ONLINE SUPPLEMENTARY MATERIALS

1. Detailed derivations of the VBEM-M algorithm for LCDM

(a) VBE-step: In this step, we update the variational density for each latent variable $\boldsymbol{z}_i\,(i = 1, \cdots, N)$.

$$q^*(\boldsymbol{z}_i) \propto \exp\left\{\mathrm{E}_{q(\boldsymbol{\pi},\boldsymbol{\lambda}^*,\boldsymbol{\lambda}_0^*)}[\log p(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\pi}, \boldsymbol{\lambda}^*, \boldsymbol{\lambda}_0^*|\boldsymbol{Q}, \boldsymbol{\Omega}, \widetilde{\mathbf{A}})]\right\}, \tag{1}$$

$$\begin{aligned}\log q^*(\boldsymbol{z}_i) \propto \sum_{l=1}^{L} z_{il} \Bigg(\sum_{j=1}^{J} \Big\{ \log(\sigma(\xi_{ijl})) + \frac{y_{ij}\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*]^{\mathrm{T}}\boldsymbol{h}_{jl}^* - \xi_{ijl}}{2} \\ -\tau(\xi_{ijl})(\boldsymbol{h}_{jl}^{*\mathrm{T}}\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*\boldsymbol{\lambda}_j^{*\mathrm{T}}]\boldsymbol{h}_{jl}^* - \xi_{ijl}^2)\Big\} + \mathrm{E}_{q(\boldsymbol{\pi})}\log(\pi_l)\Bigg).\end{aligned} \tag{2}$$

Let

$$\begin{aligned}\rho_{il} = \exp\Bigg\{\sum_{j=1}^{J} \Big\{ \log(\sigma(\xi_{ijl})) + \frac{y_{ij}\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*]^{\mathrm{T}}\boldsymbol{h}_{jl}^* - \xi_{ijl}}{2} \\ - \tau(\xi_{ijl})(\boldsymbol{h}_{jl}^{*\mathrm{T}}\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*\boldsymbol{\lambda}_j^{*\mathrm{T}}]\boldsymbol{h}_{jl}^* - \xi_{ijl}^2)\Big\} + \mathrm{E}_{q(\boldsymbol{\pi})}\log(\pi_l)\Bigg\},\end{aligned} \tag{3}$$

and

$$\pi_{il}^* = \frac{\rho_{il}}{\sum_{l=1}^{L}\rho_{il}}. \tag{4}$$

Thus, $q^*(\boldsymbol{z}_i)$ is derived as a categorical distribution with parameter $\boldsymbol{\pi}_i^*$,

$$q^*(\boldsymbol{z}_i|\boldsymbol{\pi}_i^*) = \prod_{l=1}^{L} \pi_{il}^{*z_{il}}. \tag{5}$$

(b) VBM-step: In this step, we update the variational density for $\boldsymbol{\pi}$, $\boldsymbol{\lambda}_j^*\,(j = 1, \cdots, J)$, η_0 and $\lambda_{0d}\,(d = 1, \cdots, D)$.

(b1) Update the variational density for $\boldsymbol{\pi}$

$$q^*(\boldsymbol{\pi}) \propto \exp\left\{\mathrm{E}_{q(\boldsymbol{z},\boldsymbol{\lambda}^*,\boldsymbol{\lambda}_0^*)}[\log p(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\pi}, \boldsymbol{\lambda}^*, \boldsymbol{\lambda}_0^* | \boldsymbol{Q}, \boldsymbol{\Omega}, \widetilde{\mathrm{A}})]\right\}, \tag{6}$$

$$\log q^*(\boldsymbol{\pi}) \propto \sum_{l=1}^{L} \left(\sum_{i=1}^{N} \mathrm{E}_{q(\boldsymbol{z}_i)}[z_{il}] + \delta_{0l} - 1 \right) \log \pi_l. \tag{7}$$

Let

$$\delta_l^* = \sum_{i=1}^{N} \mathrm{E}_{q(\boldsymbol{z}_i)}[z_{il}] + \delta_{0l}. \tag{8}$$

Thus, $q^*(\boldsymbol{\pi})$ is derived as a Dirichlet distribution with parameter $\boldsymbol{\delta}^*$.

$$q^*(\boldsymbol{\pi}|\boldsymbol{\delta}^*) \propto \prod_{l=1}^{L} \pi_l^{\delta_l^*-1}. \tag{9}$$

(b2) Update the variational density for $\boldsymbol{\lambda}_j^*$

$$q^*(\boldsymbol{\lambda}_j^*) \propto \exp\left\{\mathrm{E}_{q(\boldsymbol{z},\boldsymbol{\pi},\boldsymbol{\lambda}_0^*)}[\log p(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\pi}, \boldsymbol{\lambda}^*, \boldsymbol{\lambda}_0^* | \boldsymbol{Q}, \boldsymbol{\Omega}, \widetilde{\mathrm{A}})]\right\}, \tag{10}$$

$$\log q(\boldsymbol{\lambda}_j^*) \propto \sum_{i=1}^{N} \sum_{l=1}^{L} \mathrm{E}_{q(\boldsymbol{z}_i)}[z_{il}] \left\{ \frac{y_{ij} \boldsymbol{\lambda}_j^{*\mathrm{T}} \boldsymbol{h}_{jl}^*}{2} - \tau(\xi_{ijl}) (\boldsymbol{\lambda}_j^{*\mathrm{T}} \boldsymbol{h}_{jl}^* \boldsymbol{h}_{jl}^{*\mathrm{T}} \boldsymbol{\lambda}_j^*) \right\} - \frac{(\boldsymbol{\lambda}_j^* - \boldsymbol{\lambda}_0^*)^{\mathrm{T}} (\boldsymbol{\lambda}_j^* - \boldsymbol{\lambda}_0^*)}{2}. \tag{11}$$

Let

$$\begin{aligned} \boldsymbol{V}_j^{*-1} &= \mathbf{I}_{D+1}^{-1} + 2 \sum_{i=1}^{N} \sum_{l=1}^{L} \mathrm{E}_{q(\boldsymbol{z}_i)}[z_{il}] \tau(\xi_{ijl}) \boldsymbol{h}_{jl}^* \boldsymbol{h}_{jl}^{*\mathrm{T}}, \\ \boldsymbol{m}_j^* &= \boldsymbol{V}_j^* \left(\boldsymbol{\lambda}_0^* + \frac{1}{2} \sum_{i=1}^{N} \sum_{l=1}^{L} \mathrm{E}_{q(\boldsymbol{z}_i)}[z_{il}] y_{ij} \boldsymbol{h}_{jl}^* \right). \end{aligned} \tag{12}$$

Then, $q(\boldsymbol{\lambda}_j^*)$ is derived as a multivariate normal distribution with parameters $\boldsymbol{m}_j^*$ and $\boldsymbol{V}_j^*$,

$$q^*(\boldsymbol{\lambda}_j^* | \boldsymbol{m}_j^*, \boldsymbol{V}_j^*) \propto \exp\left\{ -\frac{(\boldsymbol{\lambda}_j^* - \boldsymbol{m}_j^*)^{\mathrm{T}} \boldsymbol{V}_j^{*-1} (\boldsymbol{\lambda}_j^* - \boldsymbol{m}_j^*)}{2} \right\}. \tag{13}$$

(b3) Update the variational density for η_0

$$q^*(\eta_0) \propto \exp\left\{ \mathrm{E}_{q(\boldsymbol{z},\boldsymbol{\pi},\boldsymbol{\lambda}^*)}[\log p(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\pi}, \boldsymbol{\lambda}^*, \boldsymbol{\lambda}_0^* | \boldsymbol{Q}, \boldsymbol{\Omega}, \widetilde{\mathrm{A}})] \right\}, \tag{14}$$

$$\log q(\eta_0) \propto -\sum_{j=1}^{J} \mathrm{E}_{q(\boldsymbol{\lambda}^*)} \left[\frac{(\boldsymbol{\lambda}_j^* - \boldsymbol{\lambda}_0^*)^{\mathrm{T}} (\boldsymbol{\lambda}_j^* - \boldsymbol{\lambda}_0^*)}{2} \right] - \frac{(\eta_0 - \mu_{\eta_0})^2}{2\sigma_{\eta_0}^2}. \tag{15}$$

Let

$$\begin{aligned} (\sigma_{\eta_0}^{*2})^{-1} &= J + \frac{1}{\sigma_{\eta_0}^2}, \\ \mu_{\eta_0}^* &= \sigma_{\eta_0}^{*2} \left(\frac{\mu_{\eta_0}}{\sigma_{\eta_0}^2} + \sum_{j=1}^{J} \left(\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*] \right)_{\eta} \right), \end{aligned} \tag{16}$$

where $\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*]_\eta$ is the corresponding expected value of the element η_j in the vector $\boldsymbol{\lambda}_j^*$. Then $q^*(\eta_0)$ is derived as a normal distribution with parameters $\mu_{\eta_0}^*$ and $\sigma_{\eta_0}^{*2}$,

$$q^*(\eta_0 | \mu_{\eta_0}^*, \sigma_{\eta_0}^{*2}) \propto \exp\left\{ -\frac{(\eta_0 - \mu_{\eta_0}^*)^2}{2\sigma_{\eta_0}^{*2}} \right\}. \tag{17}$$

(b4) Update the variational density for $\lambda_{0,main}$

$$q^*(\lambda_{0,main}) \propto \exp\left\{ \mathrm{E}_{q(\boldsymbol{z},\boldsymbol{\pi},\boldsymbol{\lambda}^*)}[\log p(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\pi}, \boldsymbol{\lambda}^*, \boldsymbol{\lambda}_0^* | \boldsymbol{Q}, \boldsymbol{\Omega}, \widetilde{\mathrm{A}})] \right\}, \tag{18}$$

$$\log q(\lambda_{0,main}) \propto \sum_{j=1}^{J} \mathrm{E}_{q(\boldsymbol{\lambda}^*)} \left[\frac{(\boldsymbol{\lambda}_j^* - \boldsymbol{\lambda}_0^*)^{\mathrm{T}} (\boldsymbol{\lambda}_j^* - \boldsymbol{\lambda}_0^*)}{2} \right] - \frac{(\lambda_{0,main} - \mu_{\lambda_{0,main}})^2}{2\sigma_{\lambda_{0,main}}^2} \mathcal{I}\left(\lambda_{0,main} > 0\right). \tag{19}$$

Let

$$\begin{aligned} (\sigma_{\lambda_{0,main}}^{*2})^{-1} &= J_{main}^* + \frac{1}{\sigma_{\lambda_{0,main}}^2}, \\ \mu_{\lambda_{0,main}}^* &= \sigma_{\lambda_{0,main}}^{*2} \left(\frac{\mu_{\lambda_{0,main}}}{\sigma_{\lambda_{0,main}}^2} + \sum_{d=1}^{K} \sum_{j \in J_d} \left(\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*] \right)_{\lambda_d} \right), \end{aligned} \tag{20}$$

where J_{main}^* denotes the number of all main effect terms, $J_d = \{j : \lambda_{jd} \neq 0\}$ and $\mathrm{E}_{q(\boldsymbol{\lambda}_j^*)}[\boldsymbol{\lambda}_j^*]_{\lambda_d}$ is the corresponding the expected value of the element λ_{jd} in the vector $\boldsymbol{\lambda}^*$.

Then, $q^*(\lambda_{0,main})$ is derived as a truncated normal distribution with parameters $\mu^*_{\lambda_{0,main}}$ and $\sigma^{*2}_{\lambda_{0,main}}$,

$$q^*(\lambda_{0,main}|\mu^*_{\lambda_{0,main}}, \sigma^{*2}_{\lambda_{0,main}}) \propto \exp\left\{-\frac{(\lambda_{0,main} - \mu^*_{\lambda_{0,main}})^2}{2\sigma^{*2}_{\lambda_{0,main}}}\right\}\mathcal{I}\left(c,\infty\right). \tag{21}$$

(b5) Update the variational density for $\lambda_{0,inter}$

$$q^*(\lambda_{0,inter}) \propto \exp\left\{\mathrm{E}_{q(\boldsymbol{z},\boldsymbol{\pi},\boldsymbol{\lambda}^*)}[\log p(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\pi}, \boldsymbol{\lambda}^*, \boldsymbol{\lambda}^*_0|\boldsymbol{Q}, \boldsymbol{\Omega}, \widetilde{\mathrm{A}})]\right\}, \tag{22}$$

$$\log q(\lambda_{0,inter}) \propto \sum_{j=1}^{J}\mathrm{E}_{q(\boldsymbol{\lambda}^*)}\left[\frac{(\boldsymbol{\lambda}^*_j - \boldsymbol{\lambda}^*_0)^{\mathrm{T}}(\boldsymbol{\lambda}^*_j - \boldsymbol{\lambda}^*_0)}{2}\right] - \frac{(\lambda_{0,inter} - \mu_{\lambda_{0,inter}})^2}{2\sigma^2_{\lambda}}. \tag{23}$$

Let

$$\begin{aligned}(\sigma^{*2}_{\lambda_{0,inter}})^{-1} &= J^*_{inter} + \frac{1}{\sigma^2_{\lambda_{0,inter}}},\\ \mu^*_{\lambda_{0,inter}} &= \sigma^{*2}_{\lambda_{0,inter}}\left(\frac{\mu_{\lambda_{0,inter}}}{\sigma^2_{\lambda_{0,inter}}} + \sum_{d=K+1}^{D}\sum_{j\in J_d}\left(\mathrm{E}_{q(\boldsymbol{\lambda}^*_j)}[\boldsymbol{\lambda}^*_j]\right)_{\lambda_d}\right),\end{aligned} \tag{24}$$

where J^*_{inter} denotes the number of all interaction terms. Then, $q^*(\lambda_{0,inter})$ is derived as a truncated normal distribution with parameters $\mu^*_{\lambda_{0,inter}}$ and $\sigma^{*2}_{\lambda_{0,inter}}$,

$$q^*(\lambda_{0,inter}|\mu^*_{\lambda_{0,inter}}, \sigma^{*2}_{\lambda_{0,inter}}) \propto \exp\left\{-\frac{(\lambda_{0,inter} - \mu^*_{\lambda_{0,inter}})^2}{2\sigma^{*2}_{\lambda_{0,inter}}}\right\}. \tag{25}$$

(c) M-step: In this step, we update $\xi_{ijl}\,(i = 1, \cdots, N; j = 1, \cdots, J; l = 1, \cdots, L)$. To obtain the optimal ξ_{ijl}, we maximize $\underline{\mathcal{L}}^*(q(\Theta), \boldsymbol{\xi})$, that is the derivative with respect to ξ_{ijl} is set to zero (Bishop, 2006),

$$\frac{\partial\underline{\mathcal{L}}^*(q(\Theta), \boldsymbol{\xi})}{\partial\xi_{ijl}} = 0. \tag{26}$$

Then the update fomulation is derived as

$$\xi_{ijl} = \xi_{jl} = \boldsymbol{h}^{*\mathrm{T}}_{jl}\mathrm{E}_{q(\boldsymbol{\lambda}^*_j)}[\boldsymbol{\lambda}^*_j\boldsymbol{\lambda}^{*\mathrm{T}}_j]\boldsymbol{h}^*_{jl}. \tag{27}$$

All of the expectations can be found in the main text Eq.(35), then $\underline{\mathcal{L}}^*(q(\Theta), \boldsymbol{\xi})$ in Eq.(22) can be derived as follows:

$$\begin{aligned}\underline{\mathcal{L}}^*(q(\Theta), \boldsymbol{\xi}) = & \sum_{i=1}^{N}\sum_{j=1}^{J}\sum_{l=1}^{L}\pi_{il}^*\left(\log(\sigma(\xi_{ijl}) + \tau(\xi_{ijl})\xi_{ijl}^2 - \frac{\xi_{ijl}}{2}\right) + \log\frac{\mathrm{B}(\boldsymbol{\delta}^*)}{\mathrm{B}(\boldsymbol{\delta}_0)} + \frac{1}{2}\sum_{j=1}^{J}\frac{|\boldsymbol{V}_j^*|}{|\boldsymbol{I}_{D+1}|} \\ & - \sum_{i=1}^{N}\sum_{l=1}^{L}\pi_{il}^*\log\pi_{il}^* + \frac{1}{2}\frac{\sigma_{\eta_0}^2}{\sigma_{\eta_0}^{*2}} + \frac{1}{2}\frac{\sigma_{\lambda_{0,main}}^{*2}}{\sigma_{\lambda_{0,main}}^{*2}} + \frac{1}{2}\frac{\sigma_{\lambda_{0,inter}}^{*2}}{\sigma_{\lambda_{0,inter}}^{*2}} + \log\frac{1-\Phi\left(\frac{c-\mu_{\lambda_{0,main}}^*}{\sigma_{\lambda_{0,main}}^*}\right)}{1-\Phi\left(\frac{c-\mu_{\lambda_{0,main}}}{\sigma_{\lambda_{0,main}}}\right)} \\ & + \frac{1}{2}\sigma_{\eta_0}^{*2}\left(\frac{1}{\sigma_{\eta_0}^{*2}} - \frac{1}{\sigma_{\eta_0}^2}\right) - \frac{(\mu_{\eta_0}^* - \mu_{\eta_0})^2}{\sigma_{\eta_0}^2} \\ & + \frac{1}{2}\sigma_{\lambda_{0,main}}^{*2}\left(1 - \mu_{\lambda_{0,main}}^*\frac{\phi(c)}{\Phi(c)}\right)\left(\frac{1}{\sigma_{\lambda_{0,main}}^{*2}} - \frac{1}{\sigma_{\lambda_{0,main}}^2}\right) - \frac{(\mu_{\lambda_{0,main}}^* - \mu_{\lambda_{0,main}})^2}{\sigma_{\lambda_{0,main}}^2} \\ & + \frac{1}{2}\sigma_{\lambda_{0,inter}}^{*2}\left(\frac{1}{\sigma_{\lambda_{0,inter}}^{*2}} - \frac{1}{\sigma_{\lambda_{0,inter}}^2}\right) - \frac{(\mu_{\lambda_{0,inter}}^* - \mu_{\lambda_{0,inter}})^2}{\sigma_{\lambda_{0,inter}}^2},\end{aligned} \tag{28}$$

where $\mathrm{B}(\cdot)$ is the normalizing function of the Dirichlet distribution, and $\mathrm{B}(\boldsymbol{\delta}) = \prod_{i=1}^{L}\Gamma(\delta_l)/\Gamma(\sum_{i=1}^{L}\delta_l)$. $\Gamma(x) = \int_0^\infty t^{(x-1)}\exp(-t)\,dt$. $\phi(\cdot)$ is the density function of standard normal distribution and $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

2. The settings of Q-matrices for simulation studies

Q-matrices for simulation studies 1–2 are presented in this section. Tables S1–S2 show the Q-matrices for simulation studies 1 and 2, respectively.

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Insert Tables S1–S2 about here

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3. Additional results of simulation study 2

Tables S3 – S6 present the recovery results for item parameters and attribute profiles in simulation study 2, under conditions with $\sigma = 0$ and $\sigma = 0.7$. These results demonstrate that the VBEM-M algorithm outperforms all other methods across all simulation conditions in terms of the average RMSE. Combined with the results at $\sigma = 0.3$ presented in the main text, it is evident that when the sample size, attribute dimension, and noise level are held constant, the average RMSE for parameter $\boldsymbol{\lambda}$ decreases as σ increases. Conversely, the RMSE for parameter $\boldsymbol{\eta}$ shows an increasing trend for all algorithms. Other conclusions are consistent with those in the main text and are not repeated here.

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Insert Tables S3–S6 about here

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4. Additional results of simulation study 3

Tables S7 and S8 present the recovery results for item parameters and attribute profiles in simulation study 3, and Table S9 displays the SDs for the VBEM-M and VB algorithms, as well as the SEs for the EM algorithm. Based on Table S7, it is evident that when the sample size and attribute dimension are held constant, the average RMSE for the parameter $\boldsymbol{\lambda_{inter}}$ increases as σ increases. Conversely, the RMSE for the parameter $\boldsymbol{\eta}$ shows a decreasing trend for all algorithms. However, there is no consistent trend for the parameter $\boldsymbol{\lambda_{main}}$, which shows the worst result under the $\sigma = 0.7$ condition. Additionally, Table S9 implies that the VBEM-M algorithm provides more stable estimation results since it has a lower SDs. Other conclusions are consistent with those in the main text and are not repeated here. Moreover, the computation time for the four algorithms across all simulation conditions is displayed in Table S10.

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Insert Tables S7–S10 about here

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5. Additional results of Empirical Example 1

Table S11 presents a comparison of item parameter estimates for the VBEM-M, VB, MCMC, and EM algorithms. And the the estimators of the class membership parameter, $\widehat{\pi}_l$ for $l = 1, \cdots, 2^5 = 32$ are displayed in Figure S2.

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Insert Table S11 about here

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Insert Figure S2 about here

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6. Additional results of Empirical Example 2

The item parameter estimates of VB, MCMC and EM algorithms are displayed in Tables S12-S14, respectively. Table S15 shows the estimates of $\widehat{\pi}_l (l = 1, \cdots, 2^3 = 8)$.

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Insert Tables S12–S15 about here

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7. Additional simulation studies

7.1. Simulation Study under different initial values based on DINA model

This simulation study was conducted to evaluate the performance of the VBEM-M algorithm with different initial values for the item parameters. The number of examinees was $N = 1000$, the test length was $J = 30$, the number of attributes was set as $K = 5$, and the correlation among attributes was set to $\sigma = 0.3$. The HNL case was analyzed in this simulation study. The following manipulated conditions were considered: (A) attribute dimensions $K = 1$, 2, 3, and 4; (B) initial values of η_j for different items $\eta_j = -1$ and $\eta_j = -3$, where $j = 1, 11, 14, 21$; (C) initial value of λ_j for different items $\lambda_j = 1$, $\lambda_j = 3$, and $\lambda_j = 5$, where $j = 1, 11, 14, 21$. Since the true value of η_j was -1.3863, -1 and -3 were chosen as starting points, one from above and the other from below the true value. Similarly, the true value of λ_j was set as 2.7726. We then randomly chose three starting values, 1, 3, or 5, from above or below the true value. In addition, four items—1, 11, 14, and 21—were randomly selected to evaluate different attributes. The dimensions of the attributes examined in the four items increased in the order 1, 2, 3, and 4, respectively. The Q-matrix employed in this simulation study is presented in Table S16. The estimation accuracy of the item parameters and the recovery of the attribute profile are displayed in Tables S17 and S18, respectively. Iterative plots of the VBEM-M algorithm based on the different initial values are shown in Figure S1.

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Insert Tables S16–S18 about here

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Insert Figure S1 about here

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According to the results presented in Tables S17 and S18, it was observed that the accuracy of parameter estimation and the recovery of the attribute profile under various initial value conditions were highly consistent. As illustrated in Figure S1, the VBEM-M algorithm consistently converges to a stable value for each parameter after approximately ten iterations, regardless of the initial settings of these parameters. Additionally, the convergence rates of the parameters are unaffected by their starting values, provided that they are not too distant from the true values. As a result, unlike the EM algorithm, the VBEM-M algorithm maintains a high rate of convergence even when the initial parameter values are far from the true values. However, we discovered that for estimating items with multiple attributes, the VBEM-M algorithm requires roughly the same number of iterations as our algorithm for estimating items with few attributes. Therefore, when estimating items with multiple attributes, the VBEM-M algorithm is computationally efficient since it does not require more iterations and time.

7.2. Simulation study for the deterministic input, noisy "or" gate (DINO) model

In this simulation study, we evaluate the recovery accuracy of VBEM-M algorithm for the deterministic input, noisy "or" gate (DINO) model (Templin & Henson,2006), First we reparameterize the DINO model:

$$P(x_{ij} = 1|\boldsymbol{\alpha}_i) = \frac{\exp\left(\eta_j + \lambda_j \prod_{k=1}^{K} (1 - (1 - \alpha_{ik})^{q_{jk}})\right)}{1 + \exp\left(\eta_j + \lambda_j \prod_{k=1}^{K} (1 - (1 - \alpha_{ik})^{q_{jk}})\right)}, \tag{29}$$

where

$$\begin{aligned} \eta_j &= \log\left(\frac{g_j}{1 - g_j}\right), \\ \lambda_j &= -\eta_j + \log\left(\frac{1 - s_j}{s_j}\right). \end{aligned} \tag{30}$$

The definition of g_j and s_j here is the same as in the DINA model.

7.2.1. Evaluate the performance of VBEM-M algorithm

In this simulation, we evaluate the performance of VBEM-M algorithm based on DINO model. The simulation settings are as follows. Assuming the attribute dimension $K = 5$ and the test legth $J = 30$, the manipulated factors included: The sample size (i.e., $N = 1000$ or 2000), the correlation among attributes (i.e., $\sigma = 0$, 0.3, 0.7), and the 2 noise-level (i.e., LNL, HNL). Fully crossing different levels of these factors yield 12 conditions (2 sample sizes $\times 3$ correlations $\times 2$ noisse levels). There are 100 replications for each simulation condition. The Q-matrix used here was shown in S19. The recovery result of item parameters and attribute profile was displayed in Tables S20 and S21.

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Insert Tables S19–S21 about here

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According to Tables S20 and S21, the result was similar to that in simulation study 1 in the main text. Now we briefly summarize as follows: Firstly, given the correlation and noise level, when the number of examinees increases from 1000 to 2000, the average bias, RMSE and SD for $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ parameters showed a decreasing trend. Secondly, when the number of examinees and the noise levels are given, with increasing σ, the average bias and RMSE for $\boldsymbol{\eta}$ increase somewhat. However the average bias and RMSE for $\boldsymbol{\lambda}$ tend to decrease as σ increases. Thirdly, both two parameters $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ showed a lower average RMSE and bias in HNL level than in LNL level. At last, the accuracy of attribute profile recovery performs better in LNL since the noise is the lower.

7.2.2. Compare the performance of VBEM-M algorithm with MCMC and EM algorithm

In this simulation, the performance of the VBEM-M algorithm is evaluated based on the DINO model in comparison with the VB, MCMC and EM algorithms, implemented by the R packages 'variationalDCM', 'R2jags' and 'GDINA' respectively. The correlation among attributes was set to $\sigma = 0.3$. We considered different sample sizes $N = 200$, 500, 1000, 2000 and two noise levels, labeled LNL and HNL, resulting in 4×2 simulation conditions. Each condition was replicated 30 times. The Q-matrix utilized in this study was identical to the one used in 7.2.1. The recovery results of item parameters and attribute profiles are displayed in Tables S22 to S23.

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Insert Tables S22–S23 about here

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Based on the data in Tables S22 to S23, we can draw the following conclusions: (1) In terms of the average RMSE for parameters $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$, the VBEM-M algorithm exhibits the best performance, followed by the VB and MCMC algorithm, and then the EM algorithm. This advantage is particularly pronounced with small sample sizes but diminishes as the sample size increases. (2) Among the three methods, there is no significant difference in the estimation accuracy of the $\boldsymbol{\pi}$ parameter under all simulation conditions. (3) Regarding the recovery of attribute mastery patterns, as indicated by the AAR and PAR results, the four methods perform similarly under various conditions, showing no noticeable differences.

7.3. Simulation study for linear logistic model (LLM)

In this simulation study, we evaluate the recovery accuracy for linear logistic model (LLM; Maris, 1999), the probability of a correct response for LLM can be written as

$$P(x_{ij} = 1|\boldsymbol{\alpha}_i, \eta_j, \boldsymbol{\lambda}_j, \boldsymbol{q}_j) = \frac{\exp\left(\eta_j + \sum_{k=1}^{K} \lambda_{jk}\alpha_{ik}q_{jk}\right)}{1 + \exp\left(\eta_j + \sum_{k=1}^{K} \lambda_{jk}\alpha_{ik}q_{jk}\right)}. \tag{31}$$

7.3.1. Evaluate the performance of VBEM-M algorithm

In this simulation study, we evaluate the recovery accuracy based on LLM. For the simulation conditions, we fixed attribute dimension $K = 3$ and the test length $J = 18$. The true values of $\boldsymbol{\lambda}^* = (\eta_j, \boldsymbol{\lambda}_j^{\mathrm{T}})^{\mathrm{T}}$ is shown in Table S24. The following manipulated conditions are considered: The sample size (i.e., $N = 1000$ or 2000) and the correlation among attributes (i.e., $\sigma = 0$, 0.3, 0.7). There are 100 replications for each simulation condition. The recovery result of item parameters and attribute profile was displayed in Tables S25 and S26.

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Insert Tables S23–S25 about here

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From the results in Tables S25 and S26, it is evident that the average RMSE and SD of parameters $\boldsymbol{\eta}$, $\boldsymbol{\lambda}$, and $\boldsymbol{\pi}$ decrease as the sample size N increases. Moreover, when the correlation σ increases from 0 to 0.7, the average bias for both $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ exhibits a decreasing trend. Regarding the average RMSE, the two parameters show different trends. The average RMSE for $\boldsymbol{\eta}$ decreases with the increase in σ, while $\boldsymbol{\lambda}$ displays the lowest RMSE under a low correlation case (i.e., $\sigma = 0.3$), with the average RMSE for the other two cases ($\sigma = 0$ and 0.7) being similar and slightly higher. Additionally, the recovery of the attribute profile improves as the correlation σ increases.

7.3.2. Compare the performance of VBEM-M algorithm with MCMC and EM algorithm

In this simulation, we evaluated the performance of the VBEM-M algorithm based on the LLM, in comparison with the MCMC and EM algorithms, implemented using the R packages “R2jags” and “GDINA”, respectively. The correlation among attributes was set at $\sigma = 0.3$. Different sample sizes were considered: $N = 200, 500, 1000, 2000$. Each simulation condition was replicated 30 times. The true values used in this simulation were identical to the Q-matrix employed in 7.3.1. The recovery results of item parameters and attribute profiles are displayed in Tables S27 to S28.

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Insert Tables S26–S27 about here

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According to the results displayed in Tables S27 to S28, we found that the results are similar to those in Section 7.2.2. Here, we briefly summarize our findings. Firstly, in terms of the RMSE of parameters $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$, the VBEM-M algorithm outperforms the others, with the MCMC method having a slightly higher RMSE than VBEM-M. The EM method, especially at smaller sample sizes, performs poorly. However, as the sample size increases, the differences between these algorithms gradually diminish. Secondly, there is no significant difference in the estimation of the parameter $\boldsymbol{\pi}$ among the three methods. Thirdly, concerning the estimation accuracy of the attribute profile, the three methods exhibit similar performance under all conditions, with no noticeable differences.

Table S1.

Q-matrix for simulation study 1.

Item	α_1	α_2	α_3	α_4	α_5	Item	α_1	α_2	α_3	α_4	α_5
1	1	0	0	0	0	16	0	0	1	1	0
2	0	1	0	0	0	17	1	0	1	0	1
3	0	0	1	0	0	18	1	1	0	1	0
4	0	0	0	1	0	19	0	1	1	0	0
5	0	0	0	0	1	20	1	1	1	0	1
6	1	0	0	0	0	21	0	1	1	0	1
7	0	1	0	0	0	22	1	0	0	0	0
8	0	0	1	0	0	23	0	0	1	0	0
9	0	0	0	1	0	24	0	0	0	1	1
10	0	0	0	0	1	25	0	0	1	1	0
11	1	1	1	0	0	26	0	1	1	0	1
12	0	1	0	0	1	27	0	0	0	0	1
13	1	1	1	0	0	28	0	1	1	0	1
14	1	0	1	0	1	29	0	1	0	0	1
15	0	0	0	0	1	30	0	0	1	1	0

Table S2.

Q-matrix for simulation study 2.

Item	α_1	α_2	α_3	α_4	α_5	Item	α_1	α_2	α_3	α_4	α_5
1	1	0	0	0	0	16	0	0	1	1	1
2	0	1	0	0	0	17	0	1	0	0	1
3	0	0	1	0	0	18	1	0	1	0	0
4	0	0	0	1	0	19	1	0	1	1	0
5	0	0	0	0	1	20	0	1	1	1	0
6	1	0	0	0	0	21	1	1	1	1	1
7	0	1	0	0	0	22	0	0	0	1	1
8	0	0	1	0	0	23	1	1	1	1	1
9	0	0	0	1	0	24	0	0	1	0	1
10	0	0	0	0	1	25	0	0	0	1	0
11	0	0	1	0	0	26	1	1	1	0	0
12	1	1	0	0	1	27	1	0	0	0	0
13	1	0	1	1	0	28	1	0	1	1	0
14	1	1	1	0	0	29	1	1	1	1	0
15	0	0	1	1	1	30	1	0	0	0	0

Table S3.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M, VB, MCMC-dina, MCMC-R2jags, EM-GDINA, EM-CDM algorithms for the DINA with $\sigma = 0$.

		VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
LNL	$\boldsymbol{\eta}$						
	$N = 200$	0.2701(–0.0380)	0.3022(0.0266)	0.2991(0.0325)	0.2991(0.0292)	0.3759(–0.0519)	0.3759(–0.0519)
	$N = 500$	0.1862(–0.0140)	0.1954(0.0134)	0.1950(0.0153)	0.1952(0.0142)	0.2027(–0.0140)	0.2026(–0.0139)
	$N = 1000$	0.1357(–0.0079)	0.1391(0.0064)	0.1389(0.0074)	0.1388(0.0069)	0.1415(–0.0072)	0.1414(–0.0071)
	$N = 2000$	0.0984(–0.0047)	0.0996(0.0026)	0.0996(0.0030)	0.0996(0.0027)	0.1005(–0.0043)	0.1004(–0.0042)
	$\boldsymbol{\lambda}$						
	$N = 200$	0.4727(0.1293)	0.6165(–0.2047)	0.6125(–0.2242)	0.6392(–0.2686)	1.7723(0.7008)	1.7728(0.7009)
	$N = 500$	0.3686(0.0739)	0.4115(–0.0776)	0.4101(–0.0852)	0.4167(–0.1018)	0.7319(0.2071)	0.7310(0.2069)
	$N = 1000$	0.2841(0.0473)	0.2980(–0.0318)	0.2978(–0.0351)	0.2989(–0.0439)	0.3915(0.0684)	0.3915(0.0684)
	$N = 2000$	0.2117(0.0226)	0.2171(–0.0187)	0.2171(–0.0203)	0.2177(–0.0247)	0.2286(0.0217)	0.2286(0.0217)
	$\boldsymbol{\pi}$						
	$N = 200$	0.0118(0.0000)	0.0119(0.0000)	0.0117(0.0000)	0.0117(0.0000)	0.0142(0.0000)	0.0142(0.0000)
	$N = 500$	0.0080 (0.0000)	0.0080(0.0000)	0.0080(0.0000)	0.0080(0.0000)	0.0086(0.0000)	0.0086(0.0000)
	$N = 1000$	0.0060(0.0000)	0.0060(0.0000)	0.0059(0.0000)	0.0059(0.0000)	0.0062(0.0000)	0.0062(0.0000)
	$N = 2000$	0.0042(0.0000)	0.0042(0.0000)	0.0042(0.0000)	0.0042(0.0000)	0.0043(0.0000)	0.0043(0.0000)
HNL	$\boldsymbol{\eta}$						
	$N = 200$	0.2345(–0.0219)	0.2684(–0.0007)	0.2558(0.0128)	0.2590(0.0035)	0.3553(–0.0438)	0.3584(–0.0453)
	$N = 500$	0.1555(–0.0117)	0.1639(–0.0020)	0.1616(0.0030)	0.1620(–0.0014)	0.1720(–0.0159)	0.1719(–0.0157)
	$N = 1000$	0.1082(–0.0044)	0.1109(0.0006)	0.1104(0.0029)	0.1109(0.0005)	0.1124(–0.0060)	0.1124(–0.0058)
	$N = 2000$	0.0799(–0.0003)	0.0809(0.0021)	0.0809(0.0030)	0.0808(0.0022)	0.0814(–0.0013)	0.0813(–0.0010)
	$\boldsymbol{\lambda}$						
	$N = 200$	0.4505(0.1129)	0.5577(–0.0547)	0.5266(–0.1048)	0.5516(–0.1656)	1.4935(0.4538)	1.4976(0.4539)
	$N = 500$	0.3349(0.0425)	0.3706(–0.0307)	0.3644(–0.0543)	0.3709(–0.0804)	0.5181(0.0699)	0.5181(0.0700)
	$N = 1000$	0.2463(0.0191)	0.2597(–0.0220)	0.2579(–0.0336)	0.2617(–0.0477)	0.2712(0.0136)	0.2712(0.0139)
	$N = 2000$	0.1795(0.0180)	0.1827(–0.0032)	0.1825(–0.0089)	0.1827(–0.0159)	0.1874(0.0142)	0.1874(0.0145)
	$\boldsymbol{\pi}$						
	$N = 200$	0.0145(0.0000)	0.0146(0.0000)	0.0135(0.0000)	0.0135(0.0000)	0.0197(0.0000)	0.0197(0.0000)
	$N = 500$	0.0108(0.0000)	0.0109(0.0000)	0.0107(0.0000)	0.0107(0.0000)	0.0125(0.0000)	0.0125(0.0000)
	$N = 1000$	0.0080(0.0000)	0.0080(0.0000)	0.0080(0.0000)	0.0081(0.0000)	0.0086(0.0000)	0.0085(0.0000)
	$N = 2000$	0.0060(0.0000)	0.0060(0.0000)	0.0061(0.0000)	0.0061(0.0000)	0.0062(0.0000)	0.0062(0.0000)

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all slope parameters $\boldsymbol{\lambda}$ and all class membership probability parameters $\boldsymbol{\pi}$.

TABLE S4.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M, VB, MCMC-dina, MCMC-R2jags, EM-GDINA, EM-CDM algorithms for the DINA with $\sigma = 0$.

		AAR1						AAR2					
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9863	0.9861	0.9861	0.9863	0.9863	0.9863	0.9504	0.9499	0.9501	0.9503	0.9492	0.9491
	$N = 500$	0.9860	0.9860	0.9862	0.9859	0.9860	0.9859	0.9517	0.9518	0.9515	0.9512	0.9516	0.9516
	$N = 1000$	0.9857	0.9856	0.9854	0.9856	0.9856	0.9856	0.9516	0.9517	0.9514	0.9515	0.9517	0.9516
	$N = 2000$	0.9859	0.9859	0.9858	0.9860	0.9859	0.9859	0.9518	0.9518	0.9516	0.9517	0.9518	0.9518
		AAR3						AAR4					
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
LNL	$N = 200$	0.9842	0.9840	0.9838	0.9840	0.9833	0.9833	0.9784	0.9780	0.9782	0.9783	0.9772	0.9773
	$N = 500$	0.9850	0.9850	0.9850	0.9848	0.9850	0.9850	0.9794	0.9796	0.9794	0.9797	0.9795	0.9795
	$N = 1000$	0.9837	0.9837	0.9837	0.9836	0.9837	0.9837	0.9798	0.9798	0.9797	0.9797	0.9798	0.9798
	$N = 2000$	0.9841	0.9841	0.9840	0.9841	0.9841	0.9841	0.9804	0.9804	0.9804	0.9804	0.9804	0.9804
		AAR5						PAR					
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9668	0.9667	0.9663	0.9666	0.9657	0.9657	0.8808	0.8796	0.8794	0.8805	0.8771	0.8771
	$N = 500$	0.9693	0.9690	0.9689	0.9686	0.9691	0.9691	0.8838	0.8836	0.8835	0.8829	0.8834	0.8834
	$N = 1000$	0.9681	0.9680	0.9681	0.9680	0.9681	0.9681	0.8821	0.8822	0.8818	0.8819	0.8822	0.8821
	$N = 2000$	0.9693	0.9693	0.9692	0.9691	0.9693	0.9693	0.8841	0.8840	0.8837	0.8840	0.8840	0.8840
		$\boldsymbol{\eta}$											
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9316	0.9307	0.9312	0.9313	0.9289	0.9290	0.8573	0.8534	0.8546	0.8534	0.8482	0.8480
	$N = 500$	0.9269	0.9269	0.9264	0.9268	0.9264	0.9264	0.8642	0.8629	0.8621	0.8621	0.8609	0.8610
	$N = 1000$	0.9294	0.9294	0.9293	0.9290	0.9294	0.9294	0.8671	0.8668	0.8666	0.8667	0.8667	0.8666
	$N = 2000$	0.9290	0.9291	0.9291	0.9290	0.9291	0.9291	0.8678	0.8677	0.8676	0.8679	0.8677	0.8678
		$\boldsymbol{\lambda}$											
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
HNL	$N = 200$	0.9212	0.9197	0.9199	0.9196	0.9159	0.9160	0.9058	0.9038	0.9052	0.9054	0.9000	0.9000
	$N = 500$	0.9225	0.9224	0.9218	0.9220	0.9213	0.9213	0.9119	0.9116	0.9114	0.9115	0.9106	0.9107
	$N = 1000$	0.9227	0.9227	0.9228	0.9224	0.9225	0.9224	0.9133	0.9133	0.9127	0.9130	0.9131	0.9131
	$N = 2000$	0.9233	0.9234	0.9232	0.9233	0.9234	0.9234	0.9127	0.9126	0.9126	0.9126	0.9126	0.9127
		$\boldsymbol{\pi}$											
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.8862	0.884	0.8851	0.8838	0.8802	0.8798	0.6258	0.6186	0.622	0.6208	0.6072	0.6072
	$N = 500$	0.8916	0.8914	0.8901	0.8905	0.8900	0.8901	0.6331	0.6321	0.6302	0.6313	0.6277	0.6278
	$N = 1000$	0.8912	0.8911	0.8909	0.8911	0.8909	0.8910	0.6406	0.6406	0.6401	0.6400	0.6402	0.6400
	$N = 2000$	0.8930	0.8931	0.8930	0.8929	0.8929	0.8930	0.6417	0.6418	0.6419	0.6420	0.641	6 0.6418

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute, AAR3 for the third attribute, AAR4 for the fourth attribute, and AAR5 for the fifth attribute. PAR stands for the pattern-wise agreement rate.

Table S5.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M, VB, MCMC-dina, MCMC-R2jags, EM-GDINA, EM-CDM algorithms for the DINA with $\sigma = 0.7$.

		VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
LNL	$\boldsymbol{\eta}$						
	$N = 200$	0.2820(−0.0363)	0.3165(0.0386)	0.3154(0.0402)	0.3157(0.0396)	0.3643(−0.0408)	0.3632(−0.0407)
	$N = 500$	0.1915(−0.0201)	0.2003(0.0129)	0.2006(0.0140)	0.2005(0.0132)	0.2079(−0.0164)	0.2079(−0.0164)
	$N = 1000$	0.1425(−0.0160)	0.1450(0.0011)	0.1450(0.0016)	0.1451(0.0014)	0.1481(−0.0135)	0.1481(−0.0135)
	$N = 2000$	0.1005(−0.0045)	0.1017(0.0041)	0.1018(0.0044)	0.1017(0.0043)	0.1025 (−0.0032)	0.1025(−0.0032)
	$\boldsymbol{\lambda}$	VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
	$N = 200$	0.4269(0.1027)	0.5070(−0.0986)	0.5055(−0.1088)	0.5098(−0.1245)	0.6508(0.1262)	0.6498(0.1262)
	$N = 500$	0.3076(0.0499)	0.3289(−0.0357)	0.3287(−0.0405)	0.3298(−0.0463)	0.3484(0.0455)	0.3484(0.0455)
	$N = 1000$	0.2247(0.0312)	0.2310(−0.0127)	0.2311(−0.0150)	0.2310(−0.0180)	0.2377(0.0271)	0.2377(0.0271)
	$N = 2000$	0.1570(0.0112)	0.1599(−0.0109)	0.1600(−0.0121)	0.1600(−0.0136)	0.1615(0.0088)	0.1615(0.0088)
	$\boldsymbol{\pi}$	VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
	$N = 200$	0.0063(0.0000)	0.063(0.0000)	0.0063(0.0000)	0.0064(0.0000)	0.0047(0.0000)	0.0047(0.0000)
	$N = 500$	0.0034(0.0000)	0.0034(0.0000)	0.0035(0.0000)	0.0035(0.0000)	0.0028(0.0000)	0.0028(0.0000)
	$N = 1000$	0.0023(0.0000)	0.0023(0.0000)	0.0023(0.0000)	0.0023(0.0000)	0.0020(0.0000)	0.0020(0.0000)
	$N = 2000$	0.0016(0.0000)	0.0016(0.0000)	0.0016(0.0000)	0.0016(0.0000)	0.0014(0.0000)	0.0014(0.0000)
HNL	$\boldsymbol{\eta}$	VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
	$N = 200$	0.2344(−0.0323)	0.2569(0.0014)	0.2552(0.0024)	0.2553(−0.0005)	0.2732(−0.0214)	0.2731(−0.0213)
	$N = 500$	0.1547(−0.0199)	0.1593(−0.0047)	0.1584(−0.0022)	0.1588(−0.0037)	0.1632(−0.0116)	0.1632(−0.0116)
	$N = 1000$	0.1136(−0.0069)	0.1155(0.0010)	0.1158(0.0033)	0.1156(0.0028)	0.1167(−0.0025)	0.1167(−0.0025)
	$N = 2000$	0.0802(−0.0048)	0.0808(−0.0008)	0.0807(0.0007)	0.0808(0.0002)	0.0812(−0.0028)	0.0812(−0.0027)
	$\boldsymbol{\lambda}$	VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
	$N = 200$	0.3678(0.0740)	0.4145(−0.0194)	0.4105(−0.0377)	0.4140(−0.0569)	0.4478(0.0640)	0.4477(0.0640)
	$N = 500$	0.2455(0.0343)	0.2564(−0.0049)	0.2551(−0.0156)	0.2560(−0.0227)	0.2634(0.0246)	0.2634(0.0246)
	$N = 1000$	0.1782(0.0123)	0.1828(−0.0077)	0.1830(−0.0146)	0.1835(−0.0184)	0.1848(0.0070)	0.1848(0.0070)
	$N = 2000$	0.1259(0.0067)	0.1274(−0.0034)	0.1275(−0.0073)	0.1277(−0.0090)	0.1281(0.0041)	0.1281(0.0041)
	$\boldsymbol{\pi}$	VBEM-M	VB	MCMC-dina	MCMC-R2jags	EM-GDINA	EM-CDM
	$N = 200$	0.0107(0.0000)	0.0107(0.0000)	0.0107(0.0000)	0.0108(0.0000)	0.0115(0.0000)	0.0115(0.0000)
	$N = 500$	0.0069(0.0000)	0.0069(0.0000)	0.0068(0.0000)	0.0068(0.0000)	0.0073(0.0000)	0.0072(0.0000)
	$N = 1000$	0.0050(0.0000)	0.0050(0.0000)	0.0050(0.0000)	0.0050(0.0000)	0.0053(0.0000)	0.0052(0.0000)
	$N = 2000$	0.0036(0.0000)	0.0036(0.0000)	0.0037(0.0000)	0.0037(0.0000)	0.0037(0.0000)	0.0037(0.0000)

Note: The accuracy of parameter estimation is assessed by the average bias and RMSE. The average bias of $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ are shown in parentheses. $\boldsymbol{\eta}$ represents all intercept parameters. $\boldsymbol{\lambda}$ represents all slope parameters.

Table S6.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M, VB, MCMC-dina, MCMC-R2jags, EM-GDINA, EM-CDM algorithms for the DINA with $\sigma = 0.7$.

		AAR1						AAR2					
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
LNL	$N = 200$	0.9920	0.9918	0.9918	0.9918	0.9916	0.9916	0.9729	0.9716	0.9713	0.9712	0.9703	0.9703
	$N = 500$	0.9913	0.9912	0.9914	0.9912	0.9912	0.9912	0.9736	0.9736	0.9735	0.9734	0.9735	0.9735
	$N = 1000$	0.9924	0.9923	0.9923	0.9924	0.9924	0.9924	0.9748	0.9748	0.9747	0.9748	0.9748	0.9748
	$N = 2000$	0.9923	0.9924	0.9924	0.9924	0.9923	0.9923	0.9750	0.9750	0.9750	0.9750	0.9750	0.9750
		AAR3						AAR4					
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9845	0.9842	0.9840	0.9840	0.9838	0.9838	0.9810	0.9804	0.9801	0.9806	0.9796	0.9796
	$N = 500$	0.9868	0.9868	0.9867	0.9867	0.9866	0.9866	0.9831	0.9829	0.9828	0.9829	0.9827	0.9827
	$N = 1000$	0.9855	0.9855	0.9856	0.9855	0.9855	0.9855	0.9834	0.9834	0.9835	0.9835	0.9833	0.9833
	$N = 2000$	0.9855	0.9855	0.9854	0.9855	0.9855	0.9855	0.9834	0.9834	0.9834	0.9834	0.9834	0.9834
		AAR5						PAR					
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9804	0.9800	0.9802	0.9798	0.9793	0.9793	0.9182	0.9161	0.9155	0.9157	0.9134	0.9134
	$N = 500$	0.9798	0.9797	0.9795	0.9794	0.9797	0.9797	0.9220	0.9216	0.9214	0.9212	0.9211	0.9211
	$N = 1000$	0.9801	0.9801	0.9801	0.9800	0.9801	0.9801	0.9227	0.9228	0.9229	0.9228	0.9227	0.9227
	$N = 2000$	0.9806	0.9806	0.9806	0.9805	0.9805	0.9805	0.9236	0.9237	0.9237	0.9237	0.9236	0.9236
		$\boldsymbol{\eta}$											
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
HNL	$N = 200$	0.9534	0.9526	0.9526	0.9520	0.9502	0.9502	0.9136	0.9126	0.9133	0.9129	0.9082	0.9082
	$N = 500$	0.9545	0.9540	0.9539	0.9541	0.9538	0.9538	0.9181	0.9177	0.9173	0.9176	0.9161	0.9161
	$N = 1000$	0.9564	0.9563	0.9561	0.9560	0.9561	0.9561	0.9207	0.9206	0.9195	0.9194	0.9196	0.9196
	$N = 2000$	0.9572	0.9573	0.9572	0.9571	0.9571	0.9571	0.9214	0.9214	0.9206	0.9206	0.9210	0.9211
		$\boldsymbol{\lambda}$											
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9406	0.9402	0.9401	0.9404	0.9381	0.9381	0.9370	0.9363	0.9359	0.9356	0.9338	0.9337
	$N = 500$	0.9439	0.9440	0.9439	0.9438	0.9436	0.9436	0.9400	0.9398	0.9404	0.9400	0.9392	0.9392
	$N = 1000$	0.9455	0.9455	0.9454	0.9453	0.9452	0.9453	0.9398	0.9397	0.9397	0.9397	0.9395	0.9394
	$N = 2000$	0.9461	0.9461	0.9460	0.9459	0.9460	0.9460	0.9414	0.9414	0.9411	0.9411	0.9412	0.9412
		$\boldsymbol{\pi}$											
		VBEM-M	VB	dina	R2jags	GDINA	CDM	VBEM-M	VB	dina	R2jags	GDINA	CDM
	$N = 200$	0.9266	0.9255	0.9244	0.9239	0.9228	0.9228	0.7516	0.7491	0.7491	0.7486	0.7395	0.7395
	$N = 500$	0.9285	0.9282	0.9285	0.9279	0.9266	0.9266	0.7586	0.7580	0.7581	0.7578	0.7543	0.7544
	$N = 1000$	0.9293	0.9291	0.9284	0.9284	0.9288	0.9288	0.7634	0.7632	0.7617	0.7618	0.7618	0.7619
	$N = 2000$	0.9306	0.9306	0.9298	0.9299	0.9305	0.9304	0.7678	0.7678	0.7665	0.7666	0.7671	0.7672

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute, AAR3 for the third attribute, AAR4 for the fourth attribute, and AAR5 for the fifth attribute. PAR stands for the pattern-wise agreement rate.

Table S7.

The accuracy of item parameters and class membership probability parameters using the VBEM-M, VB, MCMC-R2jags, and EM-GDINA algorithms for the LCDM model in simulation study 3.

		$\boldsymbol{\eta}$				$\boldsymbol{\lambda}_{main}$			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0$	$N = 200$	0.3291(−0.0566)	0.4389(−0.0691)	0.3573(0.1375)	0.7228(−0.1187)	0.4010(0.0909)	0.6028(−0.0696)	0.4784(−0.1421)	1.0344(0.1738)
	$N = 500$	0.2245(−0.0268)	0.2805(−0.0306)	0.2515(0.0721)	0.3241(−0.0309)	0.2927(0.0283)	0.3974(−0.0254)	0.3572(−0.0966)	0.4671(0.0406)
	$N = 1000$	0.1748(−0.0245)	0.1970(−0.0273)	0.1844(0.0275)	0.2001(−0.0247)	0.2314(0.0174)	0.2740(−0.0033)	0.2614(−0.0490)	0.2814(0.0245)
	$N = 2000$	0.1248(−0.0005)	0.1356(−0.0029)	0.1330(0.0268)	0.1366(−0.0014)	0.1694(0.0035)	0.1907(−0.0109)	0.1875(−0.0382)	0.1928(0.0026)
		$\boldsymbol{\lambda}_{inter}$				$\boldsymbol{\pi}$			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.6233(−0.0525)	1.0668(−0.0145)	1.0265(0.4202)	2.4463(0.2880)	0.0300(0.0000)	0.0318(0.0000)	0.0297(0.0000)	0.0315(0.0000)
	$N = 500$	0.4869(−0.0135)	0.7269(−0.0013)	0.7186(0.2332)	1.0164(0.0906)	0.0193(0.0000)	0.0194(0.0000)	0.0195(0.0000)	0.0195(0.0000)
	$N = 1000$	0.3787(−0.0135)	0.4903(−0.0093)	0.4869(0.1128)	0.5616(0.0207)	0.0132(0.0000)	0.0135(0.0000)	0.0132(0.0000)	0.0133(0.0000)
	$N = 2000$	0.2969(0.0115)	0.3529(0.0117)	0.3570(0.0779)	0.3657(0.0219)	0.0089(0.0000)	0.0090(0.0000)	0.0090(0.0000)	0.0090(0.0000)
		$\boldsymbol{\eta}$				$\boldsymbol{\lambda}_{main}$			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0.3$	$N = 200$	0.3044(−0.0454)	0.3929(−0.0562)	0.3312(0.1008)	0.4784(−0.0602)	0.4075(0.0720)	0.5998(−0.0909)	0.4922(−0.1235)	0.8472(0.0909)
	$N = 500$	0.2128(−0.0238	0.2431(−0.0297)	0.2263(0.0419)	0.2492(−0.0248)	0.2944(0.0293)	0.3701(−0.0272)	0.3469(−0.0708)	0.3883(0.0362)
	$N = 1000$	0.1489(0.0009)	0.1642(−0.0036)	0.1600(0.0348)	0.1661(−0.0002)	0.2244(0.0051)	0.2627(−0.0161)	0.2543(−0.0461)	0.2689(0.0146)
	$N = 2000$	0.1115(0.0030)	0.1177(0.0001)	0.1162(0.0208)	0.1183(0.0016)	0.1695(−0.0030)	0.1866(−0.0106)	0.1834(−0.0298)	0.1884(0.0044)
		$\boldsymbol{\lambda}_{inter}$				$\boldsymbol{\pi}$			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.5798(−0.0653)	1.0939(0.0215)	0.9812(0.3198)	2.3665(0.1440)	0.0177(0.0000)	0.0302(0.0000)	0.0187(0.0000)	0.0290(0.0000)
	$N = 500$	0.4880(−0.0253)	0.7015(0.0007)	0.6916(0.1476)	0.7824(0.0034)	0.0109(0.0000)	0.0179(0.0000)	0.0110(0.0000)	0.0173(0.0000)
	$N = 1000$	0.3960(−0.0014)	0.5032(0.0042)	0.5030(0.0855)	0.5311(0.0060)	0.0077(0.0000)	0.0122(0.0000)	0.0078(0.0000)	0.0119(0.0000)
	$N = 2000$	0.3140(0.0023)	0.3683(0.0019)	0.3665(0.0455)	0.3770(0.0020)	0.0054(0.0000)	0.0086(0.0000)	0.0054(0.0000)	0.0085(0.0000)
		$\boldsymbol{\eta}$				$\boldsymbol{\lambda}_{main}$			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0.7$	$N = 200$	0.2872(−0.0129)	0.3386(−0.0225)	0.3073(0.0591)	0.3599(−0.0227)	0.4428(0.0787)	0.6374(−0.1393)	0.5342(−0.1178)	1.3457(0.1212)
	$N = 500$	0.1881(−0.0078)	0.2065(−0.0131)	0.1971(0.0271)	0.2098(−0.0118)	0.3140(0.0206)	0.3960(−0.0582)	0.3777(−0.0816)	0.4229(0.0256)
	$N = 1000$	0.1387(−0.0089)	0.1475(−0.0122)	0.1438(0.0099)	0.1488(−0.0115)	0.2451(0.0043)	0.2842(−0.0257)	0.2795(−0.0497)	0.2929(0.0163)
	$N = 2000$	0.0994(−0.0023)	0.1036(−0.0047)	0.1024(0.0076)	0.1039(-0.0043)	0.1788(−0.0036)	0.1966(-0.0150)	0.1962(−0.0311)	0.1991(0.0057)
		$\boldsymbol{\lambda}_{inter}$				$\boldsymbol{\pi}$			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.5832(−0.096)	1.1817(0.1084)	0.9757 0.2396()	3.8136(−0.0491)	0.0174(0.0000)	0.0258(0.0000)	0.0173(0.0000)	0.0271(0.0000)
	$N = 500$	0.5217(-0.0321)	0.7879(0.0392)	0.7714(0.1232)	1.0967(−0.0132)	0.0094(0.0000)	0.0152(0.0000)	0.0096(0.0000)	0.0149(0.0000)
	$N = 1000$	0.4301(−0.0022)	0.5785(0.0201)	0.5721(0.0791)	0.6197(−0.0052)	0.0064(0.0000)	0.0097(0.0000)	0.0065(0.0000)	0.0093(0.0000)
	$N = 2000$	0.3501(0.0011)	0.4181(0.0076)	0.4212(0.0419)	0.4320(−0.0047)	0.0047(0.0000)	0.0070(0.0000)	0.0047(0.0000)	0.0071(0.0000)

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all main effect slope parameters $\boldsymbol{\lambda}_{main}$, all interaction slope parameters $\boldsymbol{\lambda}_{inter}$ and all class membership probability parameters $\boldsymbol{\pi}$.

Table S8.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M, VB, MCMC-R2jags and EM-GDINA Algorithms for the saturated LCDM.

		AAR1				AAR2			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0$	$N = 200$	0.9310	0.9300	0.9294	0.9282	0.9300	0.9248	0.9270	0.9242
	$N = 500$	0.9365	0.9356	0.9353	0.9352	0.9370	0.9361	0.9359	0.9360
	$N = 1000$	0.9380	0.9372	0.9373	0.9373	0.9392	0.9381	0.9386	0.9387
	$N = 2000$	0.9388	0.9388	0.9388	0.9387	0.9388	0.9389	0.9387	0.9387
		AAR3				PAR			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.9294	0.9240	0.9250	0.9234	0.8158	0.8014	0.8082	0.8024
	$N = 500$	0.9358	0.9349	0.9344	0.9346	0.8322	0.8288	0.8292	0.8287
	$N = 1000$	0.9381	0.9380	0.9377	0.9380	0.8382	0.8360	0.8370	0.8370
	$N = 2000$	0.9390	0.9389	0.9389	0.9389	0.8396	0.8392	0.8394	0.8391
		AAR1				AAR2			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0.3$	$N = 200$	0.9361	0.9306	0.9315	0.9284	0.9400	0.9334	0.9370	0.9334
	$N = 500$	0.9428	0.9425	0.9418	0.9421	0.9410	0.9404	0.9403	0.9400
	$N = 1000$	0.9427	0.9423	0.9420	0.9418	0.9441	0.9442	0.9440	0.9438
	$N = 2000$	0.9434	0.9434	0.9433	0.9431	0.9438	0.9438	0.9437	0.9436
		AAR3				PAR			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.9355	0.9310	0.9324	0.9304	0.8296	0.8126	0.8204	0.8120
	$N = 500$	0.9392	0.9391	0.9392	0.9384	0.8401	0.8387	0.8389	0.8380
	$N = 1000$	0.9434	0.9432	0.9431	0.9429	0.8468	0.8462	0.8459	0.8456
	$N = 2000$	0.9429	0.9430	0.9429	0.9428	0.8465	0.8464	0.8464	0.8461
		AAR1				AAR2			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0.7$	$N = 200$	0.9518	0.9480	0.9486	0.9460	0.9491	0.9455	0.9474	0.9436
	$N = 500$	0.9582	0.9569	0.9574	0.9576	0.9558	0.9549	0.9548	0.9548
	$N = 1000$	0.9569	0.9566	0.9567	0.9566	0.9584	0.9579	0.9583	0.9581
	$N = 2000$	0.9577	0.9574	0.9575	0.9575	0.9574	0.9574	0.9574	0.9573
		AAR3				PAR			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.9471	0.9444	0.9455	0.9410	0.8584	0.8484	0.8524	0.8422
	$N = 500$	0.9549	0.9532	0.9535	0.9532	0.8776	0.8739	0.8746	0.8748
	$N = 1000$	0.9572	0.9565	0.9568	0.9571	0.8811	0.8795	0.8803	0.8804
	$N = 2000$	0.9583	0.9582	0.9583	0.9582	0.8824	0.8819	0.8821	0.8820

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute, AAR3 for the third attribute. PAR stands for the pattern-wise agreement rate.

Table S9.

The SDs/SEs values of the VBEM-M, MCMC, and EM algorithms for the LCDM model in simulation study 3.

		$\boldsymbol{\eta}$			$\boldsymbol{\lambda}_{main}$		
		VBEM-M(SD)	MCMC(SD)	EM(SE)	VBEM-M(SD)	MCMC(SD)	EM(SE)
$\sigma = 0$	$N = 200$	0.2576	0.3642	1.1423	0.3636	0.5280	1.6786
	$N = 500$	0.1761	0.2496	0.3565	0.2534	0.3588	0.5138
	$N = 1000$	0.1289	0.1833	0.2030	0.1881	0.2640	0.2910
	$N = 2000$	0.0923	0.1311	0.1366	0.1365	0.1909	0.1968
		$\boldsymbol{\lambda}_{inter}$			$\boldsymbol{\pi}$		
		VBEM-M(SD)	MCMC(SD)	EM(SE)	VBEM-M(SD)	MCMC(SD)	EM(SE)
	$N = 200$	0.5443	1.0297	3.5766	0.0223	0.0297	0.0349
	$N = 500$	0.4016	0.6804	1.1894	0.0145	0.0188	0.0201
	$N = 1000$	0.3085	0.4886	0.6007	0.0104	0.0133	0.0134
	$N = 2000$	0.2304	0.3516	0.3843	0.0074	0.0095	0.0094
		$\boldsymbol{\eta}$			$\boldsymbol{\lambda}_{main}$		
		VBEM-M(SD)	MCMC(SD)	EM(SE)	VBEM-M(SD)	MCMC(SD)	EM(SE)
$\sigma = 0.3$	$N = 200$	0.2262	0.3046	0.5065	0.4033	0.5800	1.9391
	$N = 500$	0.1624	0.2246	0.2667	0.2547	0.3544	0.4337
	$N = 1000$	0.1069	0.1410	0.1461	0.2058	0.2775	0.2997
	$N = 2000$	0.0841	0.1156	0.1186	0.1358	0.1850	0.1878
		$\boldsymbol{\lambda}_{inter}$			$\boldsymbol{\pi}$		
		VBEM-M(SD)	MCMC(SD)	EM(SE)	VBEM-M(SD)	MCMC(SD)	EM(SE)
	$N = 200$	0.6065	1.0821	5.1583	0.0202	0.0260	0.0299
	$N = 500$	0.4126	0.6715	1.1117	0.0143	0.0183	0.0194
	$N = 1000$	0.3595	0.5529	0.7068	0.0093	0.0117	0.0119
	$N = 2000$	0.2367	0.3484	0.3690	0.0072	0.0092	0.0092
		$\boldsymbol{\eta}$			$\boldsymbol{\lambda}_{main}$		
		VBEM-M(SD)	MCMC(SD)	EM(SE)	VBEM-M(SD)	MCMC(SD)	EM(SE)
$\sigma = 0.7$	$N = 200$	0.2262	0.3046	0.5065	0.4033	0.5800	1.9391
	$N = 500$	0.1485	0.1968	0.2190	0.2790	0.3845	0.5599
	$N = 1000$	0.1069	0.1410	0.1461	0.2058	0.2775	0.2997
	$N = 2000$	0.0764	0.1008	0.1016	0.1488	0.1992	0.2023
		$\boldsymbol{\lambda}_{inter}$			$\boldsymbol{\pi}$		
		VBEM-M(SD)	MCMC(SD)	EM(SE)	VBEM-M(SD)	MCMC(SD)	EM(SE)
	$N = 200$	0.6065	1.0821	5.1583	0.0202	0.0260	0.0299
	$N = 500$	0.4623	0.7584	1.7940	0.0131	0.0165	0.0178
	$N = 1000$	0.3595	0.5529	0.7068	0.0093	0.0117	0.0119
	$N = 2000$	0.2709	0.4002	0.4332	0.0066	0.0084	0.0083

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all main effect slope parameters $\boldsymbol{\lambda}_{main}$, all interaction slope parameters $\boldsymbol{\lambda}_{inter}$ and all class membership probabilityparameters $\boldsymbol{\pi}$.

Table S10.

The computation time (in seconds) for the VBEM-M, VB, MCMC-R2jags and EM-GDINA, algorithms based on LCDM in simulation study 3.

		VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0$	$N = 200$	0.0561s	0.0795s	174.2620s	1.6987s
	$N = 500$	0.0746s	0.1151s	476.9341s	0.8055s
	$N = 1000$	0.1011s	0.1800s	1070.6123s	0.7404s
	$N = 2000$	0.1683s	0.3702s	2916.6403s	0.7988s
		VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0.3$	$N = 200$	0.0564s	0.0816s	170.1357s	1.4199s
	$N = 500$	0.0705s	0.1061s	478.8929s	0.6961s
	$N = 1000$	0.0978s	0.1827s	1072.0754s	0.6939s
	$N = 2000$	0.1680s	0.3605s	3162.5147s	0.7485s
		VBEM-M	VB	MCMC-R2jags	EM-GDINA
$\sigma = 0.7$	$N = 200$	0.0519s	0.0715s	197.4920s	2.3243s
	$N = 500$	0.0666s	0.1024s	533.1874s	0.7829s
	$N = 1000$	0.0946s	0.1547s	1219.3700s	0.6638s
	$N = 2000$	0.1715s	0.3065s	3405.8339s	0.7241s

Table S11.

The estimation results of s and g for the VBEM-M, VB, MCMC and EM algorithms in the empirical example analysis 1.

	VBEM-M		VB		MCMC		EM	
Item	g	s	g	s	g	s	g	s
1	0.0193	0.2825	0.0100	0.2782	0.0114	0.2774	0.0001	0.2778
2	0.2035	0.1120	0.2134	0.1203	0.2142	0.1209	0.2107	0.1178
3	0.1197	0.0395	0.1412	0.0395	0.14400	0.0390	0.1343	0.0383
4	0.1216	0.1252	0.1273	0.1351	0.1268	0.1356	0.1248	0.1309
5	0.1307	0.2416	0.2251	0.2482	0.2149	0.2474	0.2998	0.2474
6	0.0355	0.2247	0.0358	0.2277	0.0358	0.2288	0.0321	0.2264
7	0.0743	0.0773	0.0759	0.0803	0.0757	0.0801	0.0724	0.0779
8	0.1417	0.0480	0.1618	0.0502	0.1740	0.0501	0.1545	0.0483
9	0.0835	0.0652	0.0985	0.0632	0.1108	0.0614	0.0794	0.0628
10	0.1658	0.0664	0.1706	0.0742	0.1697	0.0749	0.1700	0.0697
11	0.0974	0.1045	0.1154	0.1040	0.1231	0.1004	0.1016	0.1051
12	0.0368	0.1347	0.0334	0.1356	0.0324	0.1367	0.0307	0.1327
13	0.1293	0.1533	0.1367	0.1602	0.1363	0.1597	0.1337	0.1528
14	0.0277	0.1993	0.0250	0.2011	0.0248	0.2000	0.0218	0.1974
15	0.0196	0.1846	0.0144	0.1841	0.0150	0.1864	0.0104	0.1824

Table S12.

The estimation results of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ using the VB algorithm in the empirical example analysis 2.

Item	$\hat{\eta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_{12}$	$\hat{\lambda}_{13}$	$\hat{\lambda}_{23}$
1	0.8321	–0.7289	0.5591	–	2.0308	–	–
2	1.0065	–	1.2450	–	–	–	–
3	–0.3432	0.7578	–	0.3436	–	0.5169	–
4	–0.1582	–	–	1.6892	–	–	–
5	1.0558	–	–	2.0021	–	–	–
6	0.8452	–	–	1.6801	–	–	–
7	–0.0966	1.8727	–	0.9283	–	0.0263	–
8	1.4501	–	1.8546	–	–	–	–
9	0.1012	–	–	1.2004	–	–	–
10	0.0549	2.0342	–	–	–	–	–
11	–0.0470	0.8213	–	0.9452	–	0.7662	–
12	–1.7631	–0.1966	–	1.2593	–	1.7195	–
13	0.6603	1.6057	–	–	–	–	–
14	0.1728	1.3702	–	–	–	–	–
15	0.9664	–	–	2.1103	–	–	–
16	–0.0961	1.5976	0.0000	0.8719	–	–0.1206	–
17	1.3379	–	0.3918	0.5408	–	–	0.5033
18	0.9032	–	–	1.3925	–	–	–
19	–0.2195	–	–	1.8518	–	–	–
20	–1.4030	0.4688	–	0.9071	–	1.1736	–
21	0.1541	0.9143	–	1.1225	–	0.1763	–
22	–0.9003	–	–	2.2475		–	–
23	0.6324	–	2.0111	–	–	–	–
24	–0.7048	–	1.5191	–	–	–	–
25	0.0930	1.1276	–	–	–	–	–
26	0.1492	–	–	1.1216	–	–	–
27	–0.8836	1.7040	–	–	–	–	–
28	0.5428	–	–	1.7469	–	–	–

Note: The values outside the parentheses represent the posterior means of the parameters, while the values inside the parentheses indicate the standard deviation.

Table S13.

The estimation results of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ using the MCMC algorithm in the empirical example analysis 2.

Item	$\hat{\eta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_{12}$	$\hat{\lambda}_{13}$	$\hat{\lambda}_{23}$	
1	0.8088(0.0875)	0.7519(0.5659)	0.6941(0.2306)	–	0.3329(0.6717)	–	–	
2	1.0381(0.0924)	–	1.2462 (0.1577)	–	–	–	–	
3	–0.3407(0.0781)	0.7923(0.4775)	–	0.3509(0.1252)	–	0.4990 (0.4976)	–	
4	–0.1411(0.0829)	–	–	1.6950(0.1101)	–	–	–	
5	1.0790(0.0948)	–	–	2.0217(0.1623)	–	–	–	
6	0.8678(0.0928)	–	–	1.6874(0.1383)	–	–	–	
7	–0.0794(0.0889)	1.7259(0.8096)	–	0.9342(0.1346)	–	0.2146(0.8389)	–	
8	1.4845(0.1034)	–	1.9309(0.2413)	–	–	–	–	
9	0.1211(0.0785)	–	–	1.1913(0.1043)	–	–	–	
10	0.0608(0.0618)	2.0513(0.1392)	–	–	–	–	–	
11	–0.0502(0.0832)	1.3260(0.8240)	–	0.9785(0.1388)	–	0.2691(0.8555)	–	
12	–1.7894(0.1093)	0.5413(0.3793)	–	1.3164(0.1640)	–	0.9635(0.4043)	–	
13	0.6679(0.0644)	1.6213(0.1528)	–	–	–	–	–	
14	0.1779(0.0616)	1.3784(0.1227)	–	–	–	–	–	
15	0.9924(0.0967)	–	–	2.1260 (0.1686)	–	–	–	
16	–0.0666(0.0873)	1.2493(0.7285)	–	0.8520(0.1374)	–	0.2744 (0.7580)	–	
17	1.3484(0.1058)	–	0.9870(0.7367)	0.6041(0.2864)	–	–	–0.1217(0.768)	
18	0.9157(0.0884)	–	–	1.4037(0.1285)	–	–	–	
19	–0.1924(0.0778)	–	–	1.8423 (0.1053)	–	–	–	
20	–1.4396(0.1028)	1.0964(0.5648)	–	0.9711(0.1474)	–	0.5333(0.5870)	–	
21	0.1506(0.0836)	1.5069(0.7123)	–	1.1555(0.1411)	–	–0.4171(0.7229)	–	
22	–0.8649(0.0912)	–	–	2.2331(0.1153)	–	–	–	
23	0.6695(0.0905)	–	2.0669(0.1977)	–	–	–	–	
24	–0.6731(0.0938)	–	1.5256(0.1166)	–	–	–	–	
25	0.0945(0.0557)	1.1459(0.1115)	–	–		–	–	–
26	0.1603(0.0781)	–	–	1.1232(0.0998)	–	–	–	
27	–0.8760(0.0627)	1.7056(0.1079)	–	–	–	–	–	
28	0.5688(0.0876)	–	–	1.7420(0.1243)	–	–	–	

Note: The values outside the parentheses represent the posterior means of the parameters, while the values inside the parentheses indicate the standard deviation.

Table S14.

The estimation results of the parameters $\boldsymbol{\eta}$ and $\boldsymbol{\lambda}$ using the EM algorithm in the empirical example analysis 2.

Item	$\hat{\eta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\lambda}_3$	$\hat{\lambda}_{12}$	$\hat{\lambda}_{13}$	$\hat{\lambda}_{23}$
1	0.8424(0.0756)	–0.9448(4.7080)	0.5653(0.2104)	–	2.2867(4.8335)	–	–
2	1.0210(0.0971)	–	1.2480(0.1610)	–	–	–	–
3	–0.3476(0.0876)	1.0324(4.4227)	–	0.3548(0.1305)	–	0.2431(4.4104)	–
4	–0.1375(0.0819)	–	–	1.6813(0.1044)	–	–	–
5	1.0789(0.0804)	–	–	2.0109(0.1663)	–	–	–
6	0.8620(0.0827)	–	–	1.6898(0.1264)	–	–	–
7	–0.0991(0.0904)	3.1707(4.8837)	–	0.9458(0.1524)	–	-1.2668(4.8807)	–
8	1.4639(0.1089)	–	1.9036(0.2891)	–	–	–	–
9	0.1133(0.0749)	–	–	1.1997(0.1100)	–	–	–
10	0.0599(0.0589)	2.0467(0.1383)	–	–	–	–	–
11	–0.0279(0.0651)	0.5312(3.5429)	–	0.9508(0.1233)	–	1.0634(3.5811)	–
12	–1.7353(0.1107)	–7.4760(4.0051)	–	1.2563(0.1706)	–	8.9954(4.0142)	–
13	0.6629(0.0584)	1.6323(0.1791)	–	–	–	–	–
14	0.1776(0.0572)	1.3735(0.1428)	–	–	–	–	–
15	0.9929(0.0764)	–	–	2.1097(0.1817)	–	–	–
16	–0.0873(0.0816)	1.8005(4.1775)	–	0.8747(0.1462)	–	–0.3122(4.2187)	–
17	1.3295(0.1192)	–	1.2410(3.5970)	0.6229(0.3866)	–	–	–0.4301(3.5452)
18	0.9248(0.0836)	–	–	1.3836(0.1171)	–	–	–
19	–0.1978(0.0800)	–	–	1.8451(0.0998)	–	–	–
20	–1.3811(0.1109)	–0.0228(4.3936)	–	0.9021(0.1813)	–	1.6674(4.4265)	–
21	0.1670(0.0934)	1.0456(2.4658)	–	1.1286(0.1435)	–	0.0410(2.4537)	–
22	–0.8772(0.0871)	–	–	2.2440(0.1031)	–	–	–
23	0.6486(0.0877)	–	2.0300(0.2279)	–	–	–	–
24	–0.6762(0.0976)	–	1.4905(0.1312)	–	–	–	–
25	0.0975(0.0522)	1.1279(0.1220)	–	–		–	– –
26	0.1644(0.0840)	–	–	1.1145(0.1063)	–	–	–
27	–0.8836(0.0736)	1.7172(0.0863)	–	–	–	–	–
28	0.5651(0.0833)	–	–	1.7410(0.1297)	–	–	–

Note: The values outside the parentheses represent the posterior means of the parameters, while the values inside the parentheses indicate the standard error.

Table S15.

The estimation results of the class membership parameters $\boldsymbol{\pi}$ using the VB, MCMC and EM algorithm in the empirical example analysis 2.

π	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	(1,1,1)
VB	0.2928	0.0040	0.0158	0.1246	0.0112	0.0157	0.1832	0.3526
MCMC	0.2964	0.0089	0.0194	0.1297	0.0077	0.0165	0.1790	0.3424
EM	0.3017	0.0021	0.0122	0.1247	0.0142	0.0159	0.1804	0.3489

Table S16.

Q-matrix for simulation study 1(b).

Item	α_1	α_2	α_3	α_4	α_5	Item	α_1	α_2	α_3	α_4	α_5
1	1	0	0	0	0	16	0	1	0	0	0
2	0	1	0	0	0	17	1	0	0	0	0
3	0	0	1	0	0	18	0	1	1	1	0
4	0	0	0	1	0	19	0	0	1	0	1
5	0	0	0	0	1	20	0	1	1	0	0
6	1	0	0	0	0	21	1	1	0	1	1
7	0	1	0	0	0	22	1	0	1	1	0
8	0	0	1	0	0	23	0	0	0	0	1
9	0	0	0	1	0	24	1	1	0	1	0
10	0	0	0	0	1	25	1	0	0	1	1
11	0	1	0	1	0	26	0	1	1	0	1
12	1	1	1	0	0	27	1	1	1	0	0
13	0	1	0	0	1	28	1	1	1	0	1
14	1	1	0	1	0	29	0	1	0	0	0
15	0	0	1	0	1	30	0	1	1	0	1

Table S17.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M under different initial values conditions.

initial value		$\boldsymbol{\eta}$ RMSE(Bias)	$\boldsymbol{\eta}$ SD	$\boldsymbol{\lambda}$ RMSE(Bias)	$\boldsymbol{\lambda}$ SD	$\boldsymbol{\pi}$ RMSE(Bias)	$\boldsymbol{\pi}$ SD
initial-I	$\eta_j = -1, \lambda_j = 1$	0.1142(–0.0043)	0.0856	0.2034(0.0155)	0.1507	0.0057(0.0000)	0.0051
initial-II	$\eta_j = -1, \lambda_j = 3$	0.1142(–0.0043)	0.0856	0.2034(0.0155)	0.1507	0.0057(0.0000)	0.0051
initial-III	$\eta_j = -1, \lambda_j = 5$	0.1142(–0.0043)	0.0856	0.2034(0.0155)	0.1507	0.0057(0.0000)	0.0051
initial-IV	$\eta_j = -3, \lambda_j = 1$	0.1142(–0.0043)	0.0856	0.2034(0.0155)	0.1507	0.0057(0.0000)	0.0051
initial-V	$\eta_j = -3, \lambda_j = 3$	0.1142(–0.0043)	0.0856	0.2034(0.0155)	0.1507	0.0057(0.0000)	0.0051
initial-VI	$\eta_j = -3, \lambda_j = 5$	0.1142(–0.0043)	0.0856	0.2034(0.0155)	0.1507	0.0057(0.0000)	0.0051

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all slope parameters $\boldsymbol{\lambda}$ and all class membership probability parameters $\boldsymbol{\pi}$.

Table S18.

Evaluation of the accuracy of attribute profile parameters usingg the VBEM-M under different initial values conditions.

		AAR1	AAR2	AAR3	AAR4	AAR5	PAR
initial-I	$\eta_j = -1, \lambda_j = 1$	0.9165	0.9522	0.9120	0.8901	0.9259	0.6889
initial-II	$\eta_j = -1, \lambda_j = 3$	0.9165	0.9522	0.9120	0.8901	0.9259	0.6889
initial-III	$\eta_j = -1, \lambda_j = 5$	0.9165	0.9522	0.9120	0.8901	0.9259	0.6889
initial-IV	$\eta_j = -3, \lambda_j = 1$	0.9165	0.9522	0.9120	0.8901	0.9259	0.6889
initial-V	$\eta_j = -3, \lambda_j = 3$	0.9165	0.9522	0.9120	0.8901	0.9259	0.6889
initial-VI	$\eta_j = -3, \lambda_j = 5$	0.9165	0.9522	0.9120	0.8901	0.9259	0.6889

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute, AAR3 for the third attribute, AAR4 for the fourth attribute, and AAR5 for the fifth attribute. PAR stands for the pattern-wise agreement rate.

Table S19.

Q-matrix for DINO model.

Item	α_1	α_2	α_3	α_4	α_5	Item	α_1	α_2	α_3	α_4	α_5
1	1	0	0	0	0	16	0	0	1	1	1
2	0	1	0	0	0	17	0	1	0	0	1
3	0	0	1	0	0	18	1	0	1	0	0
4	0	0	0	1	0	19	1	0	1	1	0
5	0	0	0	0	1	20	0	1	1	1	0
6	1	0	0	0	0	21	1	1	1	1	1
7	0	1	0	0	0	22	0	0	0	1	1
8	0	0	1	0	0	23	1	1	1	1	1
9	0	0	0	1	0	24	0	0	1	0	1
10	0	0	0	0	1	25	0	0	0	1	0
11	0	0	1	0	0	26	1	1	1	0	0
12	1	1	0	0	1	27	1	0	0	0	0
13	1	0	1	1	0	28	1	0	1	1	0
14	1	1	1	0	1	29	1	1	1	1	0
15	0	0	1	1	1	30	1	0	0	0	0

Table S20.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M algorithm under DINO model.

			$\boldsymbol{\eta}$		$\boldsymbol{\lambda}$		$\boldsymbol{\pi}$	
			RMSE(Bias)	SD	RMSE(Bias)	SD	RMSE(Bias)	SD
LNL	$N = 1000$	$\sigma = 0$	0.2167(–0.0272)	0.1538	0.2554(0.0358)	0.1811	0.0057(0.0000)	0.0053
		$\sigma = 0.3$	0.1901(–0.0228)	0.1336	0.2321(0.0368)	0.1636	0.0023(0.0000)	0.0051
		$\sigma = 0.7$	0.1692(–0.0156)	0.1179	0.2185(0.0367)	0.1521	0.0023(0.0000)	0.0045
			$\boldsymbol{\eta}$		$\boldsymbol{\lambda}$		$\boldsymbol{\pi}$	
			RMSE(Bias)	SD	RMSE(Bias)	SD	RMSE(Bias)	SD
	$N = 2000$	$\sigma = 0$	0.1602(–0.0107)	0.1099	0.1885(0.0166)	0.1296	0.0042(0.0000)	0.0038
		$\sigma = 0.3$	0.1360(–0.0100)	0.0952	0.1686(0.0142)	0.1167	0.0016(0.0000)	0.0037
		$\sigma = 0.7$	0.1202(–0.0040)	0.0836	0.1584(0.0096)	0.1080	0.0015(0.0000)	0.0032
			$\boldsymbol{\eta}$		$\boldsymbol{\lambda}$		$\boldsymbol{\pi}$	
			RMSE(Bias)	SD	RMSE(Bias)	SD	RMSE(Bias)	SD
HNL	$N = 1000$	$\sigma = 0$	0.1937(–0.0243)	0.1414	0.2291(0.0256)	0.1666	0.0077(0.0000)	0.0053
		$\sigma = 0.3$	0.1618(–0.0091)	0.1230	0.2005(0.0162)	0.1506	0.0055(0.0000)	0.0051
		$\sigma = 0.7$	0.1321(–0.0079)	0.1084	0.1754(0.0197)	0.1398	0.0051(0.0000)	0.0045
			$\boldsymbol{\eta}$		$\boldsymbol{\lambda}$		$\boldsymbol{\pi}$	
			RMSE(Bias)	SD	RMSE(Bias)	SD	RMSE(Bias)	SD
	$N = 2000$	$\sigma = 0$	0.1361(–0.0040)	0.1014	0.1647(0.0071)	0.1194	0.0057(0.0000)	0.0038
		$\sigma = 0.3$	0.1155(0.0004)	0.0874	0.1446(0.0013)	0.1071	0.0040(0.0000)	0.0037
		$\sigma = 0.7$	0.0952(-0.0011)	0.0769	0.1283(0.0062)	0.0993	0.0035(0.0000)	0.0032

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all slope parameters $\boldsymbol{\lambda}$ and all class membership probability parameters $\boldsymbol{\pi}$.

Table S21.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M algorithm under DINO model.

LNL	$N = 1000$						$N = 2000$					
	AAR1	AAR2	AAR3	AAR4	AAR5	PAR	AAR1	AAR2	AAR3	AAR4	AAR5	PAR
$\sigma = 0$	0.9721	0.9894	0.9855	0.9601	0.9789	0.8990	0.9724	0.9893	0.9857	0.9591	0.9789	0.8987
$\sigma = 0.3$	0.9749	0.9891	0.9862	0.9668	0.9807	0.9073	0.9758	0.9888	0.9859	0.967	0.9804	0.9079
$\sigma = 0.7$	0.9815	0.9875	0.9854	0.9766	0.9835	0.9230	0.9819	0.9876	0.9862	0.9775	0.9850	0.9256
HNL	$N = 1000$						$N = 2000$					
	AAR1	AAR2	AAR3	AAR4	AAR5	PAR	AAR1	AAR2	AAR3	AAR4	AAR15	PAR
$\sigma = 0$	0.8989	0.9355	0.9262	0.8825	0.9121	0.6712	0.8995	0.9364	0.9257	0.8824	0.9117	0.6700
$\sigma = 0.3$	0.9109	0.938	0.9295	0.8984	0.9176	0.6946	0.9119	0.9381	0.9309	0.9008	0.9197	0.6991
$\sigma = 0.7$	0.9333	0.952	0.9463	0.9284	0.9388	0.7707	0.935	0.9530	0.9483	0.9278	0.9398	0.7749

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute, AAR3 for the third attribute, AAR4 for the fourth attribute, and AAR5 for the fifth attribute. PAR stands for the pattern-wise agreement rate.

Table S22.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M, MCMC-R2jags and EM-GDINA algorithms for the DINO with $\sigma = 0.3$.

LNL	$\boldsymbol{\eta}$			
	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$N = 200$	0.3282(–0.0250)	0.4316(0.1183)	0.4306(0.1268)	0.7326(–0.1338)
$N = 500$	0.2544(–0.0209)	0.2905(0.0445)	0.2896(0.0480)	0.3440(–0.0360)
$N = 1000$	0.1953(–0.0189)	0.2076(0.0169)	0.2069(0.0186)	0.2165(–0.0193)
$N = 2000$	0.1440(–0.0109)	0.1479(0.0079)	0.1478(0.0090)	0.1515(–0.0102)
	$\boldsymbol{\lambda}$			
	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$N = 200$	0.4021(0.0581)	0.5495(–0.1629)	0.5516(–0.1871)	0.8527(0.1637)
$N = 500$	0.3129(0.0453)	0.3596(–0.0522)	0.3589(–0.0619)	0.4150(0.0569)
$N = 1000$	0.2373(0.0338)	0.2529(–0.0183)	0.2520(–0.0230)	0.2630(0.0319)
$N = 2000$	0.1806(0.0248)	0.1850(–0.0022)	0.1847(–0.0045)	0.1898(0.0229)
	$\boldsymbol{\pi}$			
	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$N = 200$	0.0053(0.0000)	0.0053(0.0000)	0.0054(0.0000)	0.0052(0.0000)
$N = 500$	0.0035(0.0000)	0.0035(0.0000)	0.0035(0.0000)	0.0036(0.0000)
$N = 1000$	0.0025(0.0000)	0.0025(0.0000)	0.0025(0.0000)	0.0024(0.0000)
$N = 2000$	0.0017(0.0000)	0.0017(0.0000)	0.0017(0.0000)	0.0017(0.0000)
HNL	$\boldsymbol{\eta}$			
	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$N = 200$	0.3090(–0.0491)	0.3799(0.0225)	0.3725(0.0406)	0.6309(–0.1213)
$N = 500$	0.2214(–0.0205)	0.2407(0.0132)	0.2394(0.0220)	0.2536(–0.0199)
$N = 1000$	0.1612(–0.0082)	0.1679(0.0099)	0.1680(0.0150)	0.1718(–0.0065)
$N = 2000$	0.1130(–0.0053)	0.1153(0.0041)	0.1151(0.0066)	0.1167(–0.0041)
	$\boldsymbol{\lambda}$			
	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$N = 200$	0.3963(0.0929)	0.4981(–0.0114)	0.4831(–0.0563)	0.7893(0.1717)
$N = 500$	0.2752(0.0382)	0.3013(–0.0097)	0.2993(–0.0292)	0.3153(0.0339)
$N = 1000$	0.1965(0.0143)	0.2061(–0.0107)	0.2063(–0.0215)	0.2099(0.0106)
$N = 2000$	0.1412(0.0134)	0.1442(0.0006)	0.1440(–0.0050)	0.1459(0.0113)
	$\boldsymbol{\pi}$			
	VBEM-M	VB	MCMC-R2jags	EM-GDINA
$N = 200$	0.0111(0.0000)	0.0112(0.0000)	0.0106(0.0000)	0.0142(0.0000)
$N = 500$	0.0077(0.0000)	0.0077(0.0000)	0.0076(0.0000)	0.0087(0.0000)
$N = 1000$	0.0055(0.0000)	0.0055(0.0000)	0.0056(0.0000)	0.0059(0.0000)
$N = 2000$	0.0042(0.0000)	0.0042(0.0000)	0.0042(0.0000)	0.0043(0.0000)

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all slope parameters $\boldsymbol{\lambda}$ and all class membership probability parameters $\boldsymbol{\pi}$.

Table S23.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M, MCMC-R2jags and EM-GDINA algorithms for the DINO with $\sigma = 0.3$.

		AAR1				AAR2			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.9888	0.9887	0.9887	0.9887	0.9627	0.9605	0.9592	0.95982
	$N = 500$	0.9879	0.9877	0.9877	0.9877	0.9581	0.9576	0.9581	0.9579
	$N = 1000$	0.9888	0.9887	0.9887	0.9887	0.9603	0.9604	0.9604	0.9604
	$N = 2000$	0.9885	0.9885	0.9885	0.9885	0.9626	0.9626	0.9624	0.9626
		AAR3				AAR4			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
LNL	$N = 200$	0.9865	0.9862	0.9855	0.9852	0.9843	0.9843	0.9840	0.9838
	$N = 500$	0.9857	0.9857	0.9856	0.9857	0.9823	0.9821	0.9823	0.9821
	$N = 1000$	0.9845	0.9845	0.9846	0.9845	0.9827	0.9827	0.9824	0.9827
	$N = 2000$	0.9856	0.9856	0.9857	0.9856	0.9826	0.9825	0.9825	0.9826
		AAR5				PAR			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.9707	0.9710	0.9708	0.9708	0.9025	0.8998	0.8975	0.8972
	$N = 500$	0.9727	0.9728	0.9725	0.9727	0.8970	0.8962	0.8966	0.8963
	$N = 1000$	0.9707	0.9708	0.9708	0.9707	0.8968	0.8968	0.8965	0.8969
	$N = 2000$	0.9727	0.9727	0.9728	0.9727	0.9010	0.9009	0.9009	0.9009
		AAR1				AAR2			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.9383	9373	0.9378	0.9332	0.8738	0.8718	0.8725	0.8677
	$N = 500$	0.9392	0.9389	0.9395	0.9385	0.8810	0.8803	0.8797	0.8781
	$N = 1000$	0.9388	0.9388	0.9389	0.9386	0.8871	0.8869	0.8863	0.8874
	$N = 2000$	0.9402	0.9402	0.9404	0.9402	0.8885	0.8884	0.8880	0.8883
		AAR1				AAR2			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
HNL	$N = 200$	0.9238	0.9230	0.9243	0.9217	0.9233	0.9217	0.9212	0.9163
	$N = 500$	0.9292	0.9289	0.9289	0.9279	0.9192	0.9191	0.9185	0.9183
	$N = 1000$	0.9303	0.9303	0.9301	0.9300	0.9193	0.9192	0.9190	0.9193
	$N = 2000$	0.9292	0.9291	0.9286	0.9291	0.9217	0.9216	0.9215	0.9215
		AAR5				PAR			
		VBEM-M	VB	MCMC-R2jags	EM-GDINA	VBEM-M	VB	MCMC-R2jags	EM-GDINA
	$N = 200$	0.8993	0.8972	0.8980	0.8945	0.6662	0.6605	0.6608	0.6470
	$N = 500$	0.8995	0.8990	0.8987	0.8970	0.6687	0.6675	0.6673	0.6627
	$N = 1000$	0.9051	0.9048	0.9046	0.9045	0.6782	0.6780	0.6769	0.6776
	$N = 2000$	0.9058	0.9057	0.9060	0.9058	0.6828	0.6826	0.6822	0.6824

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute, AAR3 for the third attribute, AAR4 for the fourth attribute, and AAR5 for the fifth attribute. PAR stands for the pattern-wise agreement rate.

Table S24.

True values of $\boldsymbol{\lambda}^*$ for LLM.

Item	η	λ_1	λ_2	λ_3
1	–1.5	3.5	0	0
2	–1.5	0	3.5	0
3	–1.5	0	0	3.5
4	–1.5	3.5	0	0
5	–1.5	0	3.5	0
6	–1.5	0	0	3.50
7	–1.5	0	3.5	0
8	–1.5	0	3.5	0
9	–1.5	1.5	1.5	1.5
10	–1.5	3.5	0	0
11	–1.5	1.5	1.5	1.5
12	–1.5	0	2	2
13	–1.5	3.5	0	0
14	–1.5	0	2	2
15	–1.5	2	0	2
16	–1.5	1.5	1.5	1.5
17	–1.5	0	2	2
18	–1.5	2	0	2

Table S25.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M algorithm under LLM model.

		$\boldsymbol{\eta}$		$\boldsymbol{\lambda}$		$\boldsymbol{\pi}$	
		RMSE(Bias)	SD	RMSE(Bias)	SD	RMSE(Bias)	SD
$N = 1000$	$\sigma = 0$	0.1408(–0.0103)	0.1088	0.1868(0.0162)	0.1369	0.0118(0.0000)	0.0104
	$\sigma = 0.3$	0.1348(–0.0040)	0.1043	0.1825(0.0130)	0.1383	0.0059(0.0000)	0.0102
	$\sigma = 0.7$	0.1249(0.0007)	0.1001	0.1915(0.0092)	0.1503	0.0049(0.0000)	0.0094
		$\boldsymbol{\eta}$		$\boldsymbol{\lambda}$		$\boldsymbol{\pi}$	
		RMSE(Bias)	SD	RMSE(Bias)	SD	RMSE(Bias)	SD
$N = 2000$	$\sigma = 0$	0.1047(–0.0060)	0.0772	0.1352(0.0116)	0.0973	0.0086(0.0000)	0.0074
	$\sigma = 0.3$	0.0954(–0.0040)	0.0742	0.1311(0.0062)	0.0983	0.0042(0.0000)	0.0072
	$\sigma = 0.7$	0.0906(–0.0041)	0.0711	0.1358(0.0073)	0.1067	0.0037(0.0000)	0.0066

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all slope parameters $\boldsymbol{\lambda}$ and all class membership probability parameters $\boldsymbol{\pi}$.

Table S26.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M algorithm under LLM model.

	$N = 1000$				$N = 2000$			
	AAR1	AAR2	AAR3	PAR	AAR1	AAR2	AAR3	PAR
$\sigma = 0$	0.9704	0.9701	0.9376	0.8881	0.9705	0.9709	0.9387	0.8897
$\sigma = 0.3$	0.9721	0.9711	0.9426	0.8923	0.9712	0.9713	0.9428	0.8921
$\sigma = 0.7$	0.9776	0.9785	0.9543	0.9143	0.9783	0.9784	0.9557	0.9156

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute and AAR3 for the third attribute. PAR stands for the pattern-wise agreement rate.

Table S27.

Evaluation of the accuracy of item parameters and class membership probability parameters using the VBEM-M, MCMC-R2jags and EM-GDINA algorithms for the saturated LLM with $\sigma = 0.3$.

	$\boldsymbol{\eta}$			$\boldsymbol{\lambda}$			$\boldsymbol{\pi}$		
	VBEM-M	MCMC-R2jags	EM-GDINA	VBEM-M	MCMC-R2jags	EM-GDINA	VBEM-M	MCMC-R2jags	EM-GDINA
$N = 200$	0.2896(−0.0435)	0.3042(0.0196)	0.5082(−0.0903)	0.3120(0.0533)	0.3364(−0.0070)	0.6575(0.1032)	0.0142(0.0000)	0.0142(0.0000)	0.0167(0.0000)
$N = 500$	0.1897(−0.0053)	0.1943(0.0210)	0.2150(−0.0053)	0.2064(0.0234)	0.2116(0.0016)	0.2859(0.0236)	0.0091(0.0000)	0.0091(0.0000)	0.0094(0.0000)
$N = 1000$	0.1350(0.0014)	0.1360(0.0043)	0.1546(0.0043)	0.1513(0.0153)	0.1527(0.0023)	0.2082(0.0084)	0.0062(0.0000)	0.0061(0.0000)	0.0062(0.0000)
$N = 2000$	0.0972(−0.0079)	0.0973(−0.0009)	0.1080(−0.0092)	0.1063(0.0054)	0.1067(−0.0017)	0.1368(0.0085)	0.0043(0.0000)	0.0043(0.0000)	0.0044(0.0000)

Note: The values outside the parentheses represent the RMSE, while the values inside the parentheses indicate bias. Here, RMSE and Bias denote the average RMSE and Bias, respectively, for all intercept parameters $\boldsymbol{\eta}$, all slope parameters $\boldsymbol{\lambda}$ and all class membership probability parameters $\boldsymbol{\pi}$.

Table S28.

Evaluation of the accuracy of attribute profile parameters using the VBEM-M, MCMC-R2jags and EM-GDINA algorithms for the LLM with $\sigma = 0.3$.

	AAR1			AAR2		
	VBEM-M	MCMC-R2jags	EM-GDINA	VBEM-M	MCMC-R2jags	EM-GDINA
$N = 200$	0.9687	0.9687	0.9670	0.9690	0.9692	0.9672
$N = 500$	0.9712	0.9710	0.9703	0.9705	0.9707	0.9692
$N = 1000$	0.9704	0.9702	0.9701	0.9709	0.9708	0.9713
$N = 2000$	0.9713	0.9714	0.9708	0.9721	0.9721	0.9720
	AAR3			PAR		
	VBEM-M	MCMC-R2jags	EM-GDINA	VBEM-M	MCMC-R2jags	EM-GDINA
$N = 200$	0.9353	0.9347	0.9268	0.8798	0.8798	0.8690
$N = 500$	0.9413	0.9407	0.9389	0.8905	0.8899	0.8857
$N = 1000$	0.9419	0.9422	0.9414	0.8907	0.8907	0.8902
$N = 2000$	0.9405	0.9406	0.9403	0.8914	0.8914	0.8907

Note: AAR1 represents the correct classification rate for the first attribute, AAR2 for the second attribute and AAR3 for the third attribute. PAR stands for the pattern-wise agreement rate.

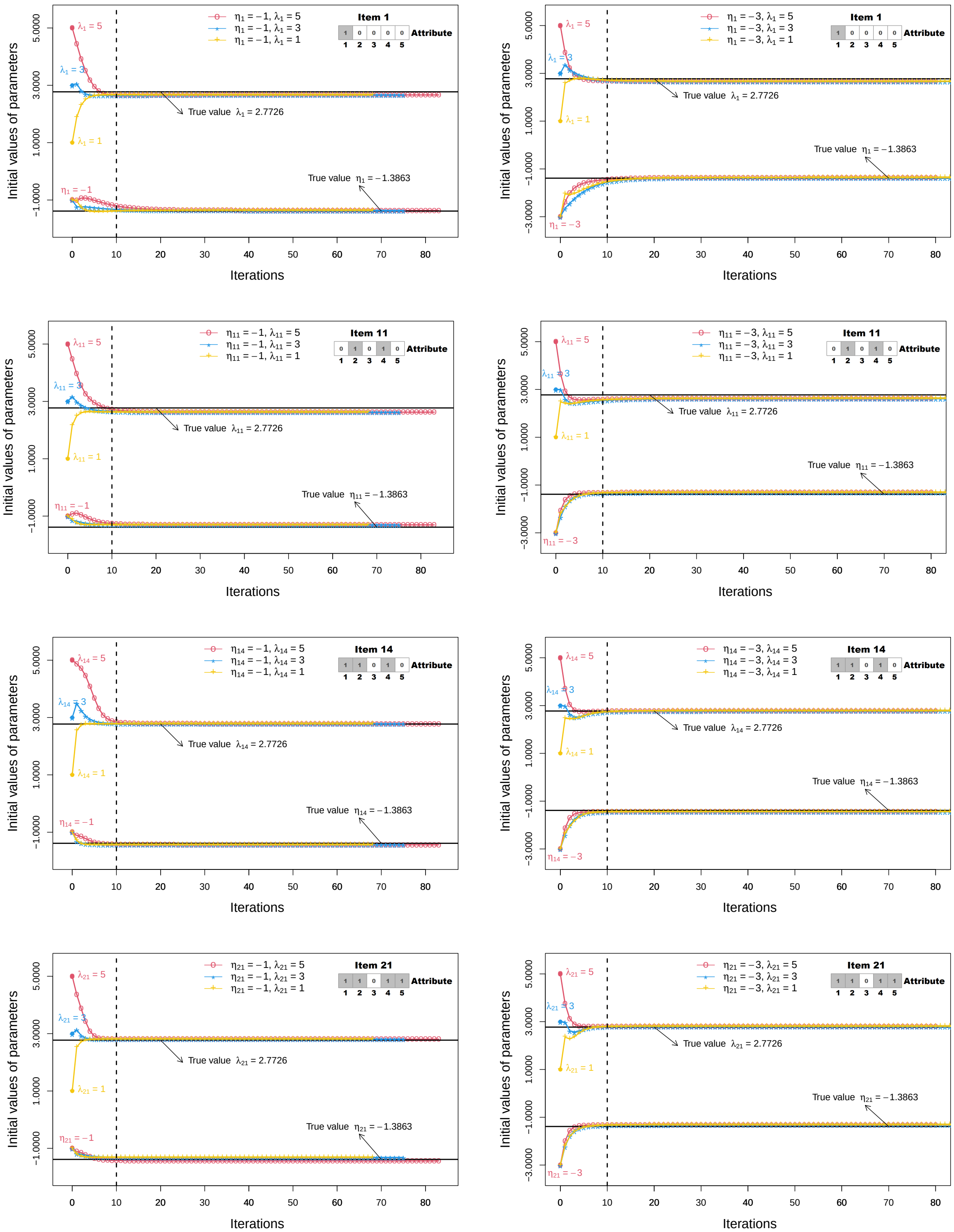

FIGURE S1.

The iteration trace plots of VBEM-M algorithm based on different initial values of parameters in the simulation study 1(b).

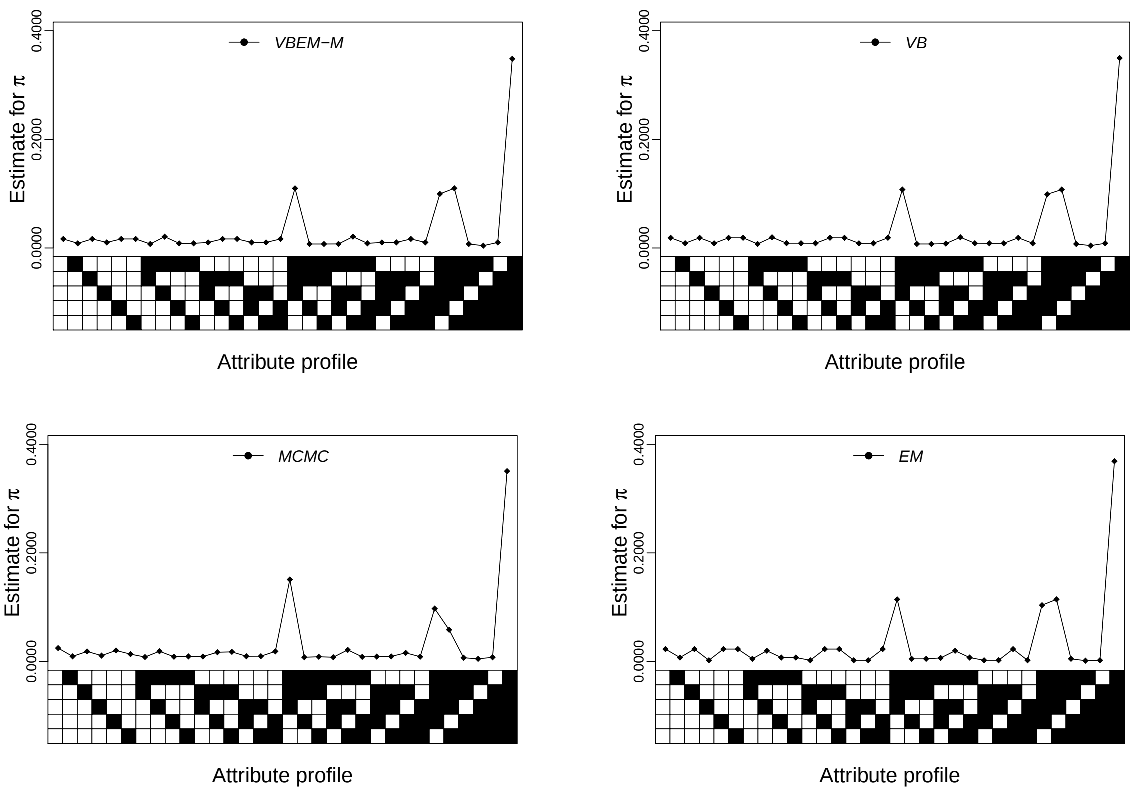

Figure S2.

The estimates of $\boldsymbol{\pi}$ in the empirical example analysis 1 based on the VBEM-M, VB, MCMC-dina and EM algorithms.