## SUPPLEMENTARY MATERIALS

### Appendix A

Figure a(a) and (b) of the main manuscript illustrate latent spaces for item response data with no dependence (a) and with strong dependence (b). The two latent spaces were obtained by fitting the proposed model (extended distance model) to the simulated data assuming no dependence and strong dependence, respectively. For both scenarios, the sample size was set to $I = 14$ and $= 200$. For Figure 2(a), we simulated binary item response data from the standard Rasch model where the 'true' values for $\beta_i$ were randomly selected from $[-2, 2]$ with replacement, while the 'true' values for $\alpha_j$ were generated from a normal distribution with a mean of 0 and a standard deviation of 2. For Figure 1(b), we generated binary item response data with local dependence based on the general interaction model (which is also presented in Equation (4) of the main manuscript):

$$\text{logit}(\mathbb{P}(y_{j,i} = 1 \mid \alpha_j, \beta_j, \zeta_{j,i})) \;=\; \alpha_j + \beta_i + \zeta_{j,i}, \tag{1}$$

where $\zeta_{j,i} \in \mathbb{R}$ represents the interaction effect between respondent $j$'s ability and item $i$'s easiness. $\zeta_{j,i}$ captures the relation or dependence between person $j$ and item $i$ after controlling for the person and item main effects $\alpha_j$ and $\beta_i$. As a consequence, $\zeta_{j,i}$ induces local dependence.

To simulate a scenario where the first 100 respondents have local dependence with Items 1–7, while the last 100 respondents have local dependence with Items 8–14, we generated $\zeta_{j,i}$ from a normal distribution with a mean of 2 and a variance of 0.2 for the first group ($j = 1, \cdots, 100$, and $i = 1, \cdots, 7$) as well as for the second group ($j = 101, \cdots, 200$, and $i = 8, \cdots, 14$); for all other cases, $\zeta_{j,i} = 0$. We set $\alpha_j = 0$ and $\beta_i = -3$ for all $j$ and $i$ for simplicity; importantly, this way, the respondents tend to give incorrect responses to the items that they have no dependence with, i.e., $\zeta_{j,i} = 0$, but they tend to give correct responses to the items that they have dependence with.

Note that generating the $\zeta_{j,i}$'s independently from a continuous distribution means that the absolute values of the generated $\zeta_{j,i}$'s almost certainly violate the triangle inequality, so the absolute values of the $\zeta_{j,i}$'s are almost certainly not distances. We discussed relations between the interaction model and our proposed model in Section 2.3 of the manuscript.

## Appendix B

Here we provide some trace plots of the MCMC draws obtained from the extended distance model applied to Example 1 (attitudes to abortion data).

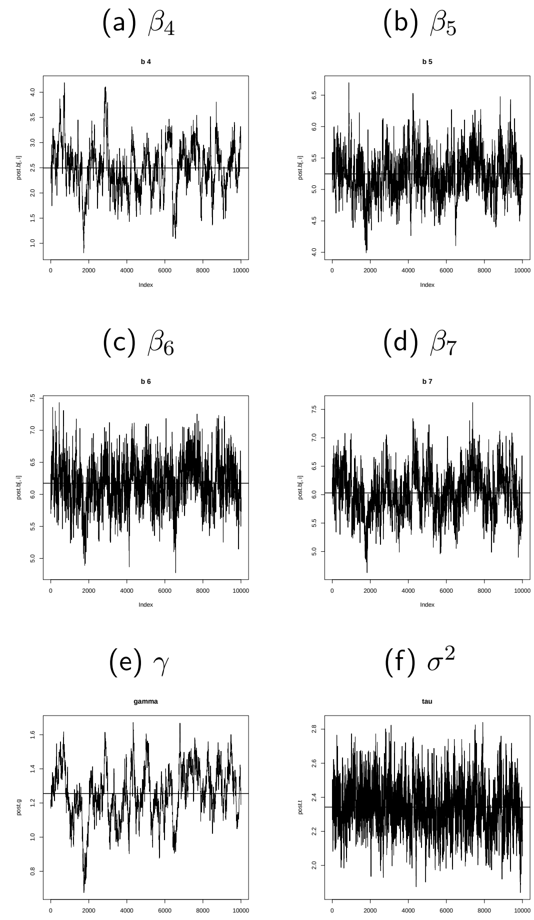


FIGURE 1.

Trace plots of the MCMC draws for the extended distance model applied to the attitudes to abortion data. (a)-(d) are for the $\beta_i$ parameters for selected items (Items 4 to 7), while (e) and (f) are for the $\gamma$ and $\sigma$ parameters, respectively. Not all parameters are shown for space limit.

## Appendix C

Here we present the posterior means and 95% credible intervals for the latent space positions estimated from the extended distance model for Example 1 (attitudes to abortion data). Table 1 lists the estimates for the item positions.

| | $X$-axis | | | $Y$-axis | | |
|---|---|---|---|---|---|---|
| Item | Post.m | 2.5% | 97.5% | Post.m | 2.5% | 97.5% |
| 1 | 0.39 | 0.22 | 0.57 | 0.09 | -0.09 | 0.29 |
| 2 | 0.35 | 0.18 | 0.54 | 0.05 | -0.09 | 0.19 |
| 3 | 0.25 | 0.11 | 0.43 | 0.03 | -0.12 | 0.18 |
| 4 | 0.29 | 0.13 | 0.46 | 0.01 | -0.18 | 0.20 |
| 5 | -0.12 | -0.25 | -0.01 | -0.01 | -0.14 | 0.12 |
| 6 | -0.12 | -0.23 | -0.02 | -0.01 | -0.13 | 0.10 |
| 7 | -0.16 | -0.28 | -0.05 | -0.02 | -0.15 | 0.10 |

TABLE 1.
The posterior means (Post.m) and 95% credible intervals (2.5% and 97.5% for lower and upper boundaries) for the item positions ($\boldsymbol{B}$) estimated from the extended distance model for Example 1.

The estimates for the respondent positions ($\boldsymbol{A}$) are not listed here due to the large sample size $N = 642$. For the $X$-axis, the posterior means ranged from -0.14 to 0.40, while the posterior standard deviations ranged from 0.15 to 0.23. For the $Y$-axis, the posterior means ranged from -0.04 to 0.07, while the posterior standard deviations ranged from 0.15 to 0.26.

## Appendix D

We fit the Rasch model to Example 1 (attitudes to abortion data) with two estimation methods: MCMC and maximum likelihood (ML) estimation. For MCMC, we selected the following priors: $\beta_i \sim \mathsf{N}\left(0, \tau_\beta^2\right)$, $\alpha_j \mid \sigma^2 \sim \mathsf{N}\left(0, \sigma^2\right)$, and $\sigma^2 \sim$ Inv-Gamma $(a_\sigma, b_\sigma)$, with $\tau_\beta^2 = 1$ , $a_\sigma = 1$, and $b_\sigma = 1$ which were the same values chosen to fit our proposed model (extended distance model). All other MCMC settings were identical to the proposed model (e.g., starting values, the number of iterations, burn-in period). For ML estimation, we used the R package lme4 (Bates, Mächler, Bolker, & Walker,

2015) which applies the Laplace approximation for binary data. Table 2 lists the point estimates as well as the 95% intervals of the estimates for the Rasch model parameters. The comparison shows that the MCMC and ML estimates are close and result in identical inferences for most parameters.

| | MCMC | | | ML | | |
|---|---|---|---|---|---|---|
| | Post.m | 2.5% | 97.5% | MLE | 2.5% | 97.5% |
| $\beta_1$ | -0.61 | -0.91 | -0.32 | -0.52 | -0.83 | -0.22 |
| $\beta_2$ | 0.09 | -0.19 | 0.38 | 0.22 | -0.08 | 0.52 |
| $\beta_3$ | -0.24 | -0.53 | 0.04 | -0.12 | -0.42 | 0.18 |
| $\beta_4$ | 0.16 | -0.12 | 0.44 | 0.30 | -0.01 | 0.60 |
| $\beta_5$ | 3.48 | 2.98 | 4.18 | 3.87 | 3.45 | 4.29 |
| $\beta_6$ | 4.39 | 3.75 | 5.18 | 4.88 | 4.36 | 5.40 |
| $\beta_7$ | 4.13 | 3.54 | 4.88 | 4.57 | 4.09 | 5.06 |
| $\sigma$ | 2.42 | 1.89 | 2.99 | 2.53 | - | - |

TABLE 2.

Result comparison between MCMC and maximum likelihood estimation (fitted with Laplace approximation of the lme4 package in R). Point estimates (posterior mean (Post.m) and MLE) and the lower and upper boundary values of the 95% credible/confidence intervals are shown.

## Appendix E

Here we provide additional details on the DRV test data applied in Example 2. The DRV test was developed with three design factors (Spiel, Gluck, & Gossler, 2001). First, the Type of inference factor concerns four inference types based on different premises and conclusions as follows:

(1) Modus Ponens (MP; A, therefore B)
(2) Modus Tollens (MT; Not B, therefore not A)
(3) Negation of Antecedent (NA; Not A, therefore B or not B)
(4) Affirmation of Consequent (AC; B, therefore A or not A).

Modus Ponens (MP) and Modus Tollens (MT) involve biconditional conclusions (with 'yes' or 'no' options), while negation of antecedent (NA) and affirmation of consequent (AC) contain 'perhaps' as an additional option. The NA and AC items are more difficult than the MP and MT items because they provoke a logically incorrect conclusion. For example, an AC item is "Tom is lying in his bed. Is Tom ill" (the correct answer is 'perhaps'). Hence, the NA and AC items are also called logical fallacy items.

Second, the Content of the Conditional factor differentiates three content types: (1) Concrete (CO), (2) Abstract (AB), and (3) Counterfactual (CF). Typically, the AB and CF item are more complex and therefore more difficult than the CO items. An example of a CF item is "If an object is put into boiling water, it becomes cold." Difficulty differences between abstract and counterfactual items are reportedly unclear (Overton, 1985).

Third, the Presentation of the Antecedent factor differentiates items presented with negation (NE) from items presented without negation (NN). For example, an item with negation goes "If the sun does not shine, Peter wears blue pants." Items with negations are often more difficult than items without negations (Roberge & Mason, 1978).

The 24 DRV items are constructed based on a combination of three design factors as follows: UN_CO_MP (1); UN_CO_NA (2); UN_CO_AC (3); UN_CO_MT (4); N_CO_MP (5); N_CO_NA (6); N_CO_AC (7); N_CO_MT (8); UN_AB_MP (9); UN_AB_NA (10); UN_AB_AC (11); UN_AB_MT (12); N_AB_MP (13); N_AB_NC (14); N_AB_AC (15); N_AB_MT (16); UN_CF_MP (17); UN_CF_NA (18); UN_CF_AC (19); UN_CF_MT (20); N_CF_MP (21); N_CF_NA (22); N_CF_MT (23); N_CF_MT (24), where the number in parenthesis represents the actual item number in the test. The acronyms in the item labels indicate the following levels of the three design factors: (1) UN vs. N: By presentation of the antecedent; no negation (UN) and Negation (N). (2) CO vs. AB vs. AC: By the content of conditional; Concrete (CO), Abstract (AB), and Counterfactual (CF). (3) MP vs. MT vs. NA vs. AC: By type of inference; Modus Ponens (MP), Modus Tollens (MT), Negation of Antecednet (NA), and Affirmation of Consequent (AC).

## Appendix F

Here we provide some trace plots of the MCMC draws obtained from the extended distance model applied to Example 2 (DRV data).

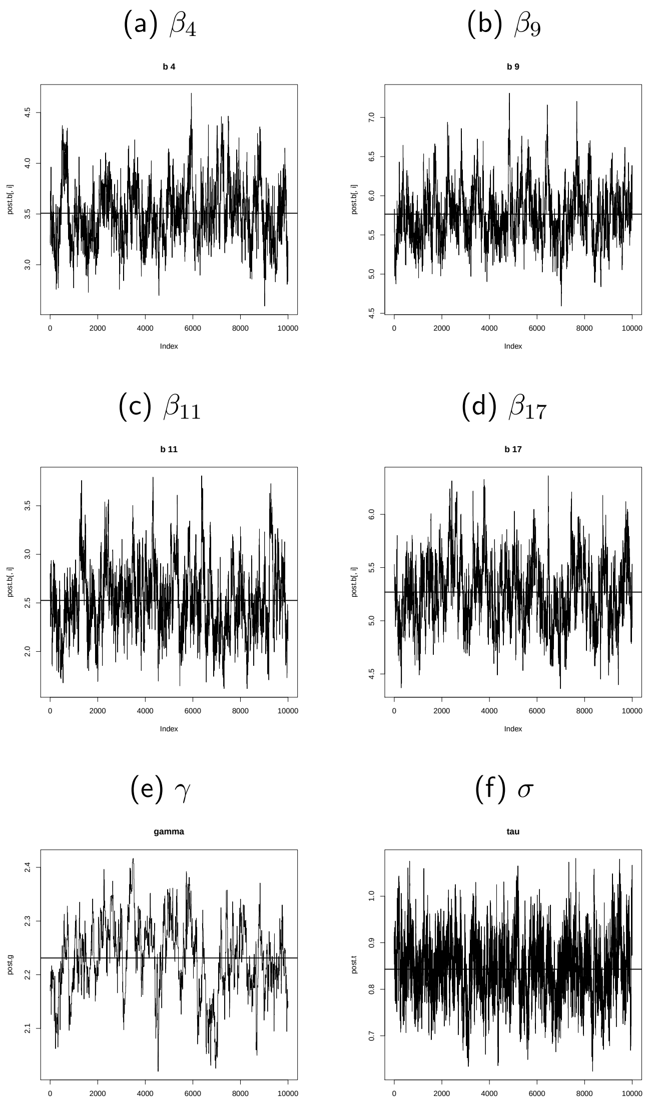


FIGURE 2.

Trace plots of the MCMC draws for the extended distance model applied to the DRV data. (a)-(d) are for the $\beta_i$ parameters for selected items (Items 4, 9, 11, and 17), while (e) and (f) are for the $\gamma$ and $\sigma$ parameters, respectively. Not all parameters are shown for space limit.

## Appendix G

Here we present the posterior means and 95% credible intervals for the latent space positions estimated from the extended distance model for Example 2 (DRV data). Table 3 lists the estimates for the item positions ($\boldsymbol{B}$).

The estimates for the respondent positions ($\boldsymbol{A}$) are not listed here due to the large sample size $N = 418$. For the $X$-axis, the posterior means ranged from -1.49 to 1.41, while the posterior standard deviations ranged from 0.28 to 0.96. For the $Y$-axis, the posterior means ranged from -1.41 to 1.72, while the posterior standard deviations ranged from 0.20 to 0.63.

| | $X$-axis | | | $Y$-axis | | |
|---|---|---|---|---|---|---|
| Item | Post.m | 2.5% | 97.5% | Post.m | 2.5% | 97.5% |
| 1 | 0.74 | 0.48 | 1.02 | -0.64 | -0.90 | -0.44 |
| 2 | -0.68 | -0.99 | -0.41 | 1.01 | 0.74 | 1.35 |
| 3 | -0.67 | -0.91 | -0.45 | 1.09 | 0.82 | 1.41 |
| 4 | 0.53 | 0.31 | 0.82 | -0.87 | -1.10 | -0.66 |
| 5 | 0.59 | 0.35 | 0.87 | -0.74 | -1.01 | -0.51 |
| 6 | -0.51 | -0.78 | -0.29 | 1.02 | 0.76 | 1.35 |
| 7 | -0.82 | -1.15 | -0.54 | 1.20 | 0.91 | 1.69 |
| 8 | 0.52 | 0.31 | 0.74 | -0.93 | -1.23 | -0.72 |
| 9 | -0.27 | -0.56 | -0.02 | -0.63 | -0.93 | -0.36 |
| 10 | 0.38 | 0.15 | 0.61 | 1.17 | 0.91 | 1.46 |
| 11 | 0.32 | 0.09 | 0.58 | 1.20 | 0.92 | 1.50 |
| 12 | -0.41 | -0.69 | -0.16 | -0.91 | -1.23 | -0.66 |
| 13 | -0.19 | -0.43 | 0.02 | -0.58 | -0.84 | -0.36 |
| 14 | 0.52 | 0.25 | 0.81 | 1.17 | 0.86 | 1.56 |
| 15 | 0.32 | 0.02 | 0.65 | 1.42 | 1.07 | 1.84 |
| 16 | -0.53 | -0.80 | -0.27 | -1.10 | -1.42 | -0.83 |
| 17 | -0.09 | -0.29 | 0.10 | -0.82 | -1.07 | -0.59 |
| 18 | 0.38 | 0.18 | 0.60 | 1.06 | 0.85 | 1.30 |
| 19 | 0.49 | 0.28 | 0.77 | 1.06 | 0.85 | 1.37 |
| 20 | -0.18 | -0.40 | 0.05 | -0.95 | -1.27 | -0.70 |
| 21 | 0.03 | -0.17 | 0.22 | -0.69 | -0.93 | -0.48 |
| 22 | 0.36 | 0.14 | 0.60 | 1.05 | 0.83 | 1.31 |
| 23 | 0.62 | 0.34 | 0.94 | 1.15 | 0.89 | 1.46 |
| 24 | -0.21 | -0.45 | 0.01 | -1.06 | -1.36 | -0.81 |

Table 3.

The posterior means (Post.m) and 95% credible intervals (2.5% and 97.5% for lower and upper boundaries) for the item positions ($\boldsymbol{B}$) estimated from the extended distance model for Example 2.

## References

Bates, D., Mächler, M., Bolker, B., & Walker, S. (2015). Fitting linear mixed-effects models using lme4. *Journal of Statistical Software*, *67*(1), 1–48. doi: 10.18637/jss.v067.i01

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Roberge, J. J., & Mason, E. J. (1978). Effects of negation on adolescents' class and conditional reasoning abilities. *The Journal of General Psychology*, *98*, 187-195. doi: 10.1080/00221309.1978.9920872

Spiel, C., Gluck, J., & Gossler, H. (2001). Stability and change of unidimensionality: The sample case of deductive reasoning. *Journal of Adolescent Research*, *16*, 150-168.