1g}, {a2m1, 0, 0, 0, a2t2, 0, a2g}, {a3m1, 0,
       0, 0, 0, a3t3, a3g}, {0, a4m2, 0, a4t1, 0, 0, a4g}, {0, a5m2, 0, 0,
      a5t2, 0, a5g}, {0, a6m2, 0, 0, 0, a6t3, a6g}, {0, 0, a7m3, a7t1, 0, 0,
      a7g}, {0, 0, a8m3, 0, a8t2, 0, a8g}, {0, 0, a9m3, 0, 0, a9t3, a9g}}

lamT = Transpose[lam]
cov1 = lam.lamT
cov2 = DiagonalMatrix[{c1, c2, c3, c4, c5, c6, c7, c8, c9}]

* Covariance matrix Sigma
cov = cov1 + cov2

* A vector of unknown parameters
parvec = {a1m1, a2m1, a3m1, a4m2, a5m2, a6m2, a7m3, a8m3, a9m3, a1t1, a2t2,
    a3t3, a4t1, a5t2, a6t3, a7t1, a8t2, a9t3, a1g, a2g, a3g, a4g, a5g, a6g,
    a7g, a8g, a9g, c1, c2, c3, c4, c5, c6, c7, c8, c9 }

* A vector of nonredundant elements of the covariance matrix
\!\(\(\(reduced\)\(\ \)\(=\)\(\ \)\({\ a1g\^2 + a1m1\^2 + a1t1\^2 + c1, \
      a1g\ a2g + a1m1\ a2m1, \ a1g\ a3g + a1m1\ a3m1, \
      a1g\ a4g + a1t1\ a4t1, \ a1g\ a5g, \ a1g\ a6g, \
      a1g\ a7g + a1t1\ a7t1, \ a1g\ a8g, \
      a1g\ a9g, \[IndentingNewLine]a2g\^2 + a2m1\^2 + a2t2\^2 + c2, \
      a2g\ a3g + a2m1\ a3m1, \ a2g\ a4g, \ a2g\ a5g + a2t2\ a5t2, \
      a2g\ a6g, \ a2g\ a7g, \ a2g\ a8g + a2t2\ a8t2, \
      a2g\ a9g, \[IndentingNewLine]a3g\^2 + a3m1\^2 + a3t3\^2 + c3, \
      a3g\ a4g, \ a3g\ a5g, \ a3g\ a6g + a3t3\ a6t3, \ a3g\ a7g, \
      a3g\ a8g, \
      a3g\ a9g + a3t3\ a9t3, \[IndentingNewLine]a4g\^2 + a4m2\^2 + a4t1\^2 +
        c4, \ a4g\ a5g + a4m2\ a5m2, \ a4g\ a6g + a4m2\ a6m2, \
      a4g\ a7g + a4t1\ a7t1, \ a4g\ a8g, \
      a4g\ a9g, \[IndentingNewLine]a5g\^2 + a5m2\^2 + a5t2\^2 + c5, \
      a5g\ a6g + a5m2\ a6m2, \ a5g\ a7g, \ a5g\ a8g + a5t2\ a8t2, \
      a5g\ a9g, \ \[IndentingNewLine]a6g\^2 + a6m2\^2 + a6t3\^2 + c6, \
      a6g\ a7g, \ a6g\ a8g, \
      a6g\ a9g + a6t3\ a9t3, \[IndentingNewLine]a7g\^2 + a7m3\^2 + a7t1\^2 +
        c7, \ a7g\ a8g + a7m3\ a8m3, \
      a7g\ a9g + a7m3\ a9m3, \[IndentingNewLine]a8g\^2 + a8m3\^2 + a8t2\^2 +
        c8, \ a8g\ a9g + a8m3\ a9m3, \ \[IndentingNewLine]a9g\^2 + a9m3\^2 +
        a9t3\^2 + c9\[IndentingNewLine]\ }\)\(\ \)\)\)

* Compute the Jacobian matrix
\!\(\(\(Jac\)\(=\)\(\ \)\(D[{a1g\^2 + a1m1\^2 + a1t1\^2 + c1, \
        a1g\ a2g + a1m1\ a2m1, \ a1g\ a3g + a1m1\ a3m1, \
        a1g\ a4g + a1t1\ a4t1, \ a1g\ a5g, \ a1g\ a6g, \
        a1g\ a7g + a1t1\ a7t1, \ a1g\ a8g, \
        a1g\ a9g, \[IndentingNewLine]a2g\^2 + a2m1\^2 + a2t2\^2 + c2, \
        a2g\ a3g + a2m1\ a3m1, \ a2g\ a4g, \ a2g\ a5g + a2t2\ a5t2, \
```

```
    a2g\ a6g, \ a2g\ a7g, \ a2g\ a8g + a2t2\ a8t2, \
    a2g\ a9g, \[IndentingNewLine]a3g\^2 + a3m1\^2 + a3t3\^2 + c3, \
    a3g\ a4g, \ a3g\ a5g, \ a3g\ a6g + a3t3\ a6t3, \ a3g\ a7g, \
    a3g\ a8g, \
    a3g\ a9g + a3t3\ a9t3, \[IndentingNewLine]a4g\^2 + a4m2\^2 +
      a4t1\^2 + c4, \ a4g\ a5g + a4m2\ a5m2, \ a4g\ a6g + a4m2\ a6m2, \
    a4g\ a7g + a4t1\ a7t1, \ a4g\ a8g, \
    a4g\ a9g, \[IndentingNewLine]a5g\^2 + a5m2\^2 + a5t2\^2 + c5, \
    a5g\ a6g + a5m2\ a6m2, \ a5g\ a7g, \ a5g\ a8g + a5t2\ a8t2, \
    a5g\ a9g, \ \[IndentingNewLine]a6g\^2 + a6m2\^2 + a6t3\^2 + c6, \
    a6g\ a7g, \ a6g\ a8g, \
    a6g\ a9g + a6t3\ a9t3, \[IndentingNewLine]a7g\^2 + a7m3\^2 +
      a7t1\^2 + c7, \ a7g\ a8g + a7m3\ a8m3, \
    a7g\ a9g + a7m3\ a9m3, \[IndentingNewLine]a8g\^2 + a8m3\^2 +
      a8t2\^2 + c8, \
    a8g\ a9g + a8m3\ a9m3, \ \[IndentingNewLine]a9g\^2 + a9m3\^2 +
      a9t3\^2 + c9}, \ {{a1m1, \ a2m1, \ a3m1, \ a4m2, \ a5m2, \ a6m2, \
      a7m3, \ a8m3, \ a9m3, \ a1t1, \ a2t2, \ a3t3, \ a4t1, \ a5t2, \
      a6t3, \ a7t1, a8t2, \ a9t3, \ a1g, \ a2g, \ a3g, \ a4g, \ a5g, \
      a6g, \ a7g, \ a8g, \ a9g, \ c1, c2, c3, c4, c5, c6, c7, c8,
      c9\ \ }\ }]\)\(\ \)\)\)

* Check whether the rank of the Jacobian matrix is full rank
MatrixRank[Jac]
```

Appendix D

We describe the simulation conditions that we selected to examine the local and empirical identification of the proposed models. For assessing identification of the proposed models, it is impractical to consider all possible combinations of N_{S_1} and N_{S_2}. If a model for a given $N^*_{S_1}$, $N^*_{S_2}$ is identified, then models with $N_{S_1} \geq N^*_{S_1}$ and $N_{S_2} \geq N^*_{S_2}$ will also be identified. We therefore start with the smallest model, $N_{S_1} = 1$ or $N_{S_2} = 1$; however, this model type is infeasible because the general factor is completely confounded with each of the secondary factors. Next we consider $N_{S_1} = 2$ and $N_{S_2} = 2$ (or $I = 4$); in this instance, neither the general nor the proportional model meet the necessary condition that $I + \frac{1}{2}I(I+1) \geq q$. (For the general model, $q = 5I = 20 > I + \frac{1}{2}I(I+1) = 14$; for the proportional model, $q = 3I + T + M = 16 > I + \frac{1}{2}I(I+1) = 14$.) When $N_{S_1} = 2$ and $N_{S_2} = 3$ (or $N_{S_1} = 3$ and $N_{S_2} = 2$; $I = 6$), the general model does not satisfy the necessary condition as $q = 5I = 30 > I + \frac{1}{2}I(I+1) = 27$, while the proportional model does ($q = 3I + N_{S_1} + N_{S_2} = 23 < I + \frac{1}{2}I(I+1) = 27$). In the case of two measurements per condition, it is reasonable to impose equality constraints on the loadings of the corresponding specific factor, $\alpha^{S_1}_{is_1} = \alpha^{S_1}_{i's_1}$, where measurements i and i' correspond to condition s_1 of Source S_1. With $N_{S_1} = 2$ and $N_{S_2} = 2$, the number of model parameters of the general model is reduced to

$q = 3I + N_{S_1} + N_{S_2} = 16$; that is, even with the equality constraint, the general model still does not meet the necessary condition ($q = 16 > I + \frac{1}{2}I(I+1) = 14$). With $N_{S_1} = 2$ and $N_{S_2} = 3$, the number of parameters of the general model is reduced to $q = 4I + N_{S_2} = 27$ with the equality constraint, thus satisfying the sample size requirement as $q = I + \frac{1}{2}I(I+1)$. Hence, we consider $N_{S_1} = 2$ and $N_{S_2} = 3$ as the minimum sample size condition by permitting the equality constraints on factor loadings for the general model. Equality constraints are not applicable for the proportional model that does not include separate sets of specific factor loadings. We then consider more conditions by increasing N_{S_1} and N_{S_2} from two to four.

Appendix E

Here we provide example Mplus code for Example 2 (Teachers' perception on students' academic abilities). There are three ability traits ($S_1 = 3$) measured at three time points ($S_2 = 3$); hence, there are in total nine observations ($I = 9$). The correlation matrix for sample size $N = 4,753$ is used as the input data.

```
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
!! General model                    !!
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

TITLE: Fitting a general model for Example 2

! Define input data and number of observations
DATA: FILE IS academic.dat;
        TYPE = CORRELATION  ;
        NOBSERVATIONS = 4753;

! Define variable names
VARIABLE:
        NAMES ARE y1-y9;

! Define estimation method
ANALYSIS:
    ESTIMATOR = ML;

! Model specification
MODEL:

! Define variables for the general factor (overall ability)
! All factor loadings are freely estimated
fg by y1-y9*;

! Define specific factors for three ability traits
! All factor loadings are freely estimated
f1 by y1 y4* y7*  ;
f2 BY y2 y5* y8* ;
```

```
f3 by y3 y6* y9* ;

! Define specific factors for three occasions
! All factor loadings are freely estimated
m1 by y1-y3*  ;
m2 BY y4-y6*  ;
m3 BY y7-y9*  ;

! Set variances for all factors to 1
fg@1;
f1-f3@1;
m1-m3@1;

! Set covariances among all factors to 0
fg with f1-f3@0;
fg with m1-m3@0;
m1 with m2-m3@0;
m2 with m3@0;
f1 with f2-f3@0;
f2 with f3@0;
f1-f3 with m1-m3@0;


!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
!! Proportional model              !!
!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!

TITLE: Fitting a proportional model for Example 2

! Define input data and number of observations
DATA: FILE IS academic.dat;
        TYPE = CORRELATION  ;
        NOBSERVATIONS = 4753;

! Define variable names
VARIABLE:
        NAMES ARE y1-y9;

! Define estimation method
ANALYSIS:
    ESTIMATOR = ML;

! Model specification
MODEL:

! Define specific factors for the general factor (overall ability)
! All factor loadings are freely estimated
fg by y1-y9* (load1-load9);

! Define specific factors for three ability traits
```

```
! All factor loadings are set equal to the general factor loadings
f1 by y1* y4* y7* (load1 load4 load7) ;
f2 BY y2* y5* y8* (load2 load5 load8);
f3 by y3* y6* y9* (load3 load6 load9);

! Define specific factors for three occasions
! All factor loadings are set equal to the general factor loadings
m1 by y1-y3* (load1-load3) ;
m2 BY y4-y6* (load4-load6) ;
m3 BY y7-y9* (load7-load9) ;

! Set variance for the general factor to 1
! Freely estimate the variances for the two sets of specific factors
fg@1;

! Set covariances among all factors to 0
fg with f1-f3@0;
fg with m1-m3@0;
m1 with m2-m3@0;
m2 with m3@0;
f1 with f2-f3@0;
f2 with f3@0;
f1-f3 with m1-m3@0;
```

Appendix F

We describe the simulation study that we conducted to evaluate parameter recovery and scalability of the proposed models with an increasing number (K) of secondary sources. Each secondary source was set to have 3 conditions ($N_{S_k} = 3$) and there was one manifest variable ($n = 1$) for each combination of conditions across the K sources. Hence, the number of secondary factors is $N_s = N_{S_k} \times K = 3K$ and the total number of manifest variables is $I = (N_{s_k})^K$. For the general model, the general and secondary factor loadings were randomly sampled from a set of $[1.0, 1.1, 1.2, ..., 2.5]$ and of $[0.5, 0.6, 0.7, ..., 2.0]$ (with replacement), respectively. For the proportional model, the proportionality constants were sampled from a set of $[0.5, 0.6, 0.7, 0.8]$ (with replacement). For both models, the residual variances were set to 1 for all I manifest variables. We used the empirical covariance matrices as input and did not provide corresponding mean vectors because the intercepts are not of interest. The total number of parameters to estimate (excluding the intercepts) is $q = 2I + I \cdot K$ for the general model and $q = 2I + N_{S_k} \cdot K$ for the proportional model.

We set the number of secondary sources to $K = 2$, $K = 3$, $K = 4$ and $K = 5$ and used a large sample size of $N = 10000$. Table 1 summarizes the number of secondary sources, the number of secondary factors, the total number of manifest variables, the number of parameters, and the computation time.

[Table 1 about here]

The number of the model parameters grows rapidly especially for the general model as the number of the secondary variance sources increases. However, computation is fast and occurs within 3 seconds when $K \leq 4$ and even when $K = 5$, which is quite an unrealistic condition in practice, the time used for the computation is still manageable (about 90 seconds) with a regular computer (a 64-bit operating system with a Intel Dual Xeon 2.4-GHz processor computer with 32 GB of memory). It is worth remarking that even though the number of parameters is markedly reduced with the proportional model, its computation time is similar to the general model's. This is because the proportional model is currently estimated with Mplus as a general model with a set of linear constraints.

Next, to evaluate the parameter recovery of the proposed models (general, proportional), we generated 50 datasets assuming a realistic condition with $K = 3$ and $N = 1000$. Figure 1 displays box plots of the error $(\hat{\varrho} - \varrho)$ for all model parameters of the two models in this setting.

[Figure1 about here]

Parameter recovery of the general model as well as the proportional model appears to be satisfactory, with no obvious bias, when the full information maximum likelihood estimation is used in Mplus.

References

Wolfram Research, Inc (2010). *Mathematica Edition: Version 8.0.* Wolfram Research, Inc, Champaign, Illinois.

TABLE 1.
Scalability of the general and proportional models in terms of model complexity and computation time. K is the number of secondary variance sources considered, N_s is the number of secondary factors, I is the total number of manifest variables, q is the number of model parameters (excluding the intercepts that are not estimated), and Time is the computation time (in seconds) with Mplus.

K	N_s	I	q	Time (sec)
General model				
$K = 2$	6	9	63	< 1
$K = 3$	9	27	189	< 1
$K = 4$	12	81	567	3
$K = 5$	15	243	1701	90
Proportional model				
$K = 2$	6	9	33	< 1
$K = 3$	9	27	69	< 1
$K = 4$	12	81	177	3
$K = 5$	15	243	501	90

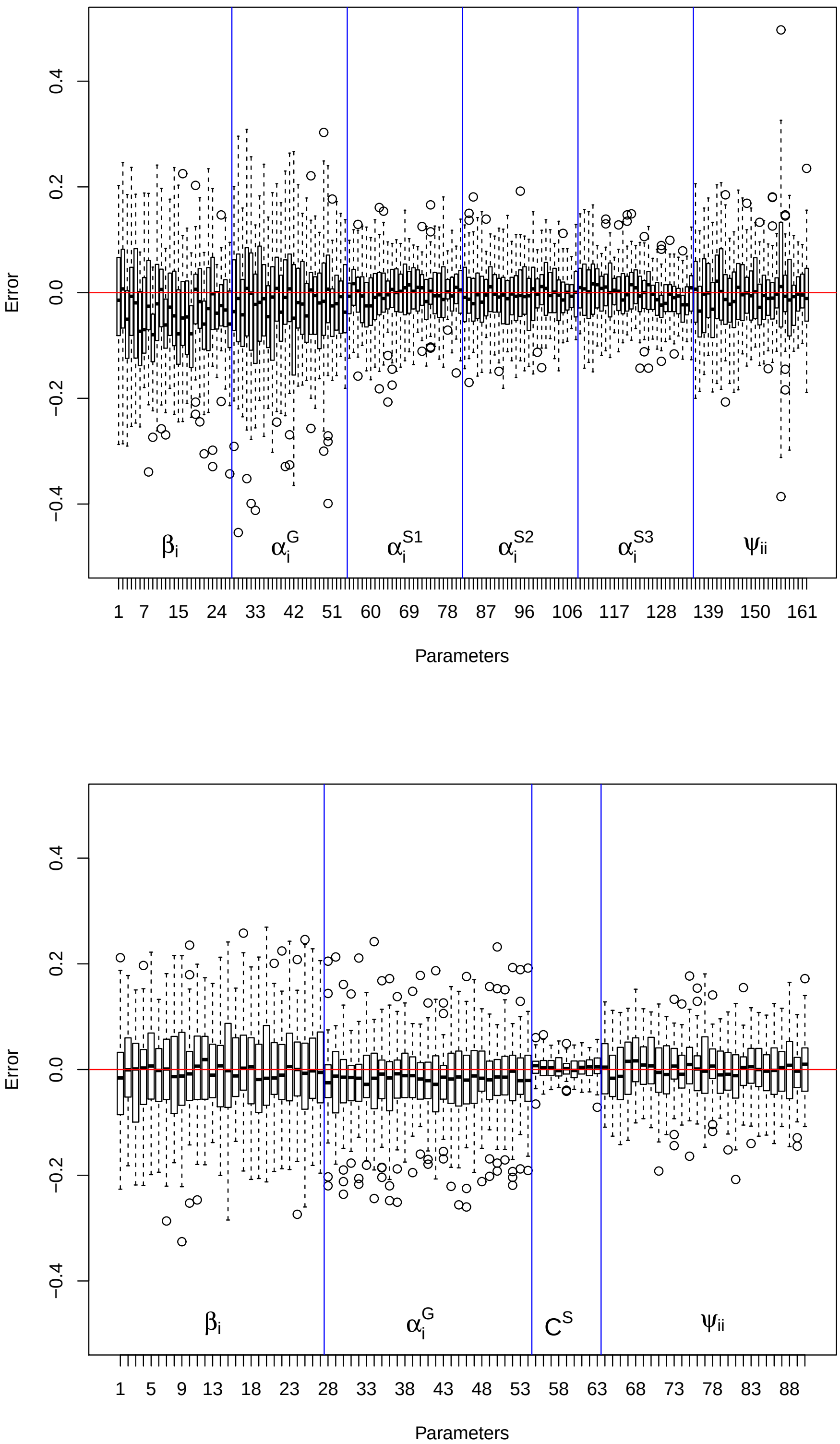

Figure 1: The error $\hat{\varrho} - \varrho$ of the parameters of the general model ($q = 162$, top panel) and the proportional model ($q = 90$, bottom panel) when $K = 3$ and $N = 1000$.