Online Supplementary Material to "A Nonparametric Test of Heterogeneity in Conditional Quantile Treatment Effects"

Zongwu Cai[a], Ying Fang[b,c], Ming Lin[b,c], Shengfang Tang[d]

[a]Department of Economics, University of Kansas, Lawrence, KS 66045, USA.

[b]Wang Yanan Institute for Studies in Economics and Fujian Key Laboratory of Statistical Sciences, Xiamen University, Xiamen, Fujian 361005, China.

[c]Department of Statistics and Data Science, School of Economics, Xiamen University, Xiamen, Fujian 361005, China.

[d]School of Finance, Jiangxi University of Finance and Economics, Nanchang, Jiangxi 330013, China.

February 8, 2024

This supplement provides some lemmas for proving the main theorems in the paper.

In the following, we consider using the Series Logit Estimator (SLE) to estimate the propensity score function $p(x)$. Specifically, let $\kappa = (\kappa_1, \ldots, \kappa_m)' \in \mathbb{N}_0^m$ be an m-dimensional vector of nonnegative integers. Define $|\kappa| = \sum_{i=1}^{m} \kappa_i$, and let $\{\kappa(\ell)\}_{\ell=1}^{\infty}$ be a sequence that includes all distinct vectors in $\mathbb{N}_0^m$ and and satisfies $|\kappa(\ell)|$ is nondecreasing in ℓ. For $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$, let x^κ denote the power function $\prod_{j=1}^{m} x_j^{\kappa_j}$. We further define $R^L(x) = (x^{\kappa(1)}, \ldots, x^{\kappa(L)})'$ as an L-vector of power functions for $L > 0$. Then the SLE for

$p(x)$ is

$$\widehat{p}_n(x) = g\big(R^L(x)'\widehat{\pi}_L\big)$$

with $g(u) = \exp(u)/(1 + \exp(u))$ and

$$\widehat{\pi}_L = \arg\max_{\pi} \sum_{i=1}^{n} \left\{ D_i \ln g\big(R^L(X_i)'\pi\big) + (1 - D_i) \ln\big[1 - g\big(R^L(X_i)'\pi\big)\big] \right\}.$$

Assume that $L = a \cdot n^{\nu}$, where a is a positive constant. According to Lemmas 1 and 2 in Hirano et al. (2003), it can be shown that under Assumption 6,

$$\sup_{x \in \mathcal{X}} |\hat{p}_n(x) - p(x)| = O_p\big(n^{3\nu/2 - 1/2}\big).$$

Define $W_0(X_i, D_i) = \frac{1-D_i}{1-p(X_i)}$ and $W_1(X_i, D_i) = \frac{D_i}{p(X_i)}$. We also let $\widehat{W}_{n,0}(X_i, D_i) = \frac{1-D_i}{1-\widehat{p}_n(X_i)}$ and $\widehat{W}_{n,1}(X_i, D_i) = \frac{D_i}{\widehat{p}_n(X_i)}$, where $\widehat{p}_n(x)$ is the SLE of the propensity score function $p(x)$ using (X_i, D_i), $i = 1, \cdots, n$. To prove Theorem 1, we first provide the following lemmas. Note that Lemma 1 provides an equivalence between the quantile estimate based on the weighted check function and the quantile estimate by inverting the weighted empirical distribution so that it can make some theoretical proofs simpler.

Lemma 1.[1] *Suppose $Y_1, \cdots, Y_n$ are observed. Let $a_i \geq 0$ for $i = 1, \cdots, n$ and $\sum_{i=1}^n a_i = 1$. For $0 < \tau < 1$, define*

$$\widetilde{q}_\tau = \inf\left\{ y : F_n(y) := \sum_{i=1}^{n} a_i I(Y_i \leq y) \geq \tau \right\},$$

and

$$\widetilde{q}^*_\tau = \inf\left\{ q : \arg\min_{q} \sum_{i=1}^{n} a_i \rho_\tau(Y_i - q) \right\},$$

where $\rho_\tau(Y_i - q) = (Y_i - q)\big(\tau - I(Y_i - q \leq 0)\big)$. Then $\widetilde{q}_\tau = \widetilde{q}^*_\tau$.

Proof of Lemma 1: Without loss of generality, we assume that $Y_1 < Y_2 < \cdots < Y_n$. Suppose that $\sum_{i=1}^{l} a_i < \tau < \sum_{i=1}^{l+1} a_i$ for some l. In this case, it is easy to see that

[1]The authors thank one of the anonymous referees for bringing our attention to this equivalent result.

$\widetilde{q}_\tau = \inf\{y : \sum_{i=1}^n a_i I(Y_i \le y) \ge \tau\} = Y_{l+1}$. Define $h(q) = \sum_{i=1}^n a_i \rho_\tau(Y_i - q)$. We consider the first derivative of $h(q)$ except for $q \in \{Y_1, \cdots, Y_n\}$ where $h(q)$ is not differentiable. We have

$$\frac{dh(q)}{dq} = \begin{cases} \sum_{i=1}^k a_i - \tau < 0, & \text{if } Y_k < q < Y_{k+1} \text{ and } k \le l; \\ \sum_{i=1}^k a_i - \tau > 0, & \text{if } Y_k < q < Y_{k+1} \text{ and } k \ge l+1. \end{cases}$$

Hence, $h(q)$ is monotonically decreasing when $q < Y_{l+1}$ and is monotonically increasing when $q > Y_{l+1}$. Therefore, $\widetilde{q}_\tau^* = \inf\{q : \arg\min_q h(q)\} = Y_{l+1} = \widetilde{q}_\tau$.

When $\sum_{i=1}^{l_1-1} a_i < \tau = \sum_{i=1}^{l_1} a_i = \sum_{i=1}^{l_2} a_i < \sum_{i=1}^{l_2+1} a_i$ for some $l_1 \le l_2$, $\widetilde{q}_\tau = \inf\big\{y : \sum_{i=1}^n a_i I(Y_i \le y) \ge \tau\big\} = Y_{l_1}$. Since

$$\frac{dh(q)}{dq} = \begin{cases} \sum_{i=1}^k a_i - \tau < 0, & \text{if } Y_k < q < Y_{k+1} \text{ and } k < l_1; \\ \sum_{i=1}^k a_i - \tau = 0, & \text{if } Y_k < q < Y_{k+1} \text{ and } l_1 \le k \le l_2; \\ \sum_{i=1}^k a_i - \tau > 0, & \text{if } Y_k < q < Y_{k+1} \text{ and } k \ge l_2+1, \end{cases}$$

any $q \in [Y_{l_1}, Y_{l_2+1}]$ is a solution to $\arg\min_q h(q)$. Hence, $\widetilde{q}_\tau^* = \inf\{q : \arg\min_q h(q)\} = Y_{l_1} = \widetilde{q}_\tau$. This completes the proof. □

Lemma 2. *Suppose Assumptions 1-5 hold. Then,*

$$\sup_{\tau\in\mathcal{A}} \sup_{z\in\mathcal{Z}} \left| \bar{q}_{j,\tau}(z) - q_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^n \left[\varrho_{n,j,\tau}(Y_i, X_i, D_i; z) - E\varrho_{n,j,\tau}(Y_i, X_i, D_i; z)\right]\right| = O_p\left(\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right)$$

for $j = 0$ and 1, where

$$\bar{q}_{j,\tau}(z) = \inf\left\{ y : \frac{\sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i)} \ge \tau \right\}$$

and

$$\varrho_{n,j,\tau}(Y_i, X_i, D_i; z) = -S_{n,j,\tau}^{-1}(z) W_j(X_i, D_i) K_h\big(Z_i - z\big)\varphi_\tau(Y_i; q_{j,\tau}(z))$$

with $\rho_\tau(y;q) = \rho_\tau(y-q) = (y-q)(\tau - I\{y \le q\})$, $\varphi_\tau(y;\theta) = \tau - I\{y \le \theta\}$ and

$$S_{n,j,\tau}(z) = \int K(u) f_{Y(j)|Z}\big(q_{j,\tau}(z)|z+hu\big) f_Z(z+hu)du = f_{Y(j)|Z}\big(q_{j,\tau}(z)|z\big) f_Z(z) + O(h^2),\ j = 0,\ 1.$$

Proof of Lemma 2: According to Lemma 1,

$$\begin{aligned}
\bar{q}_{j,\tau}(z) &= \inf\left\{ y : \frac{\sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i)} \ge \tau \right\} \\
&= \inf\left\{ q : \arg\min_q \sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i) \rho_\tau(Y_i - q) \right\}.
\end{aligned}$$

Then, following the proof of Theorem 1 in Lee et al. (2015), it can be shown that

$$\sup_{\tau\in\mathcal{A}} \sup_{z\in\mathcal{Z}} \left| \bar{q}_{j,\tau}(z) - q_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^n \left[\varrho_{n,j,\tau}(Y_i, X_i, D_i; z) - E\varrho_{n,j,\tau}(Y_i, X_i, D_i; z) \right] \right| = O_p\left(\frac{\sqrt{\ln n}}{(nh^d)^{3/4}} \right) . \square$$

Lemma 3. *Suppose that Assumptions 1-6 are satisfied, then*

$$\sup_{\tau\in\mathcal{A}} \sup_{z\in\mathcal{Z}} \left| \widehat{q}_{j,\tau}(z) - \bar{q}_{j,\tau}(z) \right| = O_p\left\{ \max\left\{ \frac{\ln n}{n^{1/2-3\nu/2}}, \frac{\sqrt{\ln n}}{(nh^d)^{3/4}} \right\} \right\}, \quad j = 0,\ 1.$$

Proof of Lemma 3: For $j = 0$ and 1, define cumulative distribution functions

$$\bar{F}_{n,j}(y \,|\, z) = \frac{\sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^n K_{h,i}(z) W_j(X_i, D_i)}$$

and

$$\widehat{F}_{n,j}(y \,|\, z) = \frac{\sum_{i=1}^n K_{h,i}(z) \widehat{W}_{n,j}(X_i, D_i) I\{Y_i \le y\}}{\sum_{i=1}^n K_{h,i}(z) \widehat{W}_{n,j}(X_i, D_i)}.$$

Then $\bar{q}_{j,\tau^*}(z) = \inf\{y : \bar{F}_{n,j}(y \,|\, z) \ge \tau^*\}$ and $\widehat{q}_{j,\tau^*}(z) = \inf\{y : \widehat{F}_{n,j}(y \,|\, z) \ge \tau^*\}$ for $0 < \tau^* <$

1. By the definition of a quantile, we have

$$\left|\bar{F}_{n,j}\big(\bar{q}_{j,\tau^*}(z)\,|\,z\big)-\tau^*\right| \le \max_{i=1,\cdots,n}\left\{\frac{K_{h,i}(z)W_j(X_i,D_i)I\{Y_i\le y\}}{\sum_{i=1}^n K_{h,i}(z)W_j(X_i,D_i)}\right\}$$

for any $0<\tau^*<1$, hence $\bar{F}_{n,j}\big(\bar{q}_{j,\tau^*}(z)\,|\,z\big)=\tau^*+O_p(1/(nh^d))$. By using $\nu\in\big[\frac{m}{s+m},\frac{d}{6\eta}\big)$, it can be shown that

$$\sup_{y\in\mathcal{Y}_j}\sup_{z\in\mathcal{Z}}\left|\widehat{F}_{n,j}(y\,|\,z)-\bar{F}_{n,j}(y\,|\,z)\right| \;=\; O_p\Big(\sup_{x\in\mathcal{X}}\big|\widehat{p}_n(x)-p(x)\big|\Big)=O_p\big(n^{\frac{3\nu-1}{2}}\big),$$

where $\mathcal{Y}_j$ is the support of $Y_i(j)$ for $j=0,1$. Let $c_n=\max\Big\{\frac{\ln n}{n^{1/2-3\nu/2}},\frac{(\ln n)^2}{nh^d}\Big\}$. Then,

$$\begin{aligned}\widehat{F}_{n,j}\big(\bar{q}_{j,\tau+c_n}(z)\,|\,z\big) &= \bar{F}_{n,j}\big(\bar{q}_{j,\tau+c_n}(z)\,|\,z\big)+O_p\big(n^{\frac{3\nu-1}{2}}\big)\\ &= \tau+c_n+O_p(1/(nh^d))+O_p\big(n^{\frac{3\nu-1}{2}}\big)>\tau\end{aligned}\tag{S.1}$$

in probability as $n\to\infty$. Similarly,

$$\begin{aligned}\widehat{F}_{n,j}\big(\bar{q}_{j,\tau-c_n}(z)\,|\,z\big) &= \bar{F}_{n,j}\big(\bar{q}_{j,\tau-c_n}(z)\,|\,z\big)+O_p\big(n^{\frac{3\nu-1}{2}}\big)\\ &= \tau-c_n+O_p(1/(nh^d))+O_p\big(n^{\frac{3\nu-1}{2}}\big)<\tau\end{aligned}\tag{S.2}$$

in probability as $n\to\infty$. Combining (S.1) and (S.2), we have

$$P\big(\bar{q}_{j,\tau-c_n}(z)\le\widehat{q}_{j,\tau}(z)\le\bar{q}_{j,\tau+c_n}(z)\big)\to 1\quad\text{as } n\to\infty.$$

Also, one has $\bar{q}_{j,\tau-c_n}(z)\le\bar{q}_{j,\tau}(z)\le\bar{q}_{j,\tau+c_n}(z)$ by definition, and it follows that

$$P\Big(\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\big|\widehat{q}_{j,\tau}(z)-\bar{q}_{j,\tau}(z)\big|\le\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\big|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\big|\Big)\to 1\quad\text{as } n\to\infty.\tag{S.3}$$

Next, we consider the order of $\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\right|$. Recall that $S_{n,j,\tau^*}(z)=\int K(u)f_{Y(j)|Z}\big(q_{j,\tau^*}(z)|z+hu\big)f_Z(z+hu)du$ for $j=0$ and 1. It is easy to show that $\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}|S_{n,j,\tau\pm c_n}(z)-S_{n,j,\tau}(z)|=O(c_n)$. By using the Bahadur representation of

$\bar{q}_{j,\tau^*}(z)$ provided by Lemma 2, it follows that

$$
\begin{aligned}
& \sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\right| \\
\leq\ & \sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\right|+O_p\left\{\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right\} \\
& +\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|S^{-1}_{n,j,\tau}(z)\frac{1}{n}\sum_{i=1}^{n}K_{h,i}(z)W_j(X_i,D_i)\times\big(I\{Y_i\leq q_{j,\tau-c_n}(z)\}-I\{Y_i\leq q_{j,\tau+c_n}(z)\}\big)+O_p(c_n)\right| \\
& +\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|S^{-1}_{n,j,\tau}(z)E\Big[K_{h,i}(z)W_j(X_i,D_i)\big(I\{Y_i\leq q_{j,\tau-c_n}(z)\}-I\{Y_i\leq q_{j,\tau+c_n}(z)\}\big)\Big]+O(c_n)\right| \\
\leq\ & \sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\right| \\
& +\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|S^{-1}_{n,j,\tau}(z)\frac{1}{n}\sum_{i=1}^{n}K_{h,i}(z)W_j(X_i,D_i)\times\big(I\{Y_i\leq q_{j,\tau-c_n}(z)\}-I\{Y_i\leq q_{j,\tau+c_n}(z)\}\big)\right| \\
& +\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|S^{-1}_{n,j,\tau}(z)E\Big[K_{h,i}(z)W_j(X_i,D_i)\big(I\{Y_i\leq q_{j,\tau-c_n}(z)\}-I\{Y_i\leq q_{j,\tau+c_n}(z)\}\big)\Big]\right| \\
& +O(c_n)+O_p\left\{\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right\} \\
:=\ & \mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3+O_p(c_n)+O_p\left\{\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right\}.
\end{aligned}
$$

For $\mathcal{M}_1$, note that $F_{Y(j)|Z}(q_{j,\tau+c_n}(z)|z)=\tau+c_n$ and $F_{Y(j)|Z}(q_{j,\tau-c_n}(z)|z)=\tau-c_n$ under Assumption 2.3. Thus,

$$
2c_n=F_{Y(j)|Z}(q_{j,\tau+c_n}(z)|z)-F_{Y(j)|Z}(q_{j,\tau-c_n}(z)|z)=f_{Y(j)|Z}(q_n^*|z)\big(q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\big),
$$

where q_n^* is a point between $q_{j,\tau-c_n}(z)$ and $q_{j,\tau+c_n}(z)$, which implies that

$$
\mathcal{M}_1=\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\right|=O(c_n)
$$

by the assumption that $f_{Y(j)|Z}(q|z)$ is uniformly bounded away from zero in a neighborhood of $q_{j,\tau}(z)$. For $\mathcal{M}_2$, since $q_{j,\tau-c_n}(z)\leq q_{j,\tau}(z)\leq q_{j,\tau+c_n}(z)$ and $\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\big|q_{j,\tau+c_n}(z)-q_{j,\tau-c_n}(z)\big|=O(c_n)$, there exists a constant A which does not rely on z and τ, such that

$$
\mathcal{M}_2\ \leq\ \sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|S^{-1}_{n,j,\tau}(z)\frac{1}{n}\sum_{i=1}^{n}K_{h,i}(z)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\right|
$$

$$
\begin{aligned}
&\leq \sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\bigg|S_{n,j,\tau}^{-1}(z)\frac{1}{n}\sum_{i=1}^{n}\Big[K_{h,i}(z)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\\
&\quad -E\Big(K_{h,i}(z)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\Big)\Big]\bigg|\\
&\quad +\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\bigg|S_{n,j,\tau}^{-1}(z)E\Big(K_{h,i}(z)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\Big)\bigg|\\
&:= \mathcal{M}_{2,1}+\mathcal{M}_{2,2}.
\end{aligned}
$$

Using Assumption 2.3, we have that

$$
\begin{aligned}
& E\bigg[\Big(K\big((Z_i-z)/h\big)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\Big)^2\bigg]\\
= \;& E\Big\{K^2\big((Z_i-z)/h\big)p(X_i)^{-j}(1-p(X_i))^{j-1}E\Big[I\{q_{j,\tau}(z)-Ac_n\leq Y_i(j)\leq q_{j,\tau}(z)+Ac_n\}\Big|X_i\Big]\Big\}\\
= \;& O(h^d c_n).
\end{aligned}
$$

Also, note that $\big\{K\big((Z_i-z)/h\big)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}:z\in\mathcal{Z},\tau\in\mathcal{A}\big\}$ is Euclidean for a constant envelope, which together with $\frac{\ln n}{nh^d c_n}=o(1)$ implies the conditions required by Theorem II.37 of Pollard (1984) are met. Hence, by Theorem II.37 of Pollard (1984),

$$
\begin{aligned}
&\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\bigg|\frac{1}{n}\sum_{i=1}^{n}\Big[K\big((Z_i-z)/h\big)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\\
&\quad -E\big(K\big((Z_i-z)/h\big)W_j(X_i,D_i)I\{q_{j,\tau}(z)-Ac_n\leq Y_i\leq q_{j,\tau}(z)+Ac_n\}\big)\Big]\bigg| = o_p(h^d c_n),
\end{aligned}
$$

together with the fact $S_{n,j,\tau}(z)$ is bounded away from zero, we have that $\mathcal{M}_{2,1}=\frac{1}{h^d}\cdot o_p(h^d c_n)=o_p(c_n)$. It is also easy to show that $\mathcal{M}_{2,2}=O(c_n)$. Hence, $\mathcal{M}_2=O_p(c_n)$. Similar to the proof of $\mathcal{M}_{2,2}=O(c_n)$, we can also show that $\mathcal{M}_3=O(c_n)$. Therefore,

$$
\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\big|\bar{q}_{j,\tau+c_n}(z)-\bar{q}_{j,\tau-c_n}(z)\big| = O_p(c_n)+O_p\bigg\{\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\bigg\}.
$$

Together with (S.3), we have

$$
\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\big|\widehat{q}_{j,\tau}(z)-\bar{q}_{j,\tau}(z)\big| = O_p(c_n)+O_p\bigg\{\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\bigg\} = O_p\bigg\{\max\bigg\{\frac{\ln n}{n^{1/2-3\nu/2}},\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\bigg\}\bigg\}.
$$

This completes the proof. □

Lemma 4. *Under Assumptions 1-6, we have*

$$\begin{aligned}&\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\Big|\widehat{q}_{j,\tau}(z)-q_{j,\tau}(z)-\frac{1}{n}\sum_{i=1}^{n}\big[\varrho_{n,j,\tau}(Y_i,X_i,D_i;z)-E\big(\varrho_{n,j,\tau}(Y_i,X_i,D_i;z)\big)\big]\Big|\\ =\;& O_p\Bigg\{\max\Big\{\frac{\ln n}{n^{1/2-3\nu/2}},\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\Big\}\Bigg\}\end{aligned}$$

for $j=0$ *and* 1*.*

Proof of Lemma 4: The result comes from Lemma 2 and Lemma 3. □

Lemma 5. *Let* $R_1,R_2,\cdots$ *be an i.i.d. sequence. Suppose that the* U*-statistic* $U_n=\sum_{1\le i<j\le n}H_n(R_i,R_j)$ *with symmetric variable function* H_n *is centered (i.e.,* $E[H_n(R_1,R_2)]=0$*) and degenerated (i.e.,* $E[H_n(R_1,R_2)|R_1=z_1]=0$ *almost surely for all* z_1*). Then, if*

$$\lim_{n\to\infty}\frac{E\Big[E^2\big[H_n(R_1,R_3)H_n(R_2,R_3)\,\big|\,R_1,R_2\big]\Big]+n^{-1}E\big[H_n^4(R_1,R_2)\big]}{E^2\big[H_n^2(R_1,R_2)\big]}=0,$$

we have that as $n\to\infty$*,*

$$\frac{2^{1/2}}{n\sigma_n}U_n\xrightarrow{D}\mathcal{N}(0,1),$$

where $\sigma_n^2=E\big[H_n^2(R_1,R_2)\big]$.

Proof of Lemma 5: The result is given by Theorem 1 in Hall (1984). □

Lemma 6. *Suppose the conditions required by Theorem 1 are satisfied. Then,*

$$\begin{aligned}nh^{d/2}\Bigg\{\int\Big[\frac{1}{n}\sum_{i=1}^{n}\big(&\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)-E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)\\ &-\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)+E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big)\Big]^2\omega(z)dz-\mu_J\Bigg\}\xrightarrow{D}\mathcal{N}(0,\sigma_J^2),\end{aligned}$$

where

$$\mu_J\;=\;\frac{1}{nh^d}\int K^2(s)ds\int\Big\{\frac{\mu_{1,\tau}(z;z)}{f^2_{Y(1)|Z}(q_{1,\tau}(z)|z)}+\frac{\mu_{0,\tau}(z;z)}{f^2_{Y(0)|Z}(q_{0,\tau}(z)|z)}\Big\}\frac{\omega(z)}{f_Z(z)}dz,$$

and

$$\sigma_J^2 = 2\int\Big(\int K(t)K(t+s)dt\Big)^2 ds\int\Big\{\frac{\mu_{1,\tau}(u;u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)}+\frac{\mu_{0,\tau}(u;u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)}\Big\}^2\frac{\omega^2(u)}{f_Z^2(u)}du,$$

with

$$\mu_{0,\tau}(z;u) = E\left[\frac{1}{1-p(X_i)}\big(I\{Y_i(0)\le q_{0,\tau}(u)\}-\tau\big)^2\big|Z_i=z\right],$$

and

$$\mu_{1,\tau}(z;u) = E\left[\frac{1}{p(X_i)}\big(I\{Y_i(1)\le q_{1,\tau}(u)\}-\tau\big)^2\big|Z_i=z\right].$$

Proof of Lemma 6: For simplicity, we let

$$\gamma_{n,\tau}(Y_i,X_i,D_i;z) = \varrho_{n,1,\tau}(Y_i,X_i,D_i;z) - E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)$$
$$-\varrho_{n,0,\tau}(Y_i,X_i,D_i;z) + E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z).$$

Then,

$$\int\Big(\frac{1}{n}\sum_{i=1}^n\gamma_{n,\tau}(Y_i,X_i,D_i;z)\Big)^2\omega(z)dz \tag{S.4}$$
$$= 2n^{-2}\sum_{1\le i<k\le n}\int\gamma_{n,\tau}(Y_i,X_i,D_i;z)\gamma_{n,\tau}(Y_k,X_k,D_k;z)\omega(z)dz + n^{-2}\sum_{i=1}^n\int\gamma_n^2(Y_i,X_i,D_i;z)\omega(z)dz$$
$$:= I_{n,1}+I_{n,2}.$$

First, we consider the term $I_{n,1}$. Let $R_i=(Y_i,X_i,D_i)$ and define

$$H_n(R_i,R_k) = \frac{2}{n^2}\int\gamma_{n,\tau}(R_i;z)\gamma_{n,\tau}(R_k;z)\omega(z)dz.$$

Then, $I_{n,1} = \sum_{1\le i<k\le n} H_n(R_i,R_k)$ is a centered U-statistic. Note that

$$E\big[H_n(R_i,R_k)^2\big] \tag{S.5}$$
$$= E\left[\frac{4}{n^4}\int\int\gamma_{n,\tau}(R_i;u)\gamma_{n,\tau}(R_k;u)\,\gamma_{n,\tau}(R_i;v)\gamma_{n,\tau}(R_k;v)\omega(u)\omega(v)dudv\right]$$
$$= \frac{4}{n^4}\int\int E\big[\gamma_{n,\tau}(R_i;u)\gamma_{n,\tau}(R_i;v)\gamma_{n,\tau}(R_k;u)\gamma_{n,\tau}(R_k;v)\big]\omega(u)\omega(v)dudv$$

$$= \frac{4}{n^4}\int\int E^2\big[\gamma_{n,\tau}(R_i;u)\gamma_{n,\tau}(R_i;v)\big]\omega(u)\omega(v)dudv.$$

It is easy to find that

$$E\big[S_{n,j,\tau}(z)^{-1}W_j(X_i,D_i)\varphi_\tau(Y_i;q_{j,\tau}(z))|Z_i=z\big]=0.$$

Hence,

$$E\big[\varrho_{n,j,\tau}(Y_i,X_i,D_i;z)\big]=E\big[K_h\big(Z_i-z\big)S_{n,j,\tau}(z)^{-1}W_j(X_i,D_i)\varphi_\tau(Y_i;q_{j,\tau}(z))\big]=O(h^2) \tag{S.6}$$

uniformly in z for $j=0$ and 1. Also note that $D_i(1-D_i)=0$, then,

$$\begin{aligned}
& E\Big[\gamma_{n,\tau}(R_i;u)\gamma_{n,\tau}(R_i;v)\Big] \qquad \text{(S.7)}\\
= \; & E\Big[\big(\varrho_{n,1,\tau}(R_i;u)-\varrho_{n,0,\tau}(R_i;u)\big)\big(\varrho_{n,1,\tau}(R_i;v)-\varrho_{n,0,\tau}(R_i;v)\big)\Big]\\
& \quad -E\Big[\varrho_{n,1,\tau}(R_i;u)-\varrho_{n,0,\tau}(R_i;u)\Big]E\Big[\varrho_{n,1,\tau}(R_i;v)-\varrho_{n,0,\tau}(R_i;v)\Big]\\
= \; & S^{-1}_{n,1,\tau}(u)S^{-1}_{n,1,\tau}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{D_i}{p^2(X_i)}\varphi_\tau(Y_i;q_{1,\tau}(u))\varphi_\tau(Y_i;q_{1,\tau}(v))\Big]\\
& +S^{-1}_{n,0,\tau}(u)S^{-1}_{n,0,\tau}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{1-D_i}{(1-p(X_i))^2}\varphi_\tau(Y_i;q_{0,\tau}(u))\varphi_\tau(Y_i;q_{0,\tau}(v))\Big]+O(h^4)\\
= \; & S^{-1}_{n,1,\tau}(u)S^{-1}_{n,1,\tau}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{1}{p(X_i)}\varphi_\tau(Y_i(1);q_{1,\tau}(u))\varphi_\tau(Y_i(1);q_{1,\tau}(v))\Big]\\
& +S^{-1}_{n,0,\tau}(u)S^{-1}_{n,0,\tau}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\frac{1}{1-p(X_i)}\varphi_\tau(Y_i(0);q_{0,\tau}(u))\varphi_\tau(Y_i(0);q_{0,\tau}(v))\Big]+O(h^4)\\
= \; & S^{-1}_{n,1,\tau}(u)S^{-1}_{n,1,\tau}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\kappa_1(Z_i;u,v)\Big]\\
& +S^{-1}_{n,0,\tau}(u)S^{-1}_{n,0,\tau}(v)E\Big[K_h\big(Z_i-u\big)K_h\big(Z_i-v\big)\kappa_0(Z_i;u,v)\Big]+O(h^4)\\
= \; & \frac{1}{h^d}S^{-1}_{n,1,\tau}(u)S^{-1}_{n,1,\tau}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_1(u+ht;u,v)f_Z(u+ht)dt\\
& +\frac{1}{h^d}S^{-1}_{n,0,\tau}(u)S^{-1}_{n,0,\tau}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_0(u+ht;u,v)f_Z(u+ht)dt+O(h^4),
\end{aligned}$$

where

$$\kappa_1(z;u,v)=E\Big[\frac{1}{p(X_i)}\varphi_\tau(Y_i(1);q_{1,\tau}(u))\varphi_\tau(Y_i(1);q_{1,\tau}(v))\big|Z_i=z\Big],$$

and

$$\kappa_0(z;u,v) = E\Big[\frac{1}{1-p(X_i)}\varphi_\tau(Y_i(0);q_{0,\tau}(u))\varphi_\tau(Y_i(0);q_{0,\tau}(v))\big|Z_i=z\Big],$$

with $\varphi_\tau(y;q) = \tau - I\{y \le q\}$. Thus,

$$\begin{aligned}
& E^2\Big[\gamma_{n,\tau}(R_i;u)\gamma_{n,\tau}(R_i;v)\Big] \qquad\qquad (S.8)\\
= \ & \frac{1}{h^{2d}}S_{n,1,\tau}^{-2}(u)S_{n,1,\tau}^{-2}(v)\Big(\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_1(u+ht;u,v)f_Z(u+ht)dt\Big)^2\\
& +\frac{1}{h^{2d}}S_{n,0,\tau}^{-2}(u)S_{n,0,\tau}^{-2}(v)\Big(\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_0(u+ht;u,v)f_Z(u+ht)dt\Big)^2\\
& +\frac{2}{h^{2d}}S_{n,1,\tau}^{-1}(u)S_{n,1,\tau}^{-1}(v)S_{n,0,\tau}^{-1}(u)S_{n,0,\tau}^{-1}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_1(u+ht;u,v)f_Z(u+ht)dt\\
& \times\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_0(u+ht;u,v)f_Z(u+ht)dt\\
& +2O(h^{4-d})S_{n,1,\tau}^{-1}(u)S_{n,1,\tau}^{-1}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_1(u+ht;u,v)f_Z(u+ht)dt\\
& +2O(h^{4-d})S_{n,0,\tau}^{-1}(u)S_{n,0,\tau}^{-1}(v)\int K(t)K\Big(t+\frac{u-v}{h}\Big)\kappa_0(u+ht;u,v)f_Z(u+ht)dt+O(h^8).
\end{aligned}$$

Combining (S.5) and (S.8), and some straightforward calculations imply that

$$\begin{aligned}
E\big[H_n(R_i,R_k)^2\big] &= \frac{4}{n^4}\int\int E^2\big[\gamma_{n,\tau}(R_i;u)\gamma_{n,\tau}(R_i;v)\big]\omega(u)\omega(v)dudv \qquad (S.9)\\
&= \frac{4}{n^4h^d}\Big\{\int\Big(\int K(t)K(t+s)dt\Big)^2 ds\cdot\Big[\int S_{n,1,\tau}^{-4}(u)\kappa_1^2(u;u,u)f_Z^2(u)\omega^2(u)du\\
&\quad +\int S_{n,0,\tau}^{-4}(u)\kappa_0^2(u;u,u)f_Z^2(u)\omega^2(u)du\\
&\quad +2\int S_{n,1,\tau}^{-2}(u)S_{n,0,\tau}^{-2}(u)\kappa_1(u;u,u)\kappa_0(u;u,u)f_Z^2(u)\omega^2(u)du\Big]+o(1)\Big\}.
\end{aligned}$$

This, coupled with $S_{n,j,\tau}(z) = f_Z(z)f_{Y(j)|Z}(q_{j,\tau}(z)|z)+O(h^2)$ for $j=0$ and 1, yields

$$\begin{aligned}
E\big[H_n(R_1,R_2)^2\big] &= \frac{4}{n^4h^d}\Big(\int\Big(\int K(t)K(t+s)dt\Big)^2 ds\\
&\quad\times\int\Big\{\frac{\kappa_1(u;u,u)}{f_{Y(1)|Z}^2(q_{1,\tau}(u)|u)}+\frac{\kappa_0(u;u,u)}{f_{Y(0)|Z}^2(q_{0,\tau}(u)|u)}\Big\}^2\frac{\omega^2(u)}{f_Z^2(u)}du+o(1)\Big)\\
&= \frac{2}{n^4h^d}\big(\sigma_J^2+o(1)\big). \qquad (S.10)
\end{aligned}$$

Similarly, by straightforward calculations, we can verify that the condition

$$\lim_{n\to\infty} \frac{E\Big[E^2\big[H_n(R_1,R_3)H_n(R_2,R_3)\,\big|\,R_1,R_2\big]\Big] + n^{-1}E\big[H_n(R_1,R_2)^4\big]}{\Big(E\big[H_n(R_1,R_2)^2\big]\Big)^2} = 0$$

in Lemma 5 is satisfied. So that

$$\frac{\sqrt{2}}{nE^{1/2}\big[H_n(R_1,R_2)^2\big]}\, I_{n,1} \xrightarrow{D} \mathcal{N}(0,1),$$

or equivalently,

$$nh^{d/2}\, I_{n,1} \xrightarrow{D} \mathcal{N}(0,\sigma_J^2). \tag{S.11}$$

Now, we move to the term $I_{n,2} = n^{-2}\sum_{i=1}^{n}\int \gamma_n^2(Y_i,X_i,D_i;z)\omega(z)dz$. Note that

$$\begin{aligned}
& E\big[\gamma_n^2(Y_i,X_i,D_i;z)\big] \\
= \;& E\big[\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)\big]^2 + E\big[\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big]^2 + O(h^4) \\
= \;& S_{n,1,\tau}^{-2}(z)E\Big[\frac{1}{h^{2d}}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{D_i}{p^2(X_i)}(I\{Y_i\le q_{1,\tau}(z)\}-\tau)^2\Big] \\
& \quad + S_{n,0,\tau}^{-2}(z)E\Big[\frac{1}{h^{2d}}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{1-D_i}{(1-p(X_i))^2}(I\{Y_i\le q_{0,\tau}(z)\}-\tau)^2\Big] + O(h^4) \\
= \;& S_{n,1,\tau}^{-2}(z)E\Big[\frac{1}{h^{2d}}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{1}{p(X_i)}(I\{Y_i(1)\le q_{1,\tau}(z)\}-\tau)^2\Big] \\
& \quad + S_{n,0,\tau}^{-2}(z)E\Big[\frac{1}{h^{2d}}K^2\Big(\frac{Z_i-z}{h}\Big)\frac{1}{1-p(X_i)}(I\{Y_i(0)\le q_{0,\tau}(z)\}-\tau)^2\Big] + O(h^4) \\
= \;& S_{n,1,\tau}^{-2}(z)E\Big[\frac{1}{h^{2d}}K^2\Big(\frac{Z_i-z}{h}\Big)\mu_1(Z_i;z)\Big] + S_{n,0,\tau}^{-2}(z)E\Big[\frac{1}{h^{2d}}K^2\Big(\frac{Z_i-z}{h}\Big)\mu_0(Z_i;z)\Big] + O(h^4) \\
= \;& S_{n,1,\tau}^{-2}(z)\Big[\frac{1}{h^d}\Big(\mu_{1,\tau}(z;z)f_Z(z)\int K^2(s)ds + O(h^2)\Big)\Big] \\
& \quad + S_{n,0,\tau}^{-2}(z)\Big[\frac{1}{h^d}\Big(\mu_{0,\tau}(z;z)f_Z(z)\int K^2(s)ds + O(h^2)\Big)\Big] + O(h^4) \\
= \;& \frac{1}{h^d}\Big\{\int K^2(s)ds\cdot\Big(S_{n,1,\tau}^{-2}(z)\mu_{1,\tau}(z;z) + S_{n,0,\tau}^{-2}(z)\mu_{0,\tau}(z;z)\Big)f_Z(z) + O(h^2)\Big\} + O(h^4),
\end{aligned}$$

coupled with $S_{n,j,\tau}(z) = f_Z(z)f_{Y(j)|Z}(q_{j,\tau}(z)|z) + O(h^2)$ for $j=0$ and 1, we have

$$\begin{aligned}
& E\big(I_{n,2}\big) \qquad\qquad \text{(S.12)} \\
= \;& \frac{1}{n}\int E\big[\gamma_n^2(Y_i,X_i,D_i;z)\big]\omega(z)dz
\end{aligned}$$

$$= \frac{1}{nh^d}\Big\{\int K^2(s)ds\cdot\int\Big(S^{-2}_{n,1,\tau}(z)\mu_{1,\tau}(z;z)+S^{-2}_{n,0,\tau}(z)\mu_{0,\tau}(z;z)\Big)f_Z(z)\omega(z)dz+O(h^2)\Big\}+O\Big(\frac{h^4}{n}\Big)$$
$$= \frac{1}{nh^d}\int K^2(s)ds\cdot\int\Big\{\frac{\mu_{1,\tau}(z;z)}{f^2_{Y(1)|Z}(q_{1,\tau}(z)|z)}+\frac{\mu_{0,\tau}(z;z)}{f^2_{Y(0)|Z}(q_{0,\tau}(z)|z)}\Big\}\frac{\omega(z)}{f_Z(z)}dz+\frac{O(h^2)}{nh^d}+O\Big(\frac{h^4}{n}\Big)$$
$$= \mu_J+O\Big(\frac{h^{2-d}}{n}\Big)+O\Big(\frac{h^4}{n}\Big).$$

Furthermore,

$$\begin{aligned}\text{Var}\big(nh^{d/2}I_{n,2}\big) &= E\Big\{nh^{d/2}\big[I_{n,2}-E(I_{n,2})\big]\Big\}^2 \qquad \text{(S.13)}\\ &= n^{-1}h^d\Big\{E\Big[\Big(\int\gamma_n^2(Y_i,X_i,D_i;z)\omega(z)dz\Big)^2\Big]-E^2\Big[\int\gamma_n^2(Y_i,X_i,D_i;z)\omega(z)dz\Big]\Big\}\\ &= n^{-1}h^d\Big\{E\Big[\Big(\int\gamma_n^2(Y_i,X_i,D_i;z)\omega(z)dz\Big)^2\Big]-\Big[\int E\big[\gamma_n^2(Y_i,X_i,D_i;z)\big]\omega(z)dz\Big]^2\Big\}\\ &= n^{-1}h^d\Big\{\int\int E\Big[\gamma_n^2(Y_i,X_i,D_i;u)\gamma_n^2(Y_i,X_i,D_i;v)\Big]\omega(u)\omega(v)dudv-O(h^{-2d})\Big\}\\ &= n^{-1}h^d\cdot O(h^{-2d})\to 0,\end{aligned}$$

together with (S.12), we have

$$I_{n,2}=\mu_J+O\Big(\frac{h^{2-d}}{n}\Big)+O\Big(\frac{h^4}{n}\Big)+o_p\Big(\frac{1}{nh^{d/2}}\Big).$$

Hence, when $1\le d\le 3$, we have

$$nh^{d/2}\big(I_{n,2}-\mu_J\big) = o_p(1). \qquad \text{(S.14)}$$

It follows by combining (S.4), (S.11) and (S.14) that

$$\begin{aligned}nh^{d/2}\Big\{\int\Big[\frac{1}{n}\sum_{i=1}^n\big(&\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)-E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z)\\ &-\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)+E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big)\Big]^2\omega(z)dz-\mu_J\Big\}\xrightarrow{D}\mathcal{N}(0,\sigma_J^2).\end{aligned}$$

The proof is completed. □

Now, we introduce some notation before considering the proofs of the following lemmas and Theorem 3. First, let P denote the distribution of $\{(Y_i(0),Y_i(1),X_i,D_i)\}_{i=1}^n$ and use

P^* to denote the Bootstrap distribution, which is the distribution of $\{(Y_i^*, X_i^*, D_i^*)\}_{i=1}^n$, conditional on $\{(Y_i, X_i, D_i)\}_{i=1}^n$. Also, we use E^* and Var* to denote the expectation and variance with respect to P^*, respectively. Finally, following Lee et al. (2015), let $S_1, S_2, \cdots$ be a sequence a random variables and $a_1, a_2, \cdots$ be a sequence of positive real numbers, define $S_n = o_{p^*}(a_n)$ if for any $\varepsilon > 0$ and $\epsilon > 0$, $\lim_{n\to\infty} P\big\{P^*(|S_n/a_n| > \epsilon) > \varepsilon\big\} = 0$. Similarly, $S_n = O_{p^*}(a_n)$ means that if for any $\varepsilon > 0$ and $\epsilon > 0$, there exists $M > 0$ such that $\limsup_{n\to\infty} P\big\{P^*(|S_n/a_n| > M) > \varepsilon\big\} < \epsilon$.

Lemma 7. *Suppose Assumptions 1-5 are satisfied. Then,*

$$\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\bar{q}^*_{j,\tau}(z) - \bar{q}_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^n \left[\varrho_{n,j,\tau}(R_i^*; z) - E^*\big(\varrho_{n,j,\tau}(R_i^*; z)\big)\right]\right| = O_{p^*}\left(\frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right)$$

for $j = 0$ and 1, where $R_i^* = (Y_i^*, X_i^*, D_i^*)'$,

$$\bar{q}^*_{j,\tau}(z) = \inf\left\{y : \frac{\sum\limits_{i=1}^n W_j(X_i^*, D_i^*) K\Big(\frac{Z_i^* - z}{h}\Big) I\{Y_i^* \le y\}}{\sum\limits_{i=1}^n W_j(X_i^*, D_i^*) K\Big(\frac{Z_i^* - z}{h}\Big)} \ge \tau\right\},$$

and

$$\varrho_{n,j,\tau}(R_i^*; z) = -S^{-1}_{n,j,\tau}(z) W_j(X_i^*, D_i^*) K_h\big(Z_i^* - z\big)\varphi_\tau(Y_i^*; q_{j,\tau}(z)).$$

Proof of Lemma 7: According to Lemma 1,

$$\begin{aligned}\bar{q}^*_{j,\tau}(z) &= \inf\left\{y : \frac{\sum\limits_{i=1}^n W_j(X_i^*, D_i^*) K\Big(\frac{Z_i^* - z}{h}\Big) I\{Y_i^* \le y\}}{\sum\limits_{i=1}^n W_j(X_i^*, D_i^*) K\Big(\frac{Z_i^* - z}{h}\Big)} \ge \tau\right\} \\ &= \inf\left\{q : \arg\min_q \sum_{i=1}^n W_j(X_i^*, D_i^*) K\Big(\frac{Z_i^* - z}{h}\Big)\rho_\tau(Y_i^* - q)\right\}.\end{aligned}$$

The result can be proved following the proof of Theorem 2 in Lee et al. (2015). □

Lemma 8. *Under Assumptions 1-6, then, we have*

$$\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\widehat{q}^*_{j,\tau}(z) - \bar{q}^*_{j,\tau}(z)\right| = O_{p^*}\left\{\max\left\{\frac{\ln n}{n^{1/2-3\nu/2}}, \frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right\}\right\}$$

for $j = 0$ *and* 1.

Proof of Lemma 8: This result can be proved by the similar arguments to that of Lemma 3 and the details are thus omitted. □

Lemma 9. *Suppose Assumptions 1-6 hold. Then, for* $j = 0$ *and* 1,

$$\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\widehat{q}^*_{j,\tau}(z) - \widehat{q}_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R_i^*;z) - E^*(\varrho_{n,j,\tau}(R_i^*;z))\right]\right| = O_{p^*}\left\{\max\left\{\frac{\ln n}{n^{1/2-3\nu/2}}, \frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right\}\right\}.$$

Proof of Lemma 9: It is easy to observe that

$$\begin{aligned}
&\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\widehat{q}^*_{j,\tau}(z) - \widehat{q}_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R_i^*;z) - E^*(\varrho_{n,j,\tau}(R_i^*;z))\right]\right| \\
\leq\ &\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\bar{q}^*_{j,\tau}(z) - \bar{q}_{j,\tau}(z) - \frac{1}{n}\sum_{i=1}^{n}\left[\varrho_{n,j,\tau}(R_i^*;z) - E^*(\varrho_{n,j,\tau}(R_i^*;z))\right]\right| \\
&+\sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\widehat{q}^*_{j,\tau}(z) - \bar{q}^*_{j,\tau}(z)\right| + \sup_{\tau\in\mathcal{A}}\sup_{z\in\mathcal{Z}}\left|\bar{q}_{j,\tau}(z) - \widehat{q}_{j,\tau}(z)\right| \\
=\ &O_{p^*}\left\{\max\left\{\frac{\ln n}{n^{1/2-3\nu/2}}, \frac{\sqrt{\ln n}}{(nh^d)^{3/4}}\right\}\right\}
\end{aligned}$$

by Lemmas 3, 7 and 8. Therefore, the proof of Lemma 9 is completed. □

Lemma 10. *Suppose the conditions required by Theorem 3 are satisfied. Then,*

$$\sup_{y\in\mathbb{R}}\left|P^*\left\{\frac{nh^{d/2}}{\sigma_J}\left[\int\left(\frac{1}{n}\sum_{i=1}^{n}\Big(\psi_{n,1,\tau}(R_i^*;z) - \psi_{n,0,\tau}(R_i^*;z)\Big)\right)^2\omega(z)dz - \mu_J\right] \leq y\right\} - \Phi(y)\right| = o_p(1),$$

where $\psi_{n,j,\tau}(R_i^*;z) = \varrho_{n,j,\tau}(Y_i^*, X_i^*, D_i^*;z) - E^*\big(\varrho_{n,j,\tau}(Y_i^*, X_i^*, D_i^*;z)\big)$ *for* $j = 0$ *and* 1. *That is,*

$$\frac{nh^{d/2}}{\sigma_J}\left[\int\left(\frac{1}{n}\sum_{i=1}^{n}\Big(\psi_{n,1,\tau}(R_i^*;z) - \psi_{n,0,\tau}(R_i^*;z)\Big)\right)^2\omega(z)dz - \mu_J\right]$$

converges to $\mathcal{N}(0,1)$ *in distribution in probability.*

Proof of Lemma 10: It is noticed that

$$\begin{aligned}
& \int \left(\frac{1}{n} \sum_{i=1}^{n} \Big(\psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \Big) \right)^2 \omega(z) dz \\
= \quad & \frac{2}{n^2} \sum_{1 \le i < j \le n} \int \Big(\psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \Big) \Big(\psi_{n,1,\tau}(R_j^*; z) - \psi_{n,0,\tau}(R_j^*; z) \Big) \omega(z) dz \\
& + \frac{1}{n^2} \sum_{i=1}^{n} \int \Big(\psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \Big)^2 \omega(z) dz \\
:= \quad & Q_{n,1}^* + Q_{n,2}^*. \qquad \text{(S.15)}
\end{aligned}$$

We first consider the term $Q_{n,1}^*$. Define

$$T_n^*(R_i^*, R_j^*) \;=\; \frac{2}{n^2} \int \Big(\psi_{n,1,\tau}(R_i^*; z) - \psi_{n,0,\tau}(R_i^*; z) \Big) \Big(\psi_{n,1,\tau}(R_j^*; z) - \psi_{n,0,\tau}(R_j^*; z) \Big) \omega(z) dz.$$

Then, $Q_{n,1}^*$ can be written as a second-order U-statistic as follows:

$$Q_{n,1}^* = \sum_{1 \le i < j \le n} T_n^*(R_i^*, R_j^*).$$

By its definition, it is easy to find that $E^*[T_n^*(R_i^*, R_j^*)] = 0$ and $E^*[T_n^*(R_i^*, R_j^*) | R_i^*] = 0$. Thus, conditional on $\{Y_i, X_i, D_i\}_{i=1}^n$, $Q_{n,1}^*$ is a second-order degenerate U-statistic. To apply Lemma 5, we need to verify the condition

$$\frac{E^*\Big[E^{*2}\big[T_n^*(R_1^*, R_3^*) T_n^*(R_2^*, R_3^*) \,\big|\, R_1^*, R_2^*\big]\Big] + n^{-1} E^*\big[T_n^{*4}(R_1^*, R_2^*)\big]}{E^{*2}\big[T_n^{*2}(R_1^*, R_2^*)\big]} = o_p(1).$$

Define

$$\begin{aligned}
\sigma_n^{*2} \quad := \quad & E^*\big(T_n^{*2}(R_1^*, R_2^*)\big) \qquad \text{(S.16)} \\
= \quad & \frac{4}{n^4} \int \int E^* \Big[\Big(\psi_{n,1,\tau}(R_1^*; u) - \psi_{n,0,\tau}(R_1^*; u) \Big) \Big(\psi_{n,1,\tau}(R_2^*; u) - \psi_{n,0,\tau}(R_2^*; u) \Big) \\
& \times \Big(\psi_{n,1,\tau}(R_1^*; v) - \psi_{n,0,\tau}(R_1^*; v) \Big) \Big(\psi_{n,1,\tau}(R_2^*; v) - \psi_{n,0,\tau}(R_2^*; v) \Big) \Big] \omega(u) \omega(v) du dv \\
= \quad & \frac{4}{n^4} \int \int E^{*2} \Big[\Big(\psi_{n,1,\tau}(R_1^*; u) - \psi_{n,0,\tau}(R_1^*; u) \Big) \Big(\psi_{n,1,\tau}(R_1^*; v) - \psi_{n,0,\tau}(R_1^*; v) \Big) \Big] \omega(u) \omega(v) du dv \\
= \quad & \frac{4}{n^4} \int \int \Big[E^*\big(\psi_{n,1,\tau}(R_1^*; u) \psi_{n,1,\tau}(R_1^*; v) \big) + E^*\big(\psi_{n,0,\tau}(R_1^*; u) \psi_{n,0,\tau}(R_1^*; v) \big)
\end{aligned}$$

$$-E^*\big(\psi_{n,1,\tau}(R_1^*;u)\psi_{n,0,\tau}(R_1^*;v)\big) - E^*\big(\psi_{n,1,\tau}(R_1^*;v)\psi_{n,0,\tau}(R_1^*;u)\big)\Big]^2\omega(u)\omega(v)dudv.$$

From (S.6) and some calculations, we have

$$\begin{aligned} E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\big) &= \frac{1}{n}\sum_{i=1}^n \varrho_{n,j,\tau}(R_i;u) \qquad \text{(S.17)} \\ &= O_p\Bigg(E\big(\varrho_{n,j,\tau}(R_i;u)\big) + \frac{1}{\sqrt{n}}\mathrm{Var}^{1/2}\big(\varrho_{n,j,\tau}(R_i;u)\big)\Bigg) = O_p\Big(h^2 + \frac{1}{\sqrt{nh^d}}\Big) \end{aligned}$$

uniformly in u. It follows that for $j = 0, 1$,

$$\begin{aligned} E^*\big(\psi_{n,j,\tau}(R_1^*;u)\psi_{n,j,\tau}(R_1^*;v)\big) &= E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\varrho_{n,j,\tau}(R_1^*;v)\big) - E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\big)\cdot E^*\big(\varrho_{n,j,\tau}(R_1^*;v)\big) \\ &= E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\varrho_{n,j,\tau}(R_1^*;v)\big) + O_p\Big(\frac{1}{nh^d} + h^4\Big), \end{aligned}$$

and

$$E^*\big(\psi_{n,1,\tau}(R_1^*;u)\psi_{n,0,\tau}(R_1^*;v)\big) = -E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\big)\cdot E^*\big(\varrho_{n,0,\tau}(R_1^*;v)\big) = O_p\Big(\frac{1}{nh^d} + h^4\Big).$$

Hence, according to (S.16), we have

$$\begin{aligned} \sigma_n^{*2} &= \frac{4}{n^4}\int\int\Big[E^*\big(\psi_{n,1,\tau}(R_1^*;u)\psi_{n,1,\tau}(R_1^*;v)\big) + E^*\big(\psi_{n,0,\tau}(R_1^*;u)\psi_{n,0,\tau}(R_1^*;v)\big) \\ &\quad -E^*\big(\psi_{n,1,\tau}(R_1^*;u)\psi_{n,0,\tau}(R_1^*;v)\big) - E^*\big(\psi_{n,1,\tau}(R_1^*;v)\psi_{n,0,\tau}(R_1^*;u)\big)\Big]^2\omega(u)\omega(v)dudv \\ &= \frac{4}{n^4}\int\int\Bigg(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big) \\ &\quad +E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big) + O_p\Big(\frac{1}{nh^d} + h^4\Big)\Bigg)^2\omega(u)\omega(v)dudv \\ &= \frac{4}{n^4}\int\int\Bigg(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big) + E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)\Bigg)^2\omega(u)\omega(v)dudv \\ &\quad +O_p\Big(\frac{1}{n^5h^d} + \frac{h^4}{n^4}\Big)\int\int\Bigg(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big) \\ &\quad +E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)\Bigg)\omega(u)\omega(v)dudv + O_p\Big(\frac{1}{n^6h^{2d}} + \frac{h^8}{n^4}\Big) \\ &:= A_1 + A_2 + o_p\Big(\frac{1}{n^4h^d}\Big). \end{aligned}$$

We focus on the term A_1. Note that

$$\begin{aligned}
& \mathrm{Var}\big[E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\varrho_{n,j,\tau}(R_1^*;v)\big)\big] \\
= \;& \mathrm{Var}\Bigg[\frac{1}{n}\sum_{i=1}^{n}\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\Bigg] \\
\leq \;& \frac{1}{n}E\Big(\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\Big)^2 \\
= \;& \frac{1}{n}E\Big(S_{n,j,\tau}^{-1}(u)S_{n,j,\tau}^{-1}(v)W_j(X_i,D_i)K_h(Z_i-u)K_h(Z_i-v)\varphi_\tau(Y_i;q_{j,\tau}(u))\varphi_\tau(Y_i;q_{j,\tau}(v))\Big)^2 \\
= \;& \frac{1}{n}\cdot\frac{1}{h^{4d}}S_{n,j,\tau}^{-2}(u)S_{n,j,\tau}^{-2}(v)E\Big\{W_j^2(X_i,D_i)K^2\Big(\frac{Z_i-u}{h}\Big)K^2\Big(\frac{Z_i-v}{h}\Big)\varphi_\tau^2(Y_i;q_{j,\tau}(u))\varphi_\tau^2(Y_i;q_{j,\tau}(v))\Big\} \\
= \;& \frac{1}{n}\cdot\frac{1}{h^{4d}}S_{n,j,\tau}^{-2}(u)S_{n,j,\tau}^{-2}(v)E\Big\{K^2\Big(\frac{Z_i-u}{h}\Big)K^2\Big(\frac{Z_i-v}{h}\Big) \\
& \times E\Big(p^{-3j}(X_i)(1-p(X_i))^{-3(1-j)}\varphi_\tau^2(Y_i(j);q_{j,\tau}(u))\varphi_\tau^2(Y_i(j);q_{j,\tau}(v))\Big|Z_i\Big)\Big\} \\
= \;& \frac{1}{n}\cdot\frac{1}{h^{3d}}S_{n,j,\tau}^{-2}(u)S_{n,j,\tau}^{-2}(v)\int K^2(s)K^2\Big(s+\frac{u-v}{h}\Big)\ell_j(u+hs;u,v)f_Z(u+hs)ds = O\Big(\frac{1}{nh^{3d}}\Big),
\end{aligned}$$

where

$$\ell_j(z;u,v) = E\Big(p^{-3j}(X_i)(1-p(X_i))^{-3(1-j)}\varphi_\tau^2(Y_i(j);q_{j,\tau}(u))\varphi_\tau^2(Y_i(j);q_{j,\tau}(v))\Big|Z_i=z\Big).$$

Therefore,

$$\begin{aligned}
E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\varrho_{n,j,\tau}(R_1^*;v)\big) &= \frac{1}{n}\sum_{i=1}^{n}\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v) \\
&= E\big(\varrho_{n,j,\tau}(R_1;u)\varrho_{n,j,\tau}(R_1;v)\big) + O_p\Big(\frac{1}{\sqrt{nh^{3d}}}\Big). \quad \text{(S.18)}
\end{aligned}$$

Then,

$$\begin{aligned}
A_1 &= \frac{4}{n^4}\int\int\Big(E^*\big(\varrho_{n,1,\tau}(R_1^*;u)\varrho_{n,1,\tau}(R_1^*;v)\big)+E^*\big(\varrho_{n,0,\tau}(R_1^*;u)\varrho_{n,0,\tau}(R_1^*;v)\big)\Big)^2\omega(u)\omega(v)dudv \\
&= \frac{4}{n^4}\int\int\Big(E\big(\varrho_{n,1,\tau}(R_1;u)\varrho_{n,1,\tau}(R_1;v)\big)+E\big(\varrho_{n,0,\tau}(R_1;u)\varrho_{n,0,\tau}(R_1;v)\big) \\
&\quad +O_p\Big(\frac{1}{\sqrt{nh^{3d}}}\Big)\Big)^2\omega(u)\omega(v)dudv.
\end{aligned}$$

Using similar arguments as in (S.7), (S.8), (S.9) and (S.10), we have

$$\begin{aligned} A_1 &= \frac{4}{n^4}\Bigg[\frac{1}{h^d}\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \\ &\quad \times\int\Bigg\{\frac{\kappa_1(u;u,u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)}+\frac{\kappa_0(u;u,u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)}\Bigg\}^2\frac{\omega^2(u)}{f^2_Z(u)}du+O_p\Big(\frac{1}{\sqrt{nh^{3d}}}\Big)+O_p\Big(\frac{1}{nh^{3d}}\Big)\Bigg] \\ &= \frac{4}{n^4h^d}\Bigg[\int\Big(\int K(t)K(t+s)dt\Big)^2 ds \\ &\quad \times\int\Bigg\{\frac{\kappa_1(u;u,u)}{f^2_{Y(1)|Z}(q_{1,\tau}(u)|u)}+\frac{\kappa_0(u;u,u)}{f^2_{Y(0)|Z}(q_{0,\tau}(u)|u)}\Bigg\}^2\frac{\omega^2(u)}{f^2_Z(u)}du+O_p\Big(\frac{1}{\sqrt{nh^{d}}}\Big)+O_p\Big(\frac{1}{nh^{2d}}\Big)\Bigg] \\ &= \frac{2}{n^4h^d}\big(\sigma_J^2+o_p(1)\big). \end{aligned}$$

It is also easy to find that $A_2 = o_p\Big(\frac{1}{n^4h^d}\Big)$. Thus,

$$\sigma_n^{*2} = E^*\big(T_n^{*2}(R_1^*,R_2^*)\big) = \frac{2\sigma_J^2}{n^4h^d}+o_p\Big(\frac{1}{n^4h^d}\Big).$$

Similarly, by some straightforward but tedious calculations, we can verify that the condition

$$\frac{E^*\Big[E^{*2}\big[T_n^*(R_1^*,R_3^*)T_n^*(R_2^*,R_3^*)\,\big|\,R_1^*,R_2^*\big]\Big]+n^{-1}E^*\big[T_n^{*4}(R_1^*,R_2^*)\big]}{E^{*2}\big[T_n^{*2}(R_1^*,R_2^*)\big]} = o_p(1)$$

is satisfied. From Lemma 5 we know that

$$\frac{\sqrt{2}Q^*_{n,1}}{n\sigma_n^*}\longrightarrow\mathcal{N}(0,1)$$

in distribution in probability. Since $\sigma_n^* = \frac{\sqrt{2}}{\sqrt{n^4h^d}}\big(\sigma_J+o_p(1)\big)$, we also have

$$\frac{nh^{d/2}Q^*_{n,1}}{\sigma_J}\longrightarrow\mathcal{N}(0,1) \qquad \text{(S.19)}$$

in distribution in probability.

Next, for the term $Q^*_{n,2}$, we have

$$Q^*_{n,2} = \frac{1}{n^2}\sum_{i=1}^n\int\Big(\psi_{n,1,\tau}(R_i^*;z)-\psi_{n,0,\tau}(R_i^*;z)\Big)^2\omega(z)dz$$

$$
\begin{aligned}
&= \frac{1}{n^2}\sum_{i=1}^{n}\int \Big(\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)-E^*\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)\Big)^2\omega(z)dz \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\int \Big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\Big)^2\omega(z)dz \\
&\quad +\frac{1}{n^2}\sum_{i=1}^{n}\int \Big(E^*\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)\Big)^2\omega(z)dz \\
&\quad -\frac{2}{n^2}\sum_{i=1}^{n}\int \Big[E^*\big(\varrho_{n,1,\tau}(R_i^*;z)-\varrho_{n,0,\tau}(R_i^*;z)\big)\Big]\Big(\psi_{n,1,\tau}(R_i^*;z)-\psi_{n,0,\tau}(R_i^*;z)\Big)\omega(z)dz \\
&:= Q_{n,2,1}^*+Q_{n,2,2}^*-2Q_{n,2,3}^*.
\end{aligned}
$$

For $Q_{n,2,1}^*$, it is easy to obtain that

$$
E^*\big(Q_{n,2,1}^*\big)=\frac{1}{n^2}\sum_{i=1}^{n}\int \Big(\varrho_{n,1,\tau}(R_i;z)-\varrho_{n,0,\tau}(R_i;z)\Big)^2\omega(z)dz,
$$

and

$$
\begin{aligned}
\mathrm{Var}^*(Q_{n,2,1}^*) &= \mathrm{Var}^*\bigg(\frac{1}{n^2}\sum_{i=1}^{n}\int \Big(\varrho_{n,1}(R_i^*;z)-\varrho_{n,0}(R_i^*;z)\Big)^2\omega(z)dz\bigg) \qquad \text{(S.20)}\\
&= \frac{1}{n^3}\mathrm{Var}^*\bigg(\int \Big(\varrho_{n,1}^2(R_i^*;z)+\varrho_{n,0}^2(R_i^*;z)\Big)\omega(z)dz\bigg) \\
&= \frac{1}{n^3}E^*\bigg(\int \Big(\varrho_{n,1}^2(R_i^*;z)+\varrho_{n,0}^2(R_i^*;z)\Big)\omega(z)dz\bigg)^2 \\
&\quad -\frac{1}{n^3}\bigg\{\int \Big(E^*\varrho_{n,1}^2(R_i^*;z)+E^*\varrho_{n,0}^2(R_i^*;z)\Big)\omega(z)dz\bigg\}^2 \\
&= \frac{1}{n^3}\int\int \Big(E^*\big(\varrho_{n,1}^2(R_i^*;u)\varrho_{n,1}^2(R_i^*;v)\big)+E^*\big(\varrho_{n,0}^2(R_i^*;u)\varrho_{n,0}^2(R_i^*;v)\big)\Big)\omega(u)\omega(v)dudv \\
&\quad -\frac{1}{n^3}\bigg\{\int \Big(E^*\varrho_{n,1}^2(R_i^*;z)+E^*\varrho_{n,0}^2(R_i^*;z)\Big)\omega(z)dz\bigg\}^2.
\end{aligned}
$$

It is easy to obtain that

$$
\begin{aligned}
&E\bigg(\int \Big(E^*\varrho_{n,1}^2(R_i^*;z)+E^*\varrho_{n,0}^2(R_i^*;z)\Big)\omega(z)dz\bigg) \\
&= \int \Big[E\Big(E^*\varrho_{n,1}^2(R_i^*;z)\Big)+E\Big(E^*\varrho_{n,0}^2(R_i^*;z)\Big)\Big]\omega(z)dz \\
&= \frac{1}{h^d}\bigg\{\int S_{n,1,\tau}^{-2}(z)\Big(\int K^2(s)\mu_{1,\tau}(z+hs;z)f_Z(z+hs)ds\Big)\omega(z)dz
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \int S_{n,0,\tau}^{-2}(z)\Big(\int K^2(s)\mu_{0,\tau}(z+hs;z)f_Z(z+hs)ds\Big)\omega(z)dz\Big\} \\
&= O(1/h^d),
\end{aligned}
$$

which implies that

$$
\int \Big(E^*\varrho_{n,1}^2(R_i^*;z) + E^*\varrho_{n,0}^2(R_i^*;z)\Big)\omega(z)dz = O_p(1/h^d). \tag{S.21}
$$

Similarly, by straightforward calculations, we can obtain that

$$
\begin{aligned}
& E\int\int \Big(E^*\big(\varrho_{n,j}^2(R_i^*;u)\varrho_{n,j}^2(R_i^*;v)\big)\Big)\omega(u)\omega(v)dudv \\
= & \int\int E\Big(E^*\big(\varrho_{n,j}^2(R_i^*;u)\varrho_{n,j}^2(R_i^*;v)\big)\Big)\omega(u)\omega(v)dudv \\
= & \frac{1}{nh^{4d}}\int\int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v)E\Big[W_j^4(X_i,D_i)K^2\Big(\frac{Z_i-u}{h}\Big)K^2\Big(\frac{Z_i-v}{h}\Big) \\
& \qquad \times\varphi_\tau^2(Y_i;q_{j,\tau}(u))\varphi_\tau^2(Y_i;q_{j,\tau}(v))\Big]\omega(u)\omega(v)dudv \\
& +\frac{1}{n^2h^4}\sum_{i\neq k}\int\int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v)E\Big[W_j^2(X_i,D_i)K^2\Big(\frac{Z_i-u}{h}\Big)\varphi_\tau^2(Y_i;q_{j,\tau}(u)\Big] \\
& \qquad \times E\Big[W_j^2(X_k,D_k)K^2\Big(\frac{Z_k-v}{h}\Big)\varphi_\tau^2(Y_k;q_{j,\tau}(v)\Big]\omega(u)\omega(v)dudv \\
= & \frac{1}{nh^{3d}}\int\int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v)\Big(\int K^2(t)K^2\Big(t+\frac{u-v}{h}\Big)\ell_j(u+ht;u,v)f_Z(u+ht)dt\Big)\omega(u)\omega(v)dudv \\
& +\frac{n(n-1)}{n^2h^{2d}}\int\int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v)\Big(\int K^2(s)\mu_j(u+hs;u)f_Z(u+hs)ds \\
& \qquad \times\int K^2(s)\mu_j(v+hs;v)f_Z(v+hs)ds\Big)\omega(u)\omega(v)dudv \\
= & \frac{n(n-1)}{n^2h^{2d}}\Big\{\Big(\int K^2(s)ds\Big)^2\int\int S_{n,j}^{-2}(u)S_{n,j}^{-2}(v)\mu_j(u;u)\mu_j(v;v)f_Z(u)f_Z(v)\omega(u)\omega(v)dudv + o(1)\Big\} \\
& +O\Big(\frac{1}{nh^{2d}}\Big) \\
= & O(1/h^{2d}),
\end{aligned}
$$

which implies

$$
\int\int \Big(E^*\big(\varrho_{n,1}^2(R_i^*;u)\varrho_{n,1}^2(R_i^*;v)\big) + E^*\big(\varrho_{n,0}^2(R_i^*;u)\varrho_{n,0}^2(R_i^*;v)\big)\Big)\omega(u)\omega(v)dudv = O_p(1/h^{2d}). \tag{S.22}
$$

From (S.20), (S.21) and (S.22), we have

$$\mathrm{Var}^*(Q^*_{n,2,1}) = \frac{1}{n^3}\cdot O_p(1/h^{2d}) - \frac{1}{n^3}\cdot O_p(1/h^{2d}) = o_p\Big(\frac{1}{n^2h^d}\Big).$$

Then, according to the results in (S.12) and (S.13), we obtain that

$$E\big(Q^*_{n,2,1}\big) = E\big[E^*\big(Q^*_{n,2,1}\big)\big] = \mu_J + O\Big(\frac{h^{2-d}}{n}\Big) + o\Big(\frac{1}{nh^{d/2}}\Big)$$

and

$$\begin{aligned}\mathrm{Var}\big(Q^*_{n,2,1}\big) &= \mathrm{Var}\big[E^*\big(Q^*_{n,2,1}\big)\big] + E\big[\mathrm{Var}^*\big(Q^*_{n,2,1}\big)\big] \\ &= O_p\Big(\frac{1}{n^3h^{2d}}\Big) + o_p\Big(\frac{1}{n^2h^d}\Big) = o_p\Big(\frac{1}{n^2h^d}\Big),\end{aligned}$$

which lead to

$$Q^*_{n,2,1} = \mu_J + O\Big(\frac{h^{2-d}}{n}\Big) + o_p\Big(\frac{1}{nh^{d/2}}\Big) = \mu_J + o_p\Big(\frac{1}{nh^{d/2}}\Big)$$

when $1 \le d \le 3$. By noting that $E^*\big(\varrho_{n,j,\tau}(R^*_1;u)\big) = O_p\Big(h^2 + \frac{1}{\sqrt{nh^d}}\Big)$ as in (S.17), it is easy to obtain that

$$Q^*_{n,2,2} = \frac{1}{n}\cdot O_p\Big(h^4 + \frac{1}{nh^d}\Big) = o_p\Big(\frac{1}{nh^{d/2}}\Big)$$

and

$$Q^*_{n,2,3} = \frac{1}{n}\cdot O_p\Big(h^2 + \frac{1}{\sqrt{nh^d}}\Big)\cdot O_p(1/h^d) = o_p\Big(\frac{1}{nh^{d/2}}\Big)$$

since $nh^{2d} \to \infty$. Thus, we have

$$Q^*_{n,2} \quad = \quad Q^*_{n,2,1} + Q^*_{n,2,2} - 2Q^*_{n,2,3} = \mu_J + o_p\Big(\frac{1}{nh^{d/2}}\Big). \tag{S.23}$$

Combining (S.15), (S.19) and (S.23), we complete the proof of Lemma 10. □

Lemma 11. *Suppose the conditions required by Theorem 3 are satisfied. Then,*

$$\int \Big(\widehat{\Delta}^*_\tau(z) - \widehat{\Delta}_\tau(z)\Big)\omega(z)dz = O_{p^*}(1/\sqrt{n}) + O_{p^*}\left\{ \max\left\{ \frac{\ln n}{n^{1/2-3\nu/2}}, \frac{\sqrt{\ln n}}{(nh^d)^{3/4}} \right\} \right\}.$$

Proof of Lemma 11: According to Lemma 9, we know that

$$\begin{aligned}
\int \Big(\widehat{\Delta}^*_\tau(z) - \widehat{\Delta}_\tau(z)\Big)\omega(z)dz &= \int \frac{1}{n}\sum_{i=1}^n \Big(\psi_{n,1,\tau}(R_i^*;z) - \psi_{n,0,\tau}(R_i^*;z)\Big)\omega(z)dz \\
&\quad + O_{p^*}\left\{ \max\left\{ \frac{\ln n}{n^{1/2-3\nu/2}}, \frac{\sqrt{\ln n}}{(nh^d)^{3/4}} \right\} \right\},
\end{aligned}$$

where $\psi_{n,j,\tau}(R_i^*;z) = \varrho_{n,j,\tau}(R_i^*;z) - E^*\big(\varrho_{n,j,\tau}(R_i^*;z)\big)$ for $j=0$ and 1. Denote

$$\mathcal{M}_n^* = \frac{1}{n}\sum_{i=1}^n \int \Big(\psi_{n,1,\tau}(R_i^*;z) - \psi_{n,0,\tau}(R_i^*;z)\Big)\omega(z)dz.$$

Obviously, $E^*(\mathcal{M}_n^*) = 0$. We then consider $E^*(\mathcal{M}_n^{*2})$. We have

$$\begin{aligned}
E^*(\mathcal{M}_n^{*2}) &= \frac{1}{n^2}\sum_{i=1}^n E^*\bigg(\int \big(\psi_{n,1,\tau}(R_i^*;z) - \psi_{n,0,\tau}(R_i^*;z)\big)\omega(z)dz\bigg)^2 \\
&= \frac{1}{n}\int\int E^*\Big(\big(\psi_{n,1,\tau}(R_i^*;u) - \psi_{n,0,\tau}(R_i^*;u)\big)\big(\psi_{n,1,\tau}(R_i^*;v) - \psi_{n,0,\tau}(R_i^*;v)\big)\Big)\omega(u)\omega(v)dudv \\
&= \frac{1}{n}\int\int E^*\big(\varrho_{n,0,\tau}(R_i^*;u)\varrho_{n,0,\tau}(R_i^*;v)\big)\omega(u)\omega(v)dudv \\
&\quad + \frac{1}{n}\int\int E^*\big(\varrho_{n,1,\tau}(R_i^*;u)\varrho_{n,1,\tau}(R_i^*;v)\big)\omega(u)\omega(v)dudv \\
&\quad - \frac{1}{n}\int\int E^*(\varrho_{n,0,\tau}(R_i^*;u)) \cdot E^*(\varrho_{n,0,\tau}(R_i^*;v))\omega(u)\omega(v)dudv \\
&\quad + \frac{1}{n}\int\int E^*(\varrho_{n,0,\tau}(R_i^*;u)) \cdot E^*(\varrho_{n,1,\tau}(R_i^*;v))\omega(u)\omega(v)dudv \\
&\quad + \frac{1}{n}\int\int E^*(\varrho_{n,1,\tau}(R_i^*;u)) \cdot E^*(\varrho_{n,0,\tau}(R_i^*;v))\omega(u)\omega(v)dudv \\
&\quad - \frac{1}{n}\int\int E^*(\varrho_{n,1,\tau}(R_i^*;u)) \cdot E^*(\varrho_{n,1,\tau}(R_i^*;v))\omega(u)\omega(v)dudv.
\end{aligned}$$

Denote

$$\begin{aligned}\mathcal{B} &= \frac{1}{n}\int\int E^*\big(\varrho_{n,j,\tau}(R_i^*;u)\varrho_{n,j,\tau}(R_i^*;v)\big)\omega(u)\omega(v)dudv\\ &= \frac{1}{n^2}\sum_{i=1}^{n}\int\int \varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\omega(u)\omega(v)dudv,\end{aligned}$$

here we have used that $E^*\big(\varrho_{n,j,\tau}(R_i^*;u)\varrho_{n,j,\tau}(R_i^*;v)\big) = n^{-1}\sum_{i=1}^{n}\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)$. Using the results in (S.7), we obtain that

$$\begin{aligned}E(\mathcal{B}) &= \frac{1}{n}\int\int\Big\{S_{n,j,\tau}^{-1}(v+hs)S_{n,j,\tau}^{-1}(v)\\ &\quad\times\int K(t)K(t+s)\kappa_j(v+hs+ht;v+hs;v)f_Z(v+hs+ht)dt\Big\}\omega(v+hs)\omega(v)dsdv\\ &= O\Big(\frac{1}{n}\Big).\end{aligned}$$

Moreover, by straightforward calculations, one can obtain that

$$\begin{aligned}&E\Big(\int\int \varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\omega(u)\omega(v)dudv\Big)^2\\ &= \int\int\int\int E\Big(\varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\varrho_{n,j,\tau}(R_i;s)\varrho_{n,j,\tau}(R_i;t)\Big)dudvdsdt\\ &= \int\int\int\int\Big\{\frac{1}{h^{4d}}\int K\Big(\frac{r-u}{h}\Big)K\Big(\frac{r-v}{h}\Big)K\Big(\frac{r-s}{h}\Big)K\Big(\frac{r-t}{h}\Big)\\ &\quad\times m_j(r;u,v,s,t)f_Z(r)\Big\}\omega(u)\omega(v)\omega(s)\omega(t)dudvdsdt\\ &= O(1),\end{aligned}$$

where

$$\begin{aligned}m_j(z;u,v,s,t) &= E\Big\{\varphi\big(Y_i(j);q_{j,\tau}(u)\big)\varphi\big(Y_i(j);q_{j,\tau}(v)\big)\varphi\big(Y_i(j);q_{j,\tau}(s)\big)\\ &\quad\times\varphi\big(Y_i(j);q_{j,\tau}(t)\big)\big(1-p(X_i)\big)^{-3(1-j)}\big(p(X_i)\big)^{-3j}\Big|Z_i=z\Big\}.\end{aligned}$$

It follows that

$$\mathrm{Var}(\mathcal{B}) = \mathrm{Var}\Big(\frac{1}{n^2}\sum_{i=1}^{n}\int\int \varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\omega(u)\omega(v)dudv\Big)$$

$$
\begin{aligned}
&= \frac{1}{n^3}\mathrm{Var}\Big(\int\int \varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\omega(u)\omega(v)dudv\Big) \\
&\leq \frac{1}{n^3}E\Big(\int\int \varrho_{n,j,\tau}(R_i;u)\varrho_{n,j,\tau}(R_i;v)\omega(u)\omega(v)dudv\Big)^2 \\
&= O\Big(\frac{1}{n^3}\Big).
\end{aligned}
$$

Hence, we have

$$
\mathcal{B} = E(\mathcal{B}) + O_p\Big(\frac{1}{n^{3/2}}\Big) = O_p\Big(\frac{1}{n}\Big).
$$

Also, by noting that

$$
E^*\big(\varrho_{n,j,\tau}(R_1^*;u)\big) = O_p\Big(h^2 + \frac{1}{\sqrt{nh^d}}\Big)
$$

as in (S.17), we have

$$
\frac{1}{n}\int\int E^*(\varrho_{n,j,\tau}(R_i^*;u)) \cdot E^*(\varrho_{n,k,\tau}(R_i^*;v))\omega(u)\omega(v)dudv = o_p\Big(n^{-1}\Big)
$$

for $j,k = 0,1$. Therefore,

$$
E^*(\mathcal{M}_n^{*2}) = O_p\big(1/n\big),
$$

which implies

$$
\mathcal{M}_n^* = O_{p^*}\big(1/\sqrt{n}\big).
$$

This completes the proof of Lemma 11. □

Lemma 12. *Suppose the conditions required by Theorem 4 are satisfied. Then,*

$$
\begin{aligned}
nh^{d/2}\Big\{ \int_{\mathcal{A}}\int \Big[\frac{1}{n}\sum_{i=1}^{n} \big(&\varrho_{n,1,\tau}(Y_i,X_i,D_i;z) - E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z) \\
&-\varrho_{n,0,\tau}(Y_i,X_i,D_i;z) + E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big)\Big]^2\omega(z,\tau)dzd\tau - \mu_S\Big\} \xrightarrow{D} \mathcal{N}(0,\sigma_S^2),
\end{aligned}
$$

where

$$
\mu_S = \frac{1}{nh^d}\int K^2(s)ds \cdot \int_{\mathcal{A}}\int \Big\{\frac{\mu_{1,\tau}(z;z)}{f^2_{Y(1)|Z}(q_{1,\tau}(z)|z)} + \frac{\mu_{0,\tau}(z;z)}{f^2_{Y(0)|Z}(q_{0,\tau}(z)|z)}\Big\}\frac{\omega(z,\tau)}{f_Z(z)}dzd\tau,
$$

and

$$\sigma_S^2 = 2\int\Big(\int K(t)K(t+s)dt\Big)^2 ds\int_{\mathcal{A}}\int_{\mathcal{A}}\int\Big\{\frac{\lambda_{1,\tau_1,\tau_2}(u;u,u)}{f_{Y(1)|Z}(q_{1,\tau_1}(u)|u)f_{Y(1)|Z}(q_{1,\tau_2}(u)|u)} + \frac{\lambda_{0,\tau_1,\tau_2}(u;u,u)}{f_{Y(0)|Z}(q_{0,\tau_1}(u)|u)f_{Y(0)|Z}(q_{0,\tau_2}(u)|u)}\Big\}^2\frac{\omega(u,\tau_1)\omega(u,\tau_2)}{f_Z^2(u)}dud\tau_1 d\tau_2.$$

Proof of Lemma 12: Define

$$\gamma_{n,\tau}(R_i;z) = \varrho_{n,1,\tau}(R_i;z) - E\varrho_{n,1,\tau}(R_i;z) - \varrho_{n,0,\tau}(R_i;z) + E\varrho_{n,0,\tau}(R_i;z),$$

where $R_i = (Y_i, X_i, D_i)$. Then,

$$\begin{aligned}
&\int_{\mathcal{A}}\int\Big(\frac{1}{n}\sum_{i=1}^n\gamma_{n,\tau}(R_i;z)\Big)^2\omega(z,\tau)dzd\tau \qquad \text{(S.24)}\\
=\;& 2n^{-2}\sum_{1\le i<k\le n}\int_{\mathcal{A}}\int\gamma_{n,\tau}(R_i;z)\gamma_{n,\tau}(R_k;z)\omega(z,\tau)dzd\tau + n^{-2}\sum_{i=1}^n\int_{\mathcal{A}}\int\gamma_{n,\tau}^2(R_i;z)\omega(z,\tau)dzd\tau\\
:=\;& B_{n,1} + B_{n,2}.
\end{aligned}$$

We first consider the term $B_{n,1}$. Define

$$G_n(R_i,R_k) = \frac{2}{n^2}\int_{\mathcal{A}}\int\gamma_{n,\tau}(R_i;z)\gamma_{n,\tau}(R_k;z)\omega(z,\tau)dzd\tau.$$

Then, $B_{n,1} = \sum_{1\le i<k\le n} G_n(R_i,R_k)$ is a centered U-statistic. Thus,

$$E\big[G_n(R_i,R_k)^2\big] = \frac{4}{n^4}\int_{\mathcal{A}}\int_{\mathcal{A}}\int\int E^2\big[\gamma_{n,\tau_1}(R_i;u)\gamma_{n,\tau_2}(R_i;v)\big]\omega(u,\tau_1)\omega(v,\tau_2)dudvd\tau_1d\tau_2.$$

It is easy to show that

$$E\big[\varrho_{n,j,\tau}(R_i;z)\big] = O(h^2)$$

uniformly in z and τ for $j = 0$ and 1. Thus, by straightforward calculations, one can obtain

$$E\Big[\gamma_{n,\tau_1}(R_i;u)\gamma_{n,\tau_2}(R_i;v)\Big]$$

$$
\begin{aligned}
= & \frac{1}{h^d} S^{-1}_{n,1,\tau_1}(u) S^{-1}_{n,1,\tau_2}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{1,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt \\
& + \frac{1}{h^d} S^{-1}_{n,0,\tau_1}(u) S^{-1}_{n,0,\tau_2}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{0,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt + O(h^4),
\end{aligned}
$$

where

$$
\lambda_{1,\tau_1,\tau_2}(z; u, v) = E\Big[\frac{1}{p(X_i)} \varphi_{\tau_1}(Y_i(1); q_{1,\tau_1}(u)) \varphi_{\tau_2}(Y_i(1); q_{1,\tau_2}(v)) \big| Z_i = z\Big],
$$

and

$$
\lambda_{0,\tau_1,\tau_2}(z; u, v) = E\Big[\frac{1}{1-p(X_i)} \varphi_{\tau_1}(Y_i(0); q_{0,\tau_1}(u)) \varphi_{\tau_2}(Y_i(0); q_{0,\tau_2}(v)) \big| Z_i = z\Big].
$$

It follows that

$$
\begin{aligned}
& E^2\Big[\gamma_{n,\tau_1}(R_i; u) \gamma_{n,\tau_2}(R_i; v)\Big] \\
= & \frac{1}{h^{2d}} S^{-2}_{n,1,\tau_1}(u) S^{-2}_{n,1,\tau_2}(v) \Big(\int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{1,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt\Big)^2 \\
& + \frac{1}{h^{2d}} S^{-2}_{n,0,\tau_1}(u) S^{-2}_{n,0,\tau_2}(v) \Big(\int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{0,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt\Big)^2 \\
& + \frac{2}{h^{2d}} S^{-1}_{n,1,\tau_1}(u) S^{-1}_{n,1,\tau_2}(v) S^{-1}_{n,0,\tau_1}(u) S^{-1}_{n,0,\tau_2}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{1,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt \\
& \times \int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{0,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt \\
& + 2O(h^{4-d}) S^{-1}_{n,1,\tau_1}(u) S^{-1}_{n,1,\tau_2}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{1,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt \\
& + 2O(h^{4-d}) S^{-1}_{n,0,\tau_1}(u) S^{-1}_{n,0,\tau_2}(v) \int K(t) K\Big(t + \frac{u-v}{h}\Big) \lambda_{0,\tau_1,\tau_2}(u+ht; u, v) f_Z(u+ht) dt + O(h^8).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
E\big[G_n(R_i, R_k)^2\big] = & \frac{4}{n^4} \int_{\mathcal{A}} \int_{\mathcal{A}} \int \int E^2\big[\gamma_{n,\tau_1}(R_i; u) \gamma_{n,\tau_2}(R_i; v)\big] \omega(u, \tau_1) \omega(v, \tau_2) du dv d\tau_1 d\tau_2 \\
= & \frac{4}{n^4 h^d} \Big\{ \int \Big(\int K(t) K(t+s) dt \Big)^2 ds \\
& \times \Big[\int_{\mathcal{A}} \int_{\mathcal{A}} \int S^{-2}_{n,1,\tau_1}(u) S^{-2}_{n,1,\tau_2}(u) \lambda^2_{1,\tau_1,\tau_2}(u; u, u) f^2_Z(u) \omega(u, \tau_1) \omega(u, \tau_2) du d\tau_1 d\tau_2 \\
& + \int_{\mathcal{A}} \int_{\mathcal{A}} \int S^{-2}_{n,0,\tau_1}(u) S^{-2}_{n,0,\tau_2}(u) \lambda^2_{0,\tau_1,\tau_2}(u; u, u) f^2_Z(u) \omega(u, \tau_1) \omega(u, \tau_2) du d\tau_1 d\tau_2
\end{aligned}
$$

$$+2\int_{\mathcal{A}}\int_{\mathcal{A}}\int S^{-1}_{n,1,\tau_1}(u)S^{-1}_{n,1,\tau_2}(u)S^{-1}_{n,0,\tau_1}(u)S^{-1}_{n,0,\tau_2}(u)$$
$$\times\lambda_{1,\tau_1,\tau_2}(u;u,u)\lambda_{0,\tau_1,\tau_2}(u;u,u)f_Z^2(u)\omega(u,\tau_1)\omega(u,\tau_2)dud\tau_1d\tau_2\Big] + o(1)\Bigg\}.$$

This, coupled with $S_{n,j,\tau}(z) = f_Z(z)f_{Y(j)|Z}(q_{j,\tau}(z)|z) + O(h^2)$ for $j = 0$ and 1, yields

$$\begin{aligned} E\big[G_n(R_1,R_2)^2\big] &= \frac{4}{n^4h^d}\Bigg(\int\Big(\int K(t)K(t+s)dt\Big)^2 ds\int_{\mathcal{A}}\int_{\mathcal{A}}\int\Bigg\{\frac{\lambda_{1,\tau_1,\tau_2}(u;u,u)}{f_{Y(1)|Z}(q_{1,\tau_1}(u)|u)f_{Y(1)|Z}(q_{1,\tau_2}(u)|u)} \\ &\quad + \frac{\lambda_{0,\tau_1,\tau_2}(u;u,u)}{f_{Y(0)|Z}(q_{0,\tau_1}(u)|u)f_{Y(0)|Z}(q_{0,\tau_2}(u)|u)}\Bigg\}^2\frac{\omega(u,\tau_1)\omega(u,\tau_2)}{f_Z^2(u)}dud\tau_1d\tau_2 + o(1)\Bigg) \\ &= \frac{2}{n^4h^d}\big(\sigma_S^2 + o(1)\big). \end{aligned}$$

Similarly, by straightforward calculations, we can verify the condition

$$\lim_{n\to\infty}\frac{E\Big[E^2\big[G_n(R_1,R_3)G_n(R_2,R_3)\,\big|\,R_1,R_2\big]\Big] + n^{-1}E\big[G_n(R_1,R_2)^4\big]}{\Big(E\big[G_n(R_1,R_2)^2\big]\Big)^2} = 0$$

in Lemma 5 is satisfied, so that

$$\frac{\sqrt{2}}{nE^{1/2}\big[G_n(R_1,R_2)^2\big]}\,B_{n,1} \xrightarrow{D} \mathcal{N}(0,1),$$

or equivalently,

$$nh^{d/2}\,B_{n,1} \xrightarrow{D} \mathcal{N}(0,\sigma_S^2). \tag{S.25}$$

Now, we move to the term $B_{n,2} = n^{-2}\sum_{i=1}^{n}\int_{\mathcal{A}}\int\gamma^2_{n,\tau}(R_i;z)\omega(z,\tau)dzd\tau$. Recall that

$$\mu_{1,\tau}(z;u) = E\Bigg[\frac{1}{p(X_i)}\big(I\{Y_i(1)\le q_{1,\tau}(u)\} - \tau\big)^2\Big|Z_i = z\Bigg],$$

and

$$\mu_{0,\tau}(z;u) = E\Bigg[\frac{1}{1-p(X_i)}\big(I\{Y_i(0)\le q_{0,\tau}(u)\} - \tau\big)^2\Big|Z_i = z\Bigg].$$

It is noted that

$$E\big[\gamma^2_{n,\tau}(R_i;z)\big] = \frac{1}{h^d}\Bigg\{\int K^2(s)ds\cdot\Big(S^{-2}_{n,1,\tau}(z)\mu_{1,\tau}(z;z) + S^{-2}_{n,0,\tau}(z)\mu_{0,\tau}(z;z)\Big)f_Z(z) + O(h^2)\Bigg\} + O(h^4),$$

coupled with $S_{n,j,\tau}(z) = f_Z(z) f_{Y(j)|Z}(q_{j,\tau}(z)|z) + O(h^2)$ for $j = 0$ and 1, we have

$$\begin{aligned}
E(B_{n,2}) &= \frac{1}{n}\int_{\mathcal{A}}\int E\big[\gamma_{n,\tau}^2(R_i;z)\big]\omega(z,\tau)dzd\tau \\
&= \frac{1}{nh^d}\Big\{\int K^2(s)ds\cdot\int_{\mathcal{A}}\int\Big(S_{n,1,\tau}^{-2}(z)\mu_{1,\tau}(z;z) + S_{n,0,\tau}^{-2}(z)\mu_{0,\tau}(z;z)\Big)f_Z(z)\omega(z,\tau)dzd\tau \\
&\quad +O(h^2)\Big\} + O\Big(\frac{h^4}{n}\Big) \\
&= \frac{1}{nh^d}\int K^2(s)ds\cdot\int_{\mathcal{A}}\int\left\{\frac{\mu_{1,\tau}(z;z)}{f_{Y(1)|Z}^2(q_{1,\tau}(z)|z)} + \frac{\mu_{0,\tau}(z;z)}{f_{Y(0)|Z}^2(q_{0,\tau}(z)|z)}\right\}\frac{\omega(z,\tau)}{f_Z(z)}dzd\tau \\
&\quad +\frac{O(h^2)}{nh^d} + O\Big(\frac{h^4}{n}\Big) \\
&= \mu_S + \frac{O(h^2)}{nh^d} + O\Big(\frac{h^4}{n}\Big).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{Var}\big(nh^{d/2}B_{n,2}\big) &= E\Big\{nh^{d/2}\big[B_{n,2} - E(B_{n,2})\big]\Big\}^2 \\
&= n^{-1}h^d\Big\{E\Big[\Big(\int_{\mathcal{A}}\int\gamma_{n,\tau}^2(R_i;z)\omega(z,\tau)dzd\tau\Big)^2\Big] - E^2\Big[\int_{\mathcal{A}}\int\gamma_{n,\tau}^2(R_i;z)\omega(z,\tau)dzd\tau\Big]\Big\} \\
&= n^{-1}h^d\Big\{E\Big[\Big(\int_{\mathcal{A}}\int\gamma_{n,\tau}^2(R_i;z)\omega(z,\tau)dzd\tau\Big)^2\Big] - \Big[\int_{\mathcal{A}}\int E\big[\gamma_{n,\tau}^2(R_i;z)\big]\omega(z,\tau)dzd\tau\Big]^2\Big\} \\
&= n^{-1}h^d\Big\{\int_{\mathcal{A}}\int_{\mathcal{A}}\int\int E\Big[\gamma_{n,\tau_1}^2(R_i;u)\gamma_{n,\tau_2}^2(R_i;v)\Big]\omega(u,\tau_1)\omega(v,\tau_2)dudvd\tau_1d\tau_2 - O(h^{-2d})\Big\} \\
&= n^{-1}h^d\cdot O(h^{-2d}) \to 0,
\end{aligned}$$

together with $E(B_{n,2}) = \mu_S + \frac{O(h^2)}{nh^d} + O\Big(\frac{h^4}{n}\Big)$, we have

$$nh^{d/2}\big[B_{n,2} - \mu_S\big] = nh^{d/2}\big[B_{n,2} - E(B_{n,2})\big] + nh^{d/2}\frac{O(h^2)}{nh^d} + o_p(1) = o_p(1) \tag{S.26}$$

when $1 \le d \le 3$.

It follows by combining (S.24), (S.25) and (S.26) that

$$\begin{aligned}
nh^{d/2}\Big\{\int_{\mathcal{A}}\int\Big[\frac{1}{n}\sum_{i=1}^n\big(\varrho_{n,1,\tau}(Y_i,X_i,D_i;z) - E\varrho_{n,1,\tau}(Y_i,X_i,D_i;z) \\
-\varrho_{n,0,\tau}(Y_i,X_i,D_i;z) + E\varrho_{n,0,\tau}(Y_i,X_i,D_i;z)\big)\Big]^2\omega(z,\tau)dzd\tau - \mu_S\Big\} \xrightarrow{D} \mathcal{N}(0,\sigma_S^2),
\end{aligned}$$

which implies the proof is completed. □

References

Hall, P. (1984). Central limit theorem for integrated square error of multivariate nonparametric density estimators. *Journal of Multivariate Analysis 14(1)*, 1-16.

Hirano, K., G. W. Imbens, and G. Ridder (2003). Efficient estimation of average treatment effects using the estimated propensity score. *Econometrica* 71(4), 1161-1189.

Lee, S., K. Song, and Y.-J. Whang (2015). Uniform asymptotics for nonparametric quantile regression with an application to testing monotonicity. *ArXiv preprint arXiv:1506.05337*, 1-30.

Pollard, D. (1984). *Convergence of Stochastic Processes.* Springer-Verlag, New York.