Online supplementary material to “Asymptotically uniformly most powerful tests for unit roots in Gaussian panels with cross-sectional dependence generated by common factors”

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Abstract

This online supplement contains two appendices to “Asymptotically uniformly most powerful tests for unit roots in Gaussian panels with cross-sectional dependence generated by common factors”. The first appendix, Supplement A, contains detailed proofs. And the second appendix, Supplement B, presents results from additional Monte Carlo simulations.

Supplement A. Detailed Proofs

A.1. Preliminaries

This section present some preliminary results that are heavily exploited in the proofs of our main results.

First, we recall some elementary results from linear algebra (throughout we only consider real matrices); see, e.g., Lütkepohl (1996) and Magnus & Neudecker (1999). Let $\mathrm{tr}[C]$ denote the trace of a square, real matrix C and let $\lambda_{\min}(C)$ (and $\lambda_{\max}(C)$) denote the minimal (maximal) eigenvalue of a symmetric, real matrix C. For any real matrix C, let $\|C\|_F = \sqrt{\mathrm{tr}\,[C'C]} = \|C'\|_F$ denote its Frobenius norm, while $\|C\|_{\mathrm{spec}} = \sqrt{\lambda_{\max}(C'C)} = \|C'\|_{\mathrm{spec}}$ denotes its spectral norm. Recall $\|C\|_{\mathrm{spec}} \leq \|C\|_F$.

The inequality $\|CD\|_F \leq \|C\|_{\mathrm{spec}} \|D\|_F$ is immediate from Raleigh's quotient. It follows that the Frobenius is submultiplicative, $\|CD\|_F \leq \|C\|_F \|D\|_F$.

Moreover, the identity $\|C \otimes D\|_F = \|C\|_F \|D\|_F$ easily follows from the alternative interpretation of the Frobenius norm being the square-root of the sum of all squared individual matrix entries. Finally, we note that for square matrices $\langle C, D\rangle_F = \operatorname{tr}[C'D]$ defines an inner product, so we have the Cauchy-Schwarz inequality $|\operatorname{tr}[C'D]| \leq \|C\|_F \|D\|_F$.

Next, we present a general lemma on approximating variances with long-run variances. The results we present in this appendix are the main keys to many proofs in Section 3. Moreover, they may be of general interest.

Lemma A.1: *Consider an indexed collection of stationary time series* $\{X_t^{(h)}\}$, $h \in \mathcal{H}$. *Denote the* $T \times T$ *covariance matrix of* $(X_1^{(h)}, \ldots, X_T^{(h)})$ *by* Σ_h, *the* m*-th autocovariance of* $\{X_t^{(h)}\}$ *by* $\gamma_h(m)$, *and its long run variance by* $\omega_h^2 < \infty$. *Also write* $\omega_{h,T}^2 = \iota'\Sigma_h\iota/T$. *If* $\sup_{h\in\mathcal{H}} \sum_{m=-\infty}^{\infty}(|m|+1)|\gamma_h(m)| < \infty$, *then (with* A *as defined in the main text),*

1. $\sup_{h\in\mathcal{H}} |\omega_{h,T}^2 - \omega_h^2| = O(T^{-1})$,
2. $\sup_{h\in\mathcal{H}} \left\|A'(\Sigma_h - \omega_h^2 I_T)\right\|_F + \sup_{h\in\mathcal{H}} \left\|A(\Sigma_h - \omega_h^2 I_T)\right\|_F = O(\sqrt{T})$,
3. $\sup_{h\in\mathcal{H}} \left\|A'(\Sigma_h - \omega_{h,T}^2 I_T)\right\|_F + \sup_{h\in\mathcal{H}} \left\|A(\Sigma_h - \omega_{h,T}^2 I_T)\right\|_F = O(\sqrt{T})$,
4. $\sup_{h\in\mathcal{H}} \|A'\Sigma_h\|_F + \sup_{h\in\mathcal{H}} \|A\Sigma_h\|_F = O(T)$.

Proof: Item 1 follows from $\omega_{h,T}^2 = \frac{1}{T}\sum_{|m|<T}(T-|m|)\gamma_h(m)$ and $\omega_h^2 = \sum_{m=-\infty}^{\infty}\gamma_h(m)$, so

$$|\omega_{h,T}^2 - \omega_h^2| = \left|\frac{1}{T}\sum_{m=-\infty}^{\infty}(\min(|m|, T)\gamma_h(m)\right|,$$

which is indeed $O(T^{-1})$ uniformly in h.

For Item 2, tedious but elementary calculations yield

$$
\begin{aligned}
&\left\|A(\Sigma_h-\omega_h^2 I_T)\right\|_F^2=\left\|A'(\Sigma_h-\omega_h^2 I_T)\right\|_F^2\\
&\quad=\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\sum_{m=s-t+1}^{T-t}\gamma_h(m)-\omega_h^2 1_{s<t}\right)^2\\
&\quad=\sum_{s=1}^{T-1}\left(\sum_{t=1}^{s}\left(\sum_{m=s+1}^{T}\gamma_h(m-t)\right)^2\right.\\
&\qquad\left.+\sum_{t=s+1}^{T}\left(\sum_{m=-\infty}^{s}\gamma_h(m-t)+\sum_{m=T+1}^{\infty}\gamma_h(m-t)\right)^2\right)\\
&\quad=\sum_{s=1}^{T-1}\sum_{t=1}^{T-s}\left(\left(\sum_{m=s}^{T-t}\gamma_h(m)\right)^2+\left(\sum_{m=s}^{\infty}\gamma_h(m)+\sum_{m=t}^{\infty}\gamma_h(m)\right)^2\right)\\
&\quad\le 5T\sum_{s=1}^{T}\left(\sum_{m=s}^{\infty}|\gamma_h(m)|\right)^2\\
&\quad\le 5T\left(\sum_{m=-\infty}^{\infty}|\gamma_h(m)|\right)\sum_{m=1}^{\infty}\min(m,T)|\gamma_h(m)|.
\end{aligned}
$$

Taking suprema, Item 2 follows immediately from this bound. Item 3 follows by combining the first two parts and $\|A\|_F=\sqrt{\frac{T(T-1)}{2}}=O(T)$. The order on $\|A\|_F$ also yields

$$
\begin{aligned}
\sup_{h\in\mathcal{H}}\|A'\Sigma_h\|_F\le&\sup_{h\in\mathcal{H}}\left\|A'(\Sigma_h-\omega_h^2 I_T)\right\|_F+\sup_{h\in\mathcal{H}}\omega_h^2\|A'\|_F\\
=&O(\sqrt{T})+O(1)O(T).
\end{aligned}
$$

Again, the second part of Item 4 is analogous. ■

Recall the covariance matrices Σ_η and Σ_ε and their rough approximations Ψ_η and Ψ_ε defined in Lemma 3.1 and (9), respectively. The following three lemmas use Lemma A.1 to show that these approximations do work well when considering partial sums.

Lemma A.2: *Under Assumption 2.1, $\left\|\Sigma_\eta^{-1}\right\|_{spec}$, $\left\|\Psi_\eta^{-1}\right\|_{spec}$, $\left\|\Sigma_\varepsilon^{-1}\right\|_{spec}$, and $\left\|\Psi_\varepsilon^{-1}\right\|_{spec}$ are all $O(1)$ as $n,T\to\infty$.*

Proof: Note that $\Sigma_\varepsilon - \Sigma_\eta$ and $\Psi_\varepsilon - \Psi_\eta$ are positive semidefinite. Hence $\lambda_{\min}(\Sigma_\varepsilon) \geq \lambda_{\min}(\Sigma_\eta) \geq \inf_{i,T} \lambda_{\min}(\Sigma_{\eta,i}) > 0$ and, using Remark 2.2 and Item 1 of Lemma A.1,

$$\begin{aligned}\lambda_{\min}(\Psi_\varepsilon) \geq & \lambda_{\min}(\Psi_\eta) = \lambda_{\min}(\Omega_\eta \otimes I_T) = \min_{i=1,\ldots,n} \omega^2_{\eta,i,T} \\ & \geq \inf_{i\in\mathbb{N}} \omega^2_{\eta,i} - \sup_{i\in\mathbb{N}} |\omega^2_{\eta,i,T} - \omega^2_{\eta,i}| \to \inf_{i\in\mathbb{N}} \omega^2_{\eta,i} > 0.\end{aligned}$$

This shows the boundedness of all four norms. ■

Lemma A.3: *Under Assumption 2.1 we have, as $n, T \to \infty$,*

$$\|\mathcal{A}'(\Sigma_\eta - \Psi_\eta)\|_F + \|\mathcal{A}(\Sigma_\eta - \Psi_\eta)\|_F = O(\sqrt{nT}) = o(\sqrt{n}T).$$

Proof: Using block diagonality and Lemma A.1, we obtain the bound

$$\begin{aligned}\|\mathcal{A}'(\Sigma_\eta - \Psi_\eta)\|_F^2 &= \sum_{i=1}^n \left\|A'(\Sigma_{\eta,i} - \omega^2_{\eta,i,T} I_T)\right\|_F^2 \\ &\leq n \sup_{i\in\mathbb{N}} \left\|A'(\Sigma_{\eta,i} - \omega^2_{\eta,i,T} I_T)\right\|_F^2 = O(nT).\end{aligned}$$

The other part is analogous; every $\mathcal{A}'$ and A' are replaced by $\mathcal{A}$ and A, respectively. ■

Lemma A.4: *Under Assumptions 2.1, 2.2, and 2.4 we have, as $n, T \to \infty$,*

$$\|\mathcal{A}'(\Sigma_\varepsilon - \Psi_\varepsilon)\|_F + \|\mathcal{A}(\Sigma_\varepsilon - \Psi_\varepsilon)\|_F = O(n\sqrt{T}) = o(\sqrt{n}T).$$

Proof: From the definitions of Σ_ε and Ψ_ε we obtain

$$\mathcal{A}'(\Sigma_\varepsilon - \Psi_\varepsilon) = \sum_{k=1}^K \mathcal{A}'\left(\lambda_k\lambda_k' \otimes \left(\Sigma_{f,k} - \omega^2_{f,k,T} I_T\right)\right) + \mathcal{A}'(\Sigma_\eta - \Omega_\eta \otimes I_T),$$

which yields the bound $\|\mathcal{A}'(\Sigma_\varepsilon - \Psi_\varepsilon)\|_F \leq I + II$ with

$$I = \sum_{k=1}^K \left\|\left(\lambda_k\lambda_k' \otimes A'\left(\Sigma_{f,k} - \omega^2_{f,k,T} I_T\right)\right)\right\|_F \text{ and } II = \|\mathcal{A}'(\Sigma_\eta - \Omega_\eta \otimes I_T)\|_F.$$

Part II is already treated in Lemma A.3. For part I, again using Lemma A.1, we get a slightly weaker bound since for the factor part there is no block diagonality:

$$\begin{aligned} I =& \sum_{k=1}^{K} \|\lambda_k \lambda_k'\|_F \left\| A' \left(\Sigma_{f,k} - \omega_{f,k,T}^2 I_T \right) \right\|_F \\ \leq& \sum_{k=1}^{K} \lambda_k' \lambda_k \left\| A' \left(\Sigma_{f,k} - \omega_{f,k,T}^2 I_T \right) \right\|_F = O(n\sqrt{T}) = o(\sqrt{n}T). \end{aligned}$$

The proof for $\|\mathcal{A}\left(\Sigma_\varepsilon - \Psi_\varepsilon\right)\|_F$ is analogous. ■

We now present a general weak convergence result for partial sums using joint asymptotics. Proposition 3.1 is a special case of Lemma A.5 with $a_{i,n,T} = 1$. We provide Lemma A.5 in general terms here as it might be of independent interest and we also use it in the proof of Proposition 5.1 to demonstrate the joint convergence of P_a and the local likelihood ratio.

Lemma A.5: *Let $a_{i,n,T}$ be a bounded sequence of non-random numbers and $\frac{1}{n}\sum_{i=1}^{n} a_{i,n,T}^2 \to \alpha$. Then, under $\mathrm{P}_{0,n,T}^{MP}$ or $\mathrm{P}_{0,n,T}^{PANIC}$, as $n, T \to \infty$,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{a_{i,n,T}}{\omega_{\eta,i,T}^2} \left(\frac{1}{T} \sum_{t=1}^{T} \sum_{s=1}^{t-1} \eta_{is}\eta_{it} - \delta_{\eta,i} \right) \xrightarrow{d} N(0, \alpha/2).$$

Proof: First consider the case of $a_{i,n,T}$ being identically equal to one and observe that this implies convergence of $\Delta_{n,T}$. Recall $A + A' = \iota\iota' - I_T$ and $2\delta_{\eta,i,T} = \omega_{\eta,i,T}^2 - \gamma_{\eta,i}(0)$, hence, with $\omega_{\eta,i,T}^2 = \frac{1}{T}\iota'\Sigma_{\eta,i}\iota$,

$$\begin{aligned} \Delta_{n,T} =& \frac{1}{\sqrt{n}T} \sum_{i=1}^{n} \frac{1}{\omega_{\eta,i,T}^2} \eta_i' \frac{A + A'}{2} \eta_i - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2} \\ =& \frac{1}{2\sqrt{n}} \sum_{i=1}^{n} \left(\left(\frac{\iota'\eta_i}{\sqrt{T}\omega_{\eta,i,T}} \right)^2 - 1 \right) - \frac{1}{2\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\omega_{\eta,i,T}^2} \left(\frac{1}{T}\eta_i'\eta_i - \gamma_{\eta,i}(0) \right). \end{aligned}$$

Observe that $X_{i,T} := \frac{\iota'\eta_i}{\sqrt{T\omega_{\eta,i,T}^2}} \sim N(0,1)$ and are independent across $i \in \mathbb{N}$. Thus, for each T, $\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}(X_{i,T}^2 - 1)$ has the same distribution as $\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}(X_i^2 - 1)$, where $X_i^2 \overset{iid}{\sim} \chi^2(1)$. Therefore, as the latter converges to a standard normal distribution as $n \to \infty$ (CLT), so does the former under joint limits. Thus, the first, leading term converges in distribution to $N(0, 1/2)$.

Asymptotic negligibility of the second, mean-zero term follows from

$$\begin{aligned}\sup_i \operatorname{var}(\frac{1}{T}\eta_i'\eta_i) =& \frac{2}{T^2}\sup_i \operatorname{tr}[\Sigma_{\eta,i}^2] = \frac{2}{T^2}\sup_i \|\Sigma_{\eta,i}\|_F^2 \\ =& \frac{2}{T}\sup_i \left| \sum_{m=-(T-1)}^{T-1} (1-\frac{|m|}{T})\gamma_{\eta,i}^2(m) \right| = O(T^{-1}).\end{aligned}$$

For general $a_{i,n,T}$ we can apply a double array CLT, see 1.9.3 in Serfling (1980), to the first (slightly adapted) term in the expansion. The Lindeberg condition is readily verified since we have a weighted sum of i.i.d. centered χ^2 variables. Asymptotic negligibility of the second remainder term follows from the boundedness condition on the $a_{i,n,T}$. ∎

Remark A.1: *We can obtain the same conclusion without requiring Gaussian innovations: as long as the Lindeberg condition holds, for example thanks to higher moment conditions, the same Theorem 1.9.3 of Serfling (1980) applies.*

We conclude this subsection by taking care of important terms that appear repeatedly in the remainder.

Lemma A.6: *Suppose that Assumptions 2.1-2.4 hold. Then, under $\mathrm{P}_{0,n,T}^{MP}$ or $\mathrm{P}_{0,n,T}^{PANIC}$ and as $n,T\to\infty$, we have*

1. $\left\|\left(\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_F = O(1)$,
2. $\left\|\sum_{t=2}^T \eta_{\cdot,t}\right\|_F = O_p(\sqrt{nT})$,
3. $\left\|\sum_{t=2}^T f_{\cdot,t}\right\|_F = O_p(\sqrt{T})$,
4. $\left\|\iota'\tilde{\eta}\Omega_\eta^{-1}\Lambda\right\|_F = O_p(\sqrt{nT})$, *and*
5. $\left\|\tilde{\eta}\Omega_\eta^{-1}\Lambda\right\|_F = O_p(\sqrt{nT})$.

Proof: For Item 1, recall that K is fixed, so that the norm we consider is irrelevant. As

$$\Lambda'\Omega_\eta^{-1}\Lambda = \sum_{i=1}^n \frac{1}{\omega_{\eta,i,T}^2}\lambda_i\lambda_i' \geq \frac{1}{\sup_{i\in\mathbb{N}}\omega_{\eta,i,T}^2}\sum_{i=1}^n \lambda_i\lambda_i',$$

the smallest eigenvalue of $\Lambda'\Omega_\eta^{-1}\Lambda$ is larger than that of $\Lambda'\Lambda$. Thus,

$$\left\|\left(\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_{\text{spec}} \leq \sup_{i\in\mathbb{N}}\omega_{\eta,i,T}^2\left\|\left(\frac{1}{n}\Lambda'\Lambda\right)^{-1}\right\|_{\text{spec}} \to \sup_{i\in\mathbb{N}}\omega_i^2\left\|\Psi_\Lambda^{-1}\right\|_{\text{spec}} < \infty,$$

thanks to Assumptions 2.1 and 2.2.

Item 2 follows from

$$\mathrm{E}\left\|\sum_{t=1}^T \eta_{\cdot,t}\right\|_F^2 = \mathrm{E}\left\|\tilde{\eta}'\iota\right\|_F^2 = \iota'\mathrm{E}\tilde{\eta}\tilde{\eta}'\iota = \iota'\sum_{i=1}^n \mathrm{E}\eta_i\eta_i'\iota = T\sum_{i=1}^n \omega_{\eta,i,T}^2 = O(nT).$$

Note that the expectation of $\left\|\sum_{t=2}^T \eta_{\cdot,t}\right\|_F^2$ is given by $(T-1)\sum_{i=1}^n \omega_{\eta,i,T-1}^2$ and is thus of the same order.

Item 3 can be obtained along a similar line of proof.

For Item 4, note $\mathrm{E}\tilde{\eta}'\iota\iota\tilde{\eta} = T\Omega_\eta$, so that

$$\begin{aligned}\mathrm{E}\left\|\iota'\tilde{\eta}\Omega_\eta^{-1}\Lambda\right\|_F^2 &= \operatorname{tr}\mathrm{E}[\tilde{\eta}'\iota\iota\tilde{\eta}]\Omega_\eta^{-1}\Lambda\Lambda'\Omega_\eta^{-1} \\ &= T\operatorname{tr}\Lambda\Lambda'\Omega_\eta^{-1} \leq T\left\|\Lambda\right\|_F^2\left\|\Omega_\eta^{-1}\right\|_{\text{spec}} = O(nT).\end{aligned}$$

Item 5 follows similarly from $\mathrm{E}\eta_{\cdot,t}\eta_{\cdot,t}' = \operatorname{diag}(\gamma_{\eta,1}(0),\ldots,\gamma_{\eta,n}(0)) =: D$, so

$$E\left\|\tilde{\eta}\Omega_\eta^{-1}\Lambda\right\|_F^2 = \operatorname{tr}(\Lambda'\Omega_\eta^{-1}\sum_{t=1}^T \mathrm{E}[\eta_{\cdot,t}\eta_{\cdot,t}']\Omega_\eta^{-1}\Lambda) \leq T\left\|\Lambda\right\|_F^2\left\|\Omega_\eta^{-1}\right\|_{\text{spec}}^2\left\|D\right\|_{\text{spec}},$$

which is indeed $O(nT)$ thanks to Assumptions 2.1 and 2.2. ∎

A.2. Proofs of Section 3

A.2.1. Proof of Lemma 3.1

Proof: In the following all probabilities and expectations are evaluated under $\mathrm{P}_{0,n,T}^{\text{PANIC}}$. To obtain the desired result, we consider the difference between the two central sequences $\Delta_{n,T} - \Delta_{n,T}^{\text{PANIC}}$ and the difference between the two Fisher informations $J_{n,T}^{\text{PANIC}} - \frac{1}{2}$. We show that expectations and variances of both differences converge to zero, implying L_2 convergence.

Part A: Under the null, $\Delta E = \eta$ and hence

$$\Delta_{n,T} - \Delta_{n,T}^{\text{PANIC}} = \frac{1}{\sqrt{nT}}\eta'\mathcal{A}'(\Psi_\eta^{-1} - \Sigma_\eta^{-1})\eta - \frac{1}{\sqrt{n}}\sum_{i=1}^n \frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2}.$$

We first show that the difference has mean zero. We have, using $\operatorname{tr}(\mathcal{A}) = 0$ and block diagonality of Σ_η,

$$\begin{aligned}
\mathrm{E}[\Delta_{n,T} - \Delta_{n,T}^{\mathrm{PANIC}}] &= \frac{1}{\sqrt{n}T}\operatorname{tr}(\mathcal{A}'(\Psi_\eta^{-1} - \Sigma_\eta^{-1})\Sigma_\eta) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2} \\
&= \frac{1}{\sqrt{n}T}\operatorname{tr}(\mathcal{A}'\Psi_\eta^{-1}\Sigma_\eta) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2} \\
&= \frac{1}{\sqrt{n}T}\operatorname{tr}((\Omega_\eta{}^{-1}\otimes A')\Sigma_\eta) - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2} \\
&= \frac{1}{\sqrt{n}}\frac{1}{T}\sum_{i=1}^{n}\frac{1}{\omega_{\eta,i,T}^2}\operatorname{tr}\left[A'\Sigma_{\eta,i}\right] - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2} = 0,
\end{aligned}$$

as $\operatorname{tr}\left[A'\Sigma_{\eta,i}\right] = T\delta_{\eta,i,T}$.

To show that the variance of $\Delta_{n,T}^{\mathrm{PANIC}} - \Delta_{n,T}$ goes to zero, observe

$$\begin{aligned}
nT^2\operatorname{var}(\Delta_{n,T}^{\mathrm{PANIC}} - \Delta_{n,T}) &= \operatorname{var}(\eta' C_\eta \eta) \\
&= \operatorname{tr}[C_\eta\Sigma_\eta C_\eta\Sigma_\eta] + \operatorname{tr}[C_\eta\Sigma_\eta C_\eta'\Sigma_\eta] \\
&\le \left\|C_\eta\Sigma_\eta\right\|_F^2 + \left\|C_\eta\Sigma_\eta\right\|_F\left\|\Sigma_\eta C_\eta\right\|_F,
\end{aligned} \tag{A.1}$$

with $C_\eta = \mathcal{A}'(\Psi_\eta^{-1} - \Sigma_\eta^{-1})$. Hence, it suffices to show $\left\|C_\eta\Sigma_\eta\right\|_F = o(\sqrt{n}T)$ and $\left\|\Sigma_\eta C_\eta\right\|_F = o(\sqrt{n}T)$. Since Ψ_η^{-1} and $\mathcal{A}'$ commute, we obtain

$$\left\|C_\eta\Sigma_\eta\right\|_F = \left\|\mathcal{A}'\Psi_\eta^{-1}(\Sigma_\eta - \Psi_\eta)\right\|_F \le \quad \left\|\Psi_\eta^{-1}\right\|_{\mathrm{spec}}\left\|\mathcal{A}'(\Sigma_\eta - \Psi_\eta)\right\|_F,$$

which is indeed $o(\sqrt{n}T)$ by Lemmas A.2 and A.3. For $\left\|\Sigma_\eta C_\eta\right\|_F$, we first have to approximate $\mathcal{A}\Sigma_\eta$ with $\mathcal{A}\Psi_\eta$ before we can use the commutativity as above:

$$\begin{aligned}
\left\|\Sigma_\eta C_\eta\right\|_F &\le \left\|\Psi_\eta C_\eta\right\|_F + \left\|C_\eta'(\Sigma_\eta - \Psi_\eta)\right\|_F \\
&= \left\|\mathcal{A}'(\Sigma_\eta - \Psi_\eta)\Sigma_\eta^{-1}\right\|_F + \left\|\left(\Psi_\eta^{-1} - \Sigma_\eta^{-1}\right)\mathcal{A}(\Sigma_\eta - \Psi_\eta)\right\|_F \\
&\le \left\|\Sigma_\eta^{-1}\right\|_{\mathrm{spec}}\left\|\mathcal{A}'(\Psi_\eta - \Sigma_\eta)\right\|_F \\
&+ \left(\left\|\Psi_\eta^{-1}\right\|_{\mathrm{spec}} + \left\|\Sigma_\eta^{-1}\right\|_{\mathrm{spec}}\right)\left\|\mathcal{A}(\Sigma_\eta - \Psi_\eta)\right\|_F = o(\sqrt{n}T).
\end{aligned}$$

Part B: First, we show that the expectation of $J_{n,T}^{\mathrm{PANIC}}$ converges to $\frac{1}{2}$. We have

$$\begin{aligned}
nT^2\mathrm{E}J_{n,T}^{\mathrm{PANIC}} &= \operatorname{tr}\left[\mathcal{A}'\Sigma_\eta^{-1}\mathcal{A}\Sigma_\eta\right] = \operatorname{tr}\left[\mathcal{A}'\Psi_\eta^{-1}\mathcal{A}\Sigma_\eta\right] - \operatorname{tr}\left[\mathcal{A}'C_\eta'\Sigma_\eta\right] \\
&= \operatorname{tr}[\mathcal{A}'\mathcal{A}] + \operatorname{tr}[\mathcal{A}'\Psi_\eta^{-1}\mathcal{A}(\Sigma_\eta - \Psi_\eta)] - \operatorname{tr}[\Sigma_\eta C_\eta\mathcal{A}].
\end{aligned}$$

This implies that the leading term is $\frac{1}{2}nT^2$, since the final two terms are $o(nT^2)$: use the arguments already presented in Part A together with the relation between the trace and the Frobenius norm and

$$\frac{1}{nT^2}\left\|\mathcal{A}\right\|_F^2 = \frac{1}{nT^2}\operatorname{tr}\left[\mathcal{A}'\mathcal{A}\right] = \frac{1}{T^2}\operatorname{tr}\left[A'A\right] = \frac{T(T-1)}{2T^2} \to \frac{1}{2}.$$

Next, we show that the variance converges to zero. By the arguments in (A.1), with $D_\eta = \mathcal{A}'\Sigma_\eta^{-1}\mathcal{A}$,

$$n^2T^4\operatorname{var}(J_{n,T}^{\mathrm{PANIC}}) \leq 2\left\|\Sigma_\eta D_\eta\right\|_F^2.$$

The required order is now easily verified, since

$$\begin{aligned}\left\|\Sigma_\eta D_\eta\right\|_F &\leq \left\|\mathcal{A}'\Psi_\eta^{-1}\mathcal{A}\Sigma_\eta\right\|_F + \left\|\Sigma_\eta C_\eta\mathcal{A}\right\|_F\\ &\leq \left\|\mathcal{A}'\mathcal{A}\right\|_F + \left\|\mathcal{A}'\Psi_\eta^{-1}\mathcal{A}(\Sigma_\eta - \Psi_\eta)\right\|_F + \left\|\Sigma_\eta C_\eta\mathcal{A}\right\|_F\end{aligned}$$

and $\left\|\mathcal{A}'\mathcal{A}\right\|_F = \sqrt{n}\left\|A'A\right\|_F \leq \sqrt{n}\left\|A\right\|_F^2 = \sqrt{n}T(T-1)/2$. ■

A.2.2. Proof of Lemma 3.2

Proof: In the following all probabilities and expectations are evaluated under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$. The proof of this lemma follows the idea of the proof of Lemma 3.1 by considering means and variances. The proof that $J_{n,T}^{\mathrm{MP}}$ converges to $\frac{1}{2}$ in L_2 is almost identical to its counterpart in the proof of Lemma 3.1: just replace η by ε, Σ_η by Σ_ε, C_η by C_ε etc. The same replacements yield that the variance of $\tilde{\Delta}_{n,T}^{\mathrm{MP}} - \Delta_{n,T}^{\mathrm{MP}}$ converges to zero, by applying them to the arguments starting at (A.1). We are left to show that the expectation of $\tilde{\Delta}_{n,T}^{\mathrm{MP}} - \Delta_{n,T}^{\mathrm{MP}}$ converges to zero. This remaining expectation is more complicated since the variance matrices Σ_ε and Ψ_ε have additional terms due to the presence of unobservable factors.

Recall, under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$, $\Delta Y = \varepsilon$ and note

$$\tilde{\Delta}_{n,T}^{\mathrm{MP}} - \Delta_{n,T}^{\mathrm{MP}} = \frac{1}{\sqrt{n}}\left(\frac{1}{T}\varepsilon'\mathcal{A}'\left(\Psi_\varepsilon^{-1} - \Sigma_\varepsilon^{-1}\right)\varepsilon - \sum_{i=1}^n \frac{\delta_{\eta,i,T}}{\omega_{\eta,i,T}^2}\right).$$

Thus, we have

$$
\begin{aligned}
\mathrm{E}[\tilde{\Delta}^{\mathrm{MP}}_{n,T} - \Delta^{\mathrm{MP}}_{n,T}] &= \frac{1}{\sqrt{n}T}\,\mathrm{tr}[\mathcal{A}'\Psi_\varepsilon^{-1}\Sigma_\varepsilon] - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\delta_{\eta,i,T}}{\omega^2_{\eta,i,T}} \\
&= \frac{1}{\sqrt{n}T}\,\mathrm{tr}[\mathcal{A}'\Psi_\eta^{-1}\Sigma_\eta] - \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\delta_{\eta,i,T}}{\omega^2_{\eta,i,T}} \\
&+ \frac{1}{\sqrt{n}T}\sum_{k=1}^{K}\mathrm{tr}\left[\psi_\varepsilon^{-1}\lambda_k\lambda_k'\otimes A'\Sigma_{f,k}\right] \\
&+ \frac{1}{\sqrt{n}T}\,\mathrm{tr}\left[\left((\psi_\varepsilon^{-1}-\Omega_\eta^{-1})\otimes A'\right)\Sigma_\eta\right] =: I + II + III.
\end{aligned}
$$

In the proof of Lemma 3.1 we have established that the first term equals zero. Therefore, the current proof is complete once we show the final two terms converge to zero.

Convergence to zero of II follows from $\frac{1}{T}\,\mathrm{tr}(A'\Sigma_{f,k}) = \delta_{f,k,T} = O(1)$ in combination with

$$
\begin{aligned}
\sum_{k=1}^{K}\mathrm{tr}\left[\psi_\varepsilon^{-1}\lambda_k\lambda_k'\right] &= \mathrm{tr}[\Lambda'\psi_\varepsilon^{-1}\Lambda] = \mathrm{tr}\left[\Omega_F^{-1} - \Omega_F^{-1}\left(\Omega_F^{-1}+\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\Omega_F^{-1}\right] \\
&\leq \mathrm{tr}\left[\Omega_F^{-1}\right] = \sum_{k=1}^{K}\frac{1}{\omega^2_{f,k,T}} \to \sum_{k=1}^{K}\frac{1}{\omega^2_{f,k}} < \infty.
\end{aligned}
$$

Convergence to zero of III follows from

$$
\begin{aligned}
|III| &\leq \frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\left(\Omega_\eta^{-1}\Lambda\left(\Omega_F^{-1}+\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\Lambda'\Omega_\eta^{-1}\right)_{i,i}|\,\mathrm{tr}\,[A'\Sigma_{\eta,i}]\,| \\
&\leq \frac{1}{\sqrt{n}T}\,\mathrm{tr}(\Omega_\eta^{-1}\Lambda\left(\Omega_F^{-1}+\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\Lambda'\Omega_\eta^{-1})\sup_i|\,\mathrm{tr}\,[A'\Sigma_{\eta,i}]\,|. \\
&\leq \frac{1}{\sqrt{n}T}\,\|\Lambda\|_F^2\left\|\Omega_\eta^{-1}\right\|^2_{\mathrm{spec}}\left\|\left(\Omega_F^{-1}+\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_{\mathrm{spec}}\sup_i|\,\mathrm{tr}\,[A'\Sigma_{\eta,i}]\,|.
\end{aligned}
$$

Observe $\sup_i \mathrm{tr}\,[A'\Sigma_{\eta,i}] = O(T)$ by Item 4 of Lemma A.1. From Assumption 2.2 we get $\|\Lambda\|_F = O(\sqrt{n})$ and

$$
\begin{aligned}
n\left\|\left(\Omega_F^{-1}+\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_{\mathrm{spec}} &= \left\|\left(\frac{1}{n}\Omega_F^{-1}+\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_{\mathrm{spec}} \\
&= \lambda^{-1}_{min}(\frac{1}{n}\Omega_F^{-1}+\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda) \leq \lambda^{-1}_{min}(\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda) \\
&\leq \lambda^{-1}_{min}(\frac{1}{n}\Lambda'\Lambda)\sup_{i\in\mathbb{N}}\omega^2_{\eta,i,T} \to \lambda^{-1}_{min}(\Psi_\Lambda)\sup_{i\in\mathbb{N}}\omega^2_{\eta,i} < \infty.
\end{aligned}
$$

A combination of these observations with the penultimate display yields $III = o(1)$. ■

A.2.3. Proof of Lemma 3.3

Proof: We have

$$
\begin{aligned}
|\Delta^*_{n,T} - \tilde{\Delta}^{\mathrm{MP}}_{n,T}| &= \frac{1}{\sqrt{nT}} |\operatorname{tr}(A\tilde{\varepsilon}(\psi_\varepsilon^{*-1} - \psi_\varepsilon{}^{-1})\tilde{\varepsilon}')| \\
&\leq \frac{1}{\sqrt{nT}} \left\| \psi_\varepsilon^{*-1} - \psi_\varepsilon{}^{-1} \right\|_F \|\tilde{\varepsilon}' A \tilde{\varepsilon}\|_F .
\end{aligned}
$$

We consider each norm separately. We have

$$
\begin{aligned}
\left\| \psi_\varepsilon^{*-1} - \psi_\varepsilon{}^{-1} \right\|_F &\leq \left\| (\Lambda'\Omega_\eta^{-1}\Lambda + \Omega_F)^{-1} - (\Lambda'\Omega_\eta^{-1}\Lambda)^{-1} \right\|_{\mathrm{spec}} \left\| \Omega_\eta^{-1} \right\|^2_{\mathrm{spec}} \|\Lambda\|_F^2 \\
&= O(n^{-2})O(1)O(n) = O(n^{-1}),
\end{aligned}
$$

as $\|\Lambda\|_F = O(\sqrt{n})$ by Assumption 2.2, $\left\|\Omega_\eta^{-1}\right\|_{\mathrm{spec}} = O(1)$ by Assumption 2.1, and

$$
\begin{aligned}
& n \left\| (\Lambda'\Omega_\eta^{-1}\Lambda + \Omega_F)^{-1} - (\Lambda'\Omega_\eta^{-1}\Lambda)^{-1} \right\|_{\mathrm{spec}} \\
&= \left\| \left(\frac{\Lambda'\Omega_\eta^{-1}\Lambda}{n} + \frac{\Omega_F}{n} \right)^{-1} - \left(\frac{\Lambda'\Omega_\eta^{-1}\Lambda}{n} \right)^{-1} \right\|_{\mathrm{spec}} \\
&= \left\| - \left(\frac{\Lambda'\Omega_\eta^{-1}\Lambda}{n} + \frac{\Omega_F}{n} \right)^{-1} \frac{\Omega_F}{n} \left(\frac{\Lambda'\Omega_\eta^{-1}\Lambda}{n} \right)^{-1} \right\|_{\mathrm{spec}} \\
&\leq \left\| \frac{\Omega_F}{n} \right\|_{\mathrm{spec}} \left\| \left(\frac{\Lambda'\Omega_\eta^{-1}\Lambda}{n} + \frac{\Omega_F}{n} \right)^{-1} \right\|_{\mathrm{spec}} \left\| \left(\frac{\Lambda'\Omega_\eta^{-1}\Lambda}{n} \right)^{-1} \right\|_{\mathrm{spec}} ,
\end{aligned}
$$

which is $O(n^{-1})$: the second norm converges to the third, which is O(1) by Item 1 of Lemma A.6. For $\|\tilde{\varepsilon}' A \tilde{\varepsilon}\|_F$, we note that $\|\tilde{\varepsilon}' A \tilde{\varepsilon}\|_F = \left\| \tilde{\varepsilon}' \frac{A+A'}{2} \tilde{\varepsilon} \right\|_F$ and recall that $A + A' = \iota\iota' - I_T$, so that

$$
2 \|\tilde{\varepsilon}' A \tilde{\varepsilon}\|_F = \|\tilde{\varepsilon}'(\iota\iota' - I_T)\tilde{\varepsilon}\|_F \leq \|\iota'\tilde{\varepsilon}\|_F^2 + \|\tilde{\varepsilon}\|_F^2 = O_p(nT),
$$

as $\|\tilde{\varepsilon}\|_F \leq \|\Lambda\|_F \left\|\tilde{f}\right\|_F + \|\tilde{\eta}\|_F = O(\sqrt{n})O_p(\sqrt{T}) + O_p(\sqrt{nT})$ and, using Items 2 and 3 of Lemma A.6, a similar bound holds for $\|\iota'\tilde{\varepsilon}\|_F$. Conclude that the central sequence difference is $O_p(n^{-1/2})$. ■

A.2.4. Proof of Lemma 3.4

Proof: As ψ_ε^{*-1} projects out the factors, we have

$$\begin{aligned}\Delta^*_{n,T} - \Delta_{n,T} &= \frac{1}{\sqrt{nT}}\operatorname{tr}(A\tilde{\varepsilon}\psi_\varepsilon^{*-1}\tilde{\varepsilon}') - \frac{1}{\sqrt{nT}}\operatorname{tr}(A\tilde{\eta}\Omega_\eta{}^{-1}\tilde{\eta}') \\ &= \frac{1}{\sqrt{nT}}\operatorname{tr}(A\tilde{\eta}(\psi_\varepsilon^{*-1} - \Omega_\eta{}^{-1})\tilde{\eta}').\end{aligned}$$

Note that for a symmetric matrix B,

$$\operatorname{tr}(A\tilde{\eta}B\tilde{\eta}') = \operatorname{tr}(\tilde{\eta}B\tilde{\eta}'A') = \operatorname{tr}(A'\tilde{\eta}B\tilde{\eta}') = \operatorname{tr}\left(\frac{A+A'}{2}\tilde{\eta}B\tilde{\eta}'\right),$$

so, as ψ_ε^{*-1} and Ω_η are symmetric and $A + A' = \iota\iota' - I_T$, we have

$$\begin{aligned}|\operatorname{tr}(A\tilde{\eta}(\psi_\varepsilon^{*-1} - \Omega_\eta{}^{-1})\tilde{\eta}')| &= \frac{1}{2}|\operatorname{tr}((\iota\iota' - I_T)\tilde{\eta}(\psi_\varepsilon^{*-1} - \Omega_\eta{}^{-1})\tilde{\eta}')| \\ &\leq |\operatorname{tr}(\iota'\tilde{\eta}(\psi_\varepsilon^{*-1} - \Omega_\eta{}^{-1})\tilde{\eta}'\iota)| + |\operatorname{tr}(\tilde{\eta}(\psi_\varepsilon^{*-1} - \Omega_\eta{}^{-1})\tilde{\eta}')| \\ &\leq \left\|\left(\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_F\left(\left\|\iota'\tilde{\eta}\Omega_\eta^{-1}\Lambda\right\|_F^2 + \left\|\tilde{\eta}\Omega_\eta^{-1}\Lambda\right\|_F^2\right) \\ &= O(n^{-1})(O_p(nT) + O_p(nT)) = O_p(T),\end{aligned}$$

using Items 1, 4, and 5 of Lemma A.6. ■

A.2.5. Proof of Proposition 3.1

Proof: Apply Lemma A.5 with $a_{i,n,T} = 1$ for all i, n, T. ■

A.3. Proofs of Section 4

A.3.1. Proof of Lemma 4.1

Remark A.2: *The proof follows along similar lines as that of Moon & Perron (2004). By treating the norm of $\tilde{\eta}'\tilde{\eta}$ differently, we obtain, under the assumptions of this paper, $\left\|\Lambda H_K - \hat{\Lambda}\right\|_F = o_p(1)$ instead of the $O_p(1)$ obtained by Moon & Perron (2004). In particular, we exploit $\|\tilde{\eta}'\tilde{\eta}\|_{spec} = o_p(\sqrt{n}T)$, whereas Moon & Perron (2004) only use $\|\tilde{\eta}'\tilde{\eta}\|_F = O_p(\sqrt{n}T)$.*

Proof: As Moon & Perron (2004), we take $H_K = \frac{\tilde{f}'\tilde{f}}{T}\frac{\Lambda'\bar{\Lambda}}{n}$. First note that from the definitions of H_K and $\hat{\Lambda}$ and using $\tilde{\varepsilon} = \tilde{f}\Lambda' + \tilde{\eta}$ we have

$$\hat{\Lambda} - \Lambda H_K = \frac{1}{nT}(\tilde{\varepsilon}'\tilde{\varepsilon} - \Lambda\tilde{f}'\tilde{f}\Lambda')\bar{\Lambda} = \frac{1}{nT}(\tilde{\eta}'\tilde{f}\Lambda' + \Lambda\tilde{f}'\tilde{\eta} + \tilde{\eta}'\tilde{\eta})\bar{\Lambda},$$

so that

$$
\begin{aligned}
\left\|\Lambda H_K-\hat{\Lambda}\right\|_F \leq & \frac{\left\|\tilde{\eta}'\tilde{f}\Lambda'\bar{\Lambda}\right\|_F}{nT}+\frac{\left\|\Lambda\tilde{f}'\tilde{\eta}\bar{\Lambda}\right\|_F}{nT}+\frac{1}{nT}\left\|\tilde{\eta}'\tilde{\eta}\bar{\Lambda}\right\|_F \\
\leq & 2\sqrt{\frac{n}{T}}\frac{\left\|\tilde{\eta}'\tilde{f}\right\|_F}{\sqrt{nT}}\frac{\|\Lambda\|_F}{\sqrt{n}}\frac{\left\|\bar{\Lambda}\right\|_F}{\sqrt{n}}+\frac{1}{nT}\left\|\tilde{\eta}'\tilde{\eta}\right\|_{\text{spec}}\left\|\bar{\Lambda}\right\|_F .
\end{aligned}
\tag{A.2}
$$

By the definition of $\bar{\Lambda}$, $\left\|\bar{\Lambda}\right\|_F=\sqrt{nK}=O(\sqrt{n})$. We have

$$
\begin{aligned}
\mathrm{E}\left\|\tilde{\eta}'\tilde{f}\right\|_F^2 = & \mathrm{E}\sum_{k=1}^{K}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}f_{kt}\eta_{it}\right)^2 \\
= & \sum_{k=1}^{K}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\gamma_{\eta,i}(t-s)\gamma_{f,k}(t-s) \\
\leq & Mn\sum_{k=1}^{K}\sum_{t=1}^{T}\sum_{s=1}^{T}|\gamma_{f,k}(t-s)| \\
= & Mn\sum_{k=1}^{K}\sum_{m=-(T-1)}^{T-1}(T-|m|)|\gamma_{f,k}(m)|=O(nT),
\end{aligned}
$$

for some finite constant M, using that, thanks to Assumption 2.1, $\gamma_{\eta,i}(t-s)$ is bounded uniformly in i and $t-s$. Thus, each term of the first summand in (A.2) is $O_p(1)$.

Finally, we consider the second summand, which is treated differently from Moon & Perron (2004). We obtain $\left\|\Lambda H_K-\hat{\Lambda}\right\|_F=o_p(1)$ if we can indeed show that $\|\tilde{\eta}'\tilde{\eta}\|_{\text{spec}}=o_p(\sqrt{n}T)$ (Moon & Perron (2004) only use $\|\tilde{\eta}'\tilde{\eta}\|_F=O_p(\sqrt{n}T)$). For this, note that $\frac{1}{T}\tilde{\eta}'\tilde{\eta}=\frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{\cdot,t}\tilde{\eta}_{\cdot,t}'$, which can be considered an approximation to $\Gamma_\eta:=\operatorname{diag}(\gamma_{\eta,1}(0),\ldots,\gamma_{\eta,n}(0))$, the $n\times n$ cross-sectional covariance matrix of the η. From Assumption 2.1, $\|\Gamma_\eta\|_{\text{spec}}<\infty$. We now show that indeed the approximation works. Using Isserlis' Theorem to write

$\mathrm{E}[\eta_{i,t}^2\eta_{i,s}^2] = 2\gamma_{\eta,i}(t-s)^2 + \mathrm{E}[\eta_{i,t}^2]\mathrm{E}[\eta_{i,s}^2]$, we have

$$\begin{aligned}
\mathrm{E}\left\|\frac{1}{T}\sum_{t=1}^{T}\tilde{\eta}_{\cdot,t}\tilde{\eta}_{\cdot,t}' - \Gamma_\eta\right\|_F^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left(\frac{1}{T}\sum_{t=1}^{T}\eta_{i,t}\eta_{j,t} - \mathrm{E}[\eta_{i,t}\eta_{j,t}]\right)^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{E}[\eta_{i,t}\eta_{j,t}\eta_{i,s}\eta_{j,s}] - \mathrm{E}[\eta_{i,t}\eta_{j,t}]\mathrm{E}[\eta_{i,s}\eta_{j,s}] \\
&= \sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}2\gamma_{\eta,i}(t-s)^2 \\
&\quad + \sum_{i\neq j}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\gamma_{\eta,i}(t-s)\gamma_{\eta,j}(t-s) \\
&= O(n/T) + O(n^2/T).
\end{aligned}$$

Conclude that the difference in Frobenius norm is $O_p(n/\sqrt{T})$.

Remark A.3: *Note that, even without Gaussianity, this conclusion holds as long as the long-run variances of the $\{\eta_{i,t}^2\}$ are uniformly bounded.*

Thus,

$$\begin{aligned}
\|\tilde{\eta}'\tilde{\eta}\|_{\text{spec}} \leq & \left\|\sum_{t=1}^{T}\tilde{\eta}_{\cdot,t}\tilde{\eta}_{\cdot,t}' - T\Gamma_\eta\right\|_F + \|T\Gamma_\eta\|_{\text{spec}} \\
=& O_p(n\sqrt{T}) + O(T) = o_p(\sqrt{n}T).
\end{aligned}$$

Finally, we show the boundedness properties of H_K. First note that

$$\|H_K\|_F \leq \frac{\left\|\tilde{f}'\tilde{f}\right\|_F}{T}\frac{\|\Lambda\|_F}{\sqrt{n}}\frac{\left\|\bar{\Lambda}\right\|_F}{\sqrt{n}} = O_p(1).$$

To show boundedness of the inverse, we will show that the limiting eigenvalues of H_K are positive. Introduce $\Gamma_f := \mathrm{diag}(\gamma_{f,1}(0), \ldots, \gamma_{f,K}(0))$, the $K \times K$ covariance matrix of the f, and write

$$\left\|H_K - \Gamma_f\frac{\Lambda'\bar{\Lambda}}{n}\right\|_{\text{spec}} \leq \left\|\frac{\Lambda'\bar{\Lambda}}{n}\right\|_F\left\|\frac{\tilde{f}'\tilde{f}}{T} - \Gamma_F\right\|_F = O_p(1)o_p(1),$$

where the latter follows from Assumption 2.1. As Γ_F has full rank, it is sufficient to show that the eigenvalues of $\frac{\Lambda'\bar{\Lambda}}{n}$ are bounded away from zero. $\bar{\Lambda}$ is defined through the eigenvectors of $\tilde{\varepsilon}'\tilde{\varepsilon}/(nT)$. As the eigenvalues of $\tilde{\varepsilon}'\tilde{\varepsilon}$ are closely related to those of $\Lambda\tilde{f}'\tilde{f}\Lambda'$, we can use this relation to learn about the rank of

$\Lambda'\bar{\Lambda}$. Formally, define D to be the $K \times K$ matrix with the K largest eigenvalues of $\tilde{\varepsilon}'\tilde{\varepsilon}/(nT)$. Then, from the definition of $\bar{\Lambda}$,

$$D = \frac{\bar{\Lambda}'}{\sqrt{n}} \frac{\tilde{\varepsilon}'\tilde{\varepsilon}}{nT} \frac{\bar{\Lambda}}{\sqrt{n}}.$$

Recalling some of the above results we obtain

$$\left\| \frac{\tilde{\varepsilon}'\tilde{\varepsilon}}{nT} - \frac{\Lambda \tilde{f}'\tilde{f}\Lambda'}{nT} \right\|_{\text{spec}} = o_p(n^{-1/2}), \tag{A.3}$$

so that

$$D = \frac{\bar{\Lambda}'}{\sqrt{n}} \frac{\Lambda \tilde{f}'\tilde{f}\Lambda'}{nT} \frac{\bar{\Lambda}}{\sqrt{n}} + o_p(n^{-1/2}) = \frac{\bar{\Lambda}'\Lambda}{n} \Gamma_f \frac{\Lambda'\bar{\Lambda}}{n} + o_p(1).$$

As the Kth largest eigenvalue of $\tilde{\varepsilon}'\tilde{\varepsilon}/(nT)$ is bounded away from zero (using (A.3) the nonzero limiting eigenvalues are given by those of $\Psi_\Lambda \Gamma_F$, a product of two rank K matrices), so must the limit of $\frac{\Lambda'\bar{\Lambda}}{n}$ and thus H_K. ■

A.3.2. Proof of Lemma 4.2

Proof: First note that

$$\mathrm{E}\left\|\hat{\Omega}_\eta - \Omega_\eta\right\|_F^2 = \sum_{i=1}^n \mathrm{E}(\hat{\omega}_{\eta,i}^2 - \omega_{\eta,i}^2)^2 \le n \max_{i=1,\dots,n} \mathrm{E}(\hat{\omega}_{\eta,i}^2 - \omega_{\eta,i}^2)^2 = o(1),$$

from Assumption 4.1. Thus both $\left\|\hat{\Omega}_\eta - \Omega_\eta\right\|_F$ and $\left\|\hat{\Omega}_\eta - \Omega_\eta\right\|_{\mathrm{spec}}$ are $o_p(1)$. Together with Assumption 2.1 this also implies, with probability converging to one,

$$0 < \frac{\inf_{i\in\mathbb{N}} \omega_{\eta,i}^2}{2} < \min_{i=1,\dots,n} \hat{\omega}_{\eta,i}^2 \le \max_{i=1,\dots,n} \hat{\omega}_{\eta,i}^2 < 2 \sup_{i\in\mathbb{N}} \omega_{\eta,i}^2 < \infty.$$

Therefore, $\left\|\hat{\Omega}_\eta^{-1}\right\|_{\mathrm{spec}} = O_p(1)$, so that finally also $\left\|\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1}\right\|_F$ and $\left\|\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1}\right\|_{\mathrm{spec}}$ are $o_p(1)$. Similarly, we note for the one-sided long-run variances that $\sum_{i=1}^n (\hat{\delta}_{\eta,i} - \delta_{\eta,i})^2 = o_p(1)$ follows from Assumption 4.1, so that, along the same lines, we obtain $\max_{i=1,\dots,n} \hat{\delta}_{\eta,i} = O_p(1)$.

We split the central sequence difference in three parts: one for replacing ψ_ε^* with $\hat{\psi}_\varepsilon$, one to take care of the initial value, and one for estimating the correction term. Thus $\hat{\Delta}_{n,T} - \Delta_{n,T}^* = I - II - III$, with

$$I = \frac{1}{\sqrt{nT}} \operatorname{tr}(A'\tilde{\varepsilon}\left(\hat{\psi}_\varepsilon^{-1} - \psi_\varepsilon^{*-1}\right)\tilde{\varepsilon}')$$

$$II = \frac{1}{\sqrt{nT}} \sum_{t=2}^T \varepsilon_{\cdot,1}' \hat{\psi}_\varepsilon^{-1} \varepsilon_{\cdot,t}$$

$$III = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{\hat{\delta}_{\eta,i}}{\hat{\omega}_{\eta,i}^2} - \frac{\delta_{\eta,i}}{\omega_{\eta,i}^2} \right).$$

For part I, insert Equations (11) and (13) to find

$$\begin{aligned}
|I| &= \frac{1}{\sqrt{nT}} |\operatorname{tr}(\tilde{\varepsilon}' A' \tilde{\varepsilon}(\hat{\psi}_\varepsilon^{-1} - \psi_\varepsilon^{*-1}))| \\
&\le \frac{1}{\sqrt{nT}} |\operatorname{tr}(\tilde{\varepsilon}' A' \tilde{\varepsilon}(\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1})| \\
&+ \frac{1}{\sqrt{nT}} \left|\operatorname{tr}\left(\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\tilde{\varepsilon}'A'\tilde{\varepsilon}\hat{\Omega}_\eta^{-1}\hat{\Lambda}\left(\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right)^{-1} - \Lambda'\Omega_\eta^{-1}\tilde{\varepsilon}'A'\tilde{\varepsilon}\Omega_\eta^{-1}\Lambda\left(\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right)\right| \\
&\le \frac{1}{\sqrt{nT}} |\operatorname{tr}(\tilde{\varepsilon}' A' \tilde{\varepsilon}(\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1})| \\
&+ \frac{1}{\sqrt{nT}} \left\|\hat{\Omega}_\eta^{-1}\hat{\Lambda}\left(\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right)^{-1}\hat{\Lambda}'\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1}\Lambda\left(\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\Lambda'\Omega_\eta^{-1}\right\|_F \|\tilde{\varepsilon}'A'\tilde{\varepsilon}\|_F.
\end{aligned}$$

As $\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1}$ is diagonal, the first summand is bounded by (using Cauchy-Schwarz)

$$\frac{1}{\sqrt{nT}} \left(\sum_{i=1}^{n} (\varepsilon_i' A \varepsilon_i)^2 \right)^{1/2} \left\| \hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1} \right\|_F = \frac{1}{\sqrt{nT}} O_p(\sqrt{n}T) o_p(1) = o_p(1).$$

For II, we have

$$\begin{aligned}
\sqrt{n}TII &\leq \left\| \hat{\psi}_\varepsilon^{-1} \right\|_{\text{spec}} \|\varepsilon_{\cdot,1}\|_F \left(\|\iota' \tilde{\varepsilon}\|_F + \|\varepsilon_{\cdot,1}\|_F \right) \\
&= O_p(1) O_p(\sqrt{n}) (O_p(\sqrt{nT}) + O_p(\sqrt{n}) = O_p(n\sqrt{T}),
\end{aligned}$$

where $\|\varepsilon_{\cdot,1}\|_F \leq \|\Lambda\|_F \left\| \tilde{f}_{\cdot,1} \right\|_F + \|\tilde{\eta}_\cdot, 1\|_F = O_p(\sqrt{n})$ and $\left\| \hat{\psi}_\varepsilon^{-1} \right\|_{\text{spec}} = O_p(1)$ follows from Assumption 4.1 and Item 2 of Lemma A.7 implying

$$\begin{aligned}
\left\| \hat{\psi}_\varepsilon^{-1} - \psi_\varepsilon^{*-1} \right\|_{\text{spec}} &= O_p(n^{-1/2}) \text{ and} \\
\left\| \psi_\varepsilon^{*-1} \right\|_{\text{spec}} &\leq \left\| \Omega_\eta^{-1} \right\|_{\text{spec}} + \left\| \Omega_\eta^{-1} \right\|_{\text{spec}}^2 \|\Lambda\|_F^2 \left\| \left(\Lambda' \Omega_\eta^{-1} \Lambda \right)^{-1} \right\|_F \\
&= O(1) + O(1) O(n) O(n^{-1}) = O(1),
\end{aligned}$$

using Assumptions 2.1 and 2.2 and Item 1 of Lemma A.6. We conclude that $II = O_p\left(\frac{\sqrt{n}}{\sqrt{T}} \right) = o_p(1)$.

Finally, we obtain for III:

$$\begin{aligned}
III =& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\omega_{\eta,i}^2} (\hat{\delta}_{\eta,i} - \delta_{\eta,i}) + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\hat{\delta}_{\eta,i}}{\hat{\omega}_{\eta,i}^2 \omega_{\eta,i}^2} (\omega_{\eta,i}^2 - \hat{\omega}_{\eta,i}^2) \\
\leq& \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{(\omega_{\eta,i}^2)^2} \right)^{1/2} \left(\sum_{i=1}^{n} (\hat{\delta}_{\eta,i} - \delta_{\eta,i})^2 \right)^{1/2} \\
&+ \left(\frac{1}{n} \sum_{i=1}^{n} \frac{\hat{\delta}_{\eta,i}^2}{(\hat{\omega}_{\eta,i}^2 \omega_{\eta,i}^2)^2} \right)^{1/2} \left(\sum_{i=1}^{n} (\hat{\omega}_{\eta,i}^2 - \omega_{\eta,i}^2)^2 \right)^{1/2},
\end{aligned}$$

which is indeed $o_p(1)$ thanks to the observations at the beginning of this proof. ■

A.4. Auxiliary Lemmas

Lemma A.7: *Consider the factor estimates and the H_K from Lemma 4.1. Then, under Assumptions 2.1, 2.2, 2.4, 2.5, and 4.1, under $\mathrm{P}_{0,n,T}^{MP}$ or $\mathrm{P}_{0,n,T}^{PANIC}$ and as $n, T \to \infty$, we have*

1. $\left\| \left(\hat{\Lambda}' \hat{\Omega}_\eta^{-1} \hat{\Lambda} \right)^{-1} - \left(H_K' \Lambda' \Omega_\eta^{-1} \Lambda H_K \right)^{-1} \right\|_F = o_p(n^{-3/2})$, *and*

2. $\left\|\hat{\Omega}_\eta^{-1}\hat{\Lambda}\left(\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right)^{-1}\hat{\Lambda}'\hat{\Omega}_\eta^{-1}-\Omega_\eta^{-1}\Lambda\left(\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\Lambda'\Omega_\eta^{-1}\right\|_F=o_p(n^{-1/2})$.

Proof: We start by noting that $\left\|H_K'\Lambda'\Omega_\eta^{-1}\Lambda H_K-\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right\|_F=o_p(\sqrt{n})$: The terms for approximating the loadings are negligible thanks to $\left\|\Lambda H_K-\hat{\Lambda}\right\|_F$ (Lemma 4.1) and $\left\|\Omega_\eta^{-1}-\hat{\Omega}_\eta^{-1}\right\|_{\text{spec}}$ being $o_p(1)$ in combination with H_K being bounded and $\left\|\Omega_\eta^{-1}\right\|_{\text{spec}}=O(1)$. The term due to approximating the long-run variances, $H_K'\Lambda'(\Omega_\eta^{-1}-\hat{\Omega}_\eta^{-1})\hat{\Lambda}$, can again be treated using Cauchy-Schwarz: Ignoring H_K, its (k,l)th entry is given by

$$\begin{aligned}\sum_{i=1}^n\lambda_{ik}\hat{\lambda}_{il}((\hat{\omega}_{\eta,i}^2)^{-1}-(\omega_{\eta,i}^2)^{-1})\le&\left(\sum_{i=1}^n\lambda_{ik}\hat{\lambda}_{il}\right)^{1/2}\left\|\Omega_\eta^{-1}-\hat{\Omega}_\eta^{-1}\right\|_F\\=&O_p(\sqrt{n})o_p(1),\end{aligned}$$

thanks to the discussion at the beginning of Section A.3.2.

Next, we have that

$$\left\|\frac{1}{n}H_K'\Lambda'\Omega_\eta^{-1}\Lambda H_K\right\|_F\le\left\|H_K\right\|_F^2\frac{\|\Lambda\|_F^2}{n}\left\|\Omega_\eta^{-1}\right\|_{\text{spec}}=O(1),$$

and

$$\begin{aligned}\lambda_{\min}\left(\frac{1}{n}H_K'\Lambda'\Omega_\eta^{-1}\Lambda H_K\right)=&\left\|H_K^{-1}\left(\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}(H_K')^{-1}\right\|_{\text{spec}}^{-1}\\\ge&\left\|H_K^{-1}\right\|_F^{-2}\left\|\left(\frac{1}{n}\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\right\|_{\text{spec}}^{-1},\end{aligned}$$

which is bounded away from zero thanks to $\left\|H_K^{-1}\right\|_F$ being bounded and Item 1 of Lemma A.6. Thus, we can restrict attention to a compact subset of the invertible matrices on $\mathbb{R}^K$, on which the matrix inverse is uniformly continuous. Therefore, $\left\|\frac{1}{n}H_K'\Lambda'\Omega_\eta^{-1}\Lambda H_K-\frac{1}{n}\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right\|_F=o_p(n^{-1/2})$ implies the same for $\left\|\left(\frac{1}{n}H_K'\Lambda'\Omega_\eta^{-1}\Lambda H_K\right)^{-1}-\left(\frac{1}{n}\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right)^{-1}\right\|_F$.

For Item 2, let $a=\Omega_\eta^{-1}\Lambda H_K$ and $b=\left(H_K'\Lambda'\Omega_\eta^{-1}\Lambda H_K\right)^{-1}$ and define $\hat{a}=$

$\hat{\Omega}_\eta^{-1}\hat{\Lambda}$ and $\hat{b} = \left(\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right)^{-1}$ analogously. Thus

$$\begin{aligned}
&\left\|\hat{\Omega}_\eta^{-1}\hat{\Lambda}\left(\hat{\Lambda}'\hat{\Omega}_\eta^{-1}\hat{\Lambda}\right)^{-1}\hat{\Lambda}'\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1}\Lambda\left(\Lambda'\Omega_\eta^{-1}\Lambda\right)^{-1}\Lambda'\Omega_\eta^{-1}\right\|_F \\
&\quad = \left\|\hat{a}\hat{b}\hat{a}' - aba'\right\|_F \\
&\quad \leq \left\|\hat{a}-a\right\|_F \left\|\hat{b}\right\|_F \left\|\hat{a}\right\|_F + \left\|a\right\|_F \left\|\hat{b}-b\right\|_F \left\|\hat{a}\right\|_F + \left\|a\right\|_F \left\|b\right\|_F \left\|\hat{a}-a\right\|_F .
\end{aligned}$$

From Assumption 2.2 and H_K being bounded it follows that $\|b\|_F = O_p(n^{-1})$ and in combination with Assumption 2.1 we obtain

$$\|a\|_F \leq \left\|\Omega_\eta^{-1}\right\|_{\text{spec}} \|\Lambda\|_F \|H_K\|_F = O_p(\sqrt{n}).$$

From Item 1, $\left\|\hat{b}-b\right\|_F = o_p(n^{-3/2})$ so that also $\left\|\hat{b}\right\|_F = O_p(n^{-1})$. Finally, we have

$$\begin{aligned}
\|\hat{a}-a\|_F \leq& \left\|\hat{\Omega}_\eta^{-1} - \Omega_\eta^{-1}\right\|_{\text{spec}} \left\|\hat{\Lambda}\right\|_F \|H_K\|_F + \left\|\Omega_\eta^{-1}\right\|_{\text{spec}} \left\|\hat{\Lambda} - \Lambda H_K\right\|_F \\
=& o_p(1)O_p(\sqrt{n})O_p(1) + O(1)o_p(1) = o_p(\sqrt{n}),
\end{aligned}$$

where $\left\|\hat{\Lambda} - \Lambda H_K\right\|_F = o_p(1)$ by Lemma 4.1. Combining all these results indeed yields the correct rate. ■

Independent proof of Proposition 5.1: Here we demonstrate the joint asymptotic normality required to apply the second part of Corollary 5.1. We divide the proof into two parts. In Part A, we prove the theorem for P_a while in Part B we discuss t_a. We omit the proofs concerning P_b and t_b as they follow along the same lines.

Part A: First, we establish the joint convergence, under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$ and $\mathrm{P}_{0,n,T}^{\mathrm{PANIC}}$, of P_a and the local likelihood ratio. As already hinted at in Remark 3.4, the results in Sections 3.1 and 3.2 imply that we only have to show this convergence once to get the powers in both experiments, as both likelihood ratios are asymptotically equivalent and the models coincide under the hypothesis. Having established this joint convergence, an application of Le Cam's third lemma will lead to the asymptotic distribution of P_a under $\mathrm{P}_{h,n,T}^{\mathrm{MP}}$ and $\mathrm{P}_{h,n,T}^{\mathrm{PANIC}}$.

Specifically, Lemmas 3.1 and 3.4 imply that the limiting distributions of $\left(P_a, \log\frac{\mathrm{dP}_{h,n,T}^{\mathrm{PANIC}}}{\mathrm{dP}_{0,n,T}^{\mathrm{PANIC}}}\right)$ and $\left(P_a, \log\frac{\mathrm{dP}_{h,n,T}^{\mathrm{MP}}}{\mathrm{dP}_{0,n,T}^{\mathrm{MP}}}\right)$ are equal to that of $\left(P_a, h\Delta_{n,T} - \frac{1}{4}h^2\right)$, under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$ and $\mathrm{P}_{0,n,T}^{\mathrm{PANIC}}$. From Lemma 1 and Lemma 2 in Bai & Ng (2010) we see that P_a is adaptive with respect to the estimation of nuisance parameters while Lemma A.2 in Moon & Perron (2004) shows that $\frac{1}{nT^2}\sum_{i=1}^n E'_{i,-1}E_{i,-1}$ converges in probability to $\frac{1}{2}\omega^2$. Therefore, P_a is asymptotically equivalent to $\tilde{P}_a = \frac{\frac{1}{\sqrt{n}T}\sum_{i=1}^n E'_{i,-1}\Delta E_i - \frac{1}{\sqrt{n}}\sum_{i=1}^n \delta_{\eta,i}}{\sqrt{\phi^4/2}}$.

Under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$ or $\mathrm{P}_{0,n,T}^{\mathrm{PANIC}}$, we can compute the asymptotic distribution of all possible linear combinations of $\tilde{P}_a$ and $\Delta_{n,T}$ by an application of Lemma A.5. For all α, β in $\mathbb{R}$, we find, using $a_{i,n,T} = \alpha\frac{\omega^2_{\eta,i,T}}{\sqrt{\phi^4/2}} + \beta$ in Lemma A.5,

$$\alpha\tilde{P}_a + \beta\Delta_{n,T} \xrightarrow{d} N\left(0, \left(\alpha^2 + \alpha\beta\sqrt{\frac{2\omega^4}{\phi^4}} + \frac{\beta^2}{2}\right)\right).$$

Thus, the Cramér-Wold theorem and the asymptotic equivalence of P_a and $\tilde{P}_a$, yield, still under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$ or $\mathrm{P}_{0,n,T}^{\mathrm{PANIC}}$,

$$(P_a, \Delta_{n,T}) \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{\frac{\omega^4}{2\phi^4}} \\ \sqrt{\frac{\omega^4}{2\phi^4}} & 1/2 \end{pmatrix}\right).$$

Equivalently,

$$\left(P_a, \log\frac{\mathrm{dP}_{h,n,T}}{\mathrm{dP}_{0,n,T}}\right) \xrightarrow{d} N\left(\begin{pmatrix} 0 \\ -\frac{1}{4}h^2 \end{pmatrix}, \begin{pmatrix} 1 & h\sqrt{\frac{\omega^4}{2\phi^4}} \\ h\sqrt{\frac{\omega^4}{2\phi^4}} & 1/2h^2 \end{pmatrix}\right).$$

Applying Le Cam's third lemma, we obtain $P_a \xrightarrow{d} N\left(h\sqrt{\frac{\omega^4}{2\phi^4}}, 1\right)$ under $\mathrm{P}_{h,n,T}^{\mathrm{MP}}$ or $\mathrm{P}_{h,n,T}^{\mathrm{PANIC}}$.

Part B: As far as t_a is concerned, we recall that t_a is adaptive with respect to the estimation of nuisance parameters (see proofs of Theorem 2a) and b) in Moon & Perron (2004)) and that $\frac{1}{nT^2}\sum_{t=1}^T Y'_{\cdot,t-1}Q_\gamma Y_{\cdot,t-1}$ converges in probability to $\frac{1}{2}\omega^2$ under $\mathrm{P}_{0,n,T}^{\mathrm{MP}}$. Thus, t_a is asymptotically equivalent to

$$\tilde{t}_a = \frac{\frac{1}{\sqrt{n}T}\sum_{i=1}^n Y'_{\cdot,t-1}Q_\Lambda\Delta Y_{\cdot,t-1} - \sqrt{n}\sum_{i=1}^n \delta_{\eta,i}}{\sqrt{\phi^4/2}}.$$

Moreover, we have

$$\begin{aligned}
\frac{1}{\sqrt{n}T}\sum_{t=1}^T Y'_{\cdot,t}Q_\Lambda\Delta Y_{\cdot,t-1} &= \frac{1}{\sqrt{n}T}\sum_{t=1}^T E'_{\cdot,t}Q_\Lambda\Delta E_{\cdot,t-1} \\
&= \frac{1}{\sqrt{n}T}\sum_{t=1}^T E'_{\cdot,t}\Delta E_{\cdot,t-1} - \frac{1}{\sqrt{n}T}\sum_{t=1}^T E'_{\cdot,t}\Lambda(\Lambda'\Lambda)^{-1}\Lambda\Delta E_{\cdot,t-1} \\
&= \frac{1}{\sqrt{n}T}\sum_{i=1}^n E'_{-1,i}\Delta E_i + o_p(1),
\end{aligned}$$

where the last equality follows from the proof of Lemma 2 c) in Moon & Perron (2004). Therefore, t_a is asymptotically equivalent to $\tilde{P}_a$. Thus, following the same steps as in Part A, we find $t_a \xrightarrow{d} N\left(h\sqrt{\frac{\omega^4}{2\phi^4}}, 1\right)$ under $\mathrm{P}_{h,n,T}^{\mathrm{MP}}$ or $\mathrm{P}_{h,n,T}^{\mathrm{PANIC}}$. $\blacksquare$

Supplement B. Additional Monte-Carlo Results

In this supplement we present sizes and powers for additional DGPs and additional long-run variance estimates. The first subsection provides sizes and powers for additional DGPs. In the second subsection, we consider the same DGPs as in Section 6 of the main text and Section B.1, but now with long-run variances estimated using the Newey & West (1994) bandwidth.

B.1. Sizes and Powers in Additional DGPs

First, Table B.1 complements Table 1 in the main text with the sizes for $\sqrt{\omega^4/\phi^4} \in \{0.6, 1\}$. Next, Figures B.1 and B.2 consider the powers in the presence of MA and AR serial correlation, respectively. The results are similar to those for i.i.d innovations. Figure B.3 shows the results when the factor innovations are overdifferenced, i.e., the factor is stationary under the hypothesis. The powers appear to be unaffected. Figure B.4 considers the case of the dependence being generated by three factors, with the corresponding sizes reported in Table B.2. For very small sample sizes, powers of both tests are affected, but generally the results are similar also here.

We now consider deviations from our assumptions. Figure B.5 reports the size-corrected powers of our tests against heterogeneous alternatives of the form

$$\rho_i = 1 + \frac{hU_i}{\sqrt{n}T},$$

where the U_i are i.i.d. random variables with mean one. We draw the U_i from a Uniform(0.2,1.8) distribution. Once again, the finite-sample behavior does not appear to be affected significantly, for both small and large samples.

Finally, we consider non-Gaussian innovations. Figure B.6 reports size corrected powers with the innovations drawn from a t distribution with five degrees of freedom. The corresponding sizes are reported in Table B.3. Also here, the conclusions remain the same.

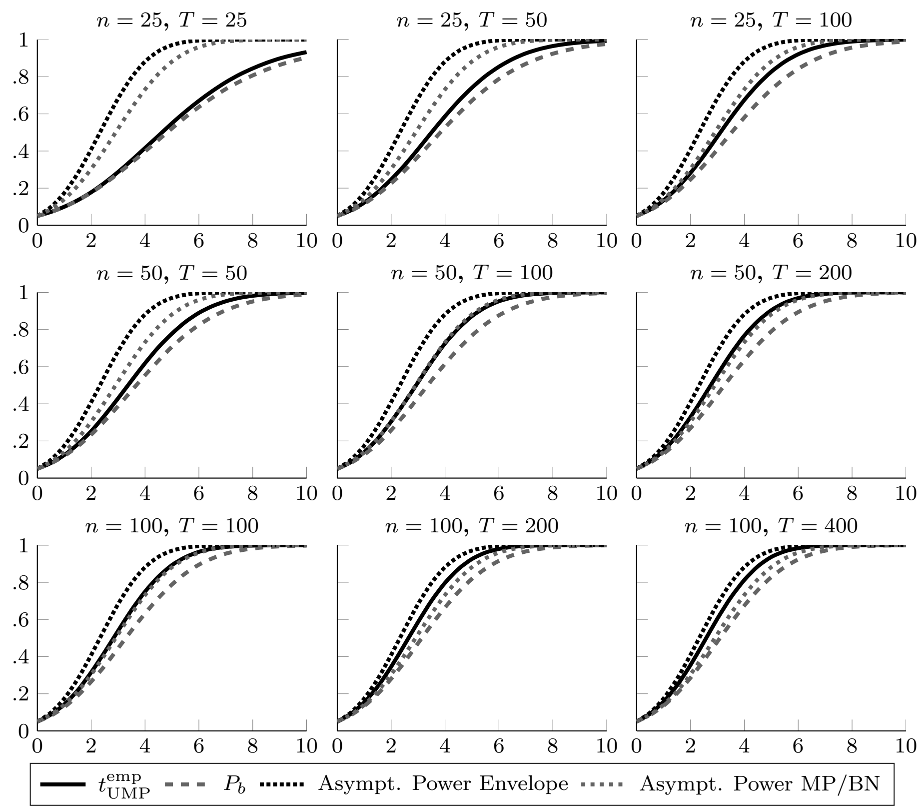

Figure B.1: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with MA factor innovations and MA idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Based on 100 000 replications.

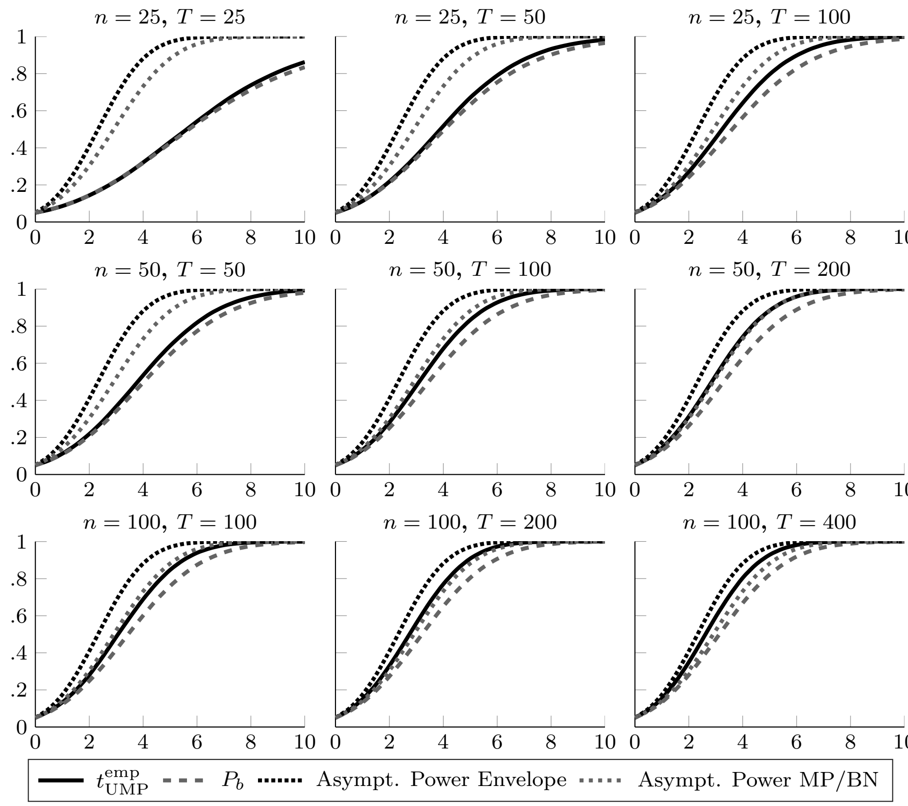

Figure B.2: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with AR factor innovations and AR idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Based on 100 000 replications.

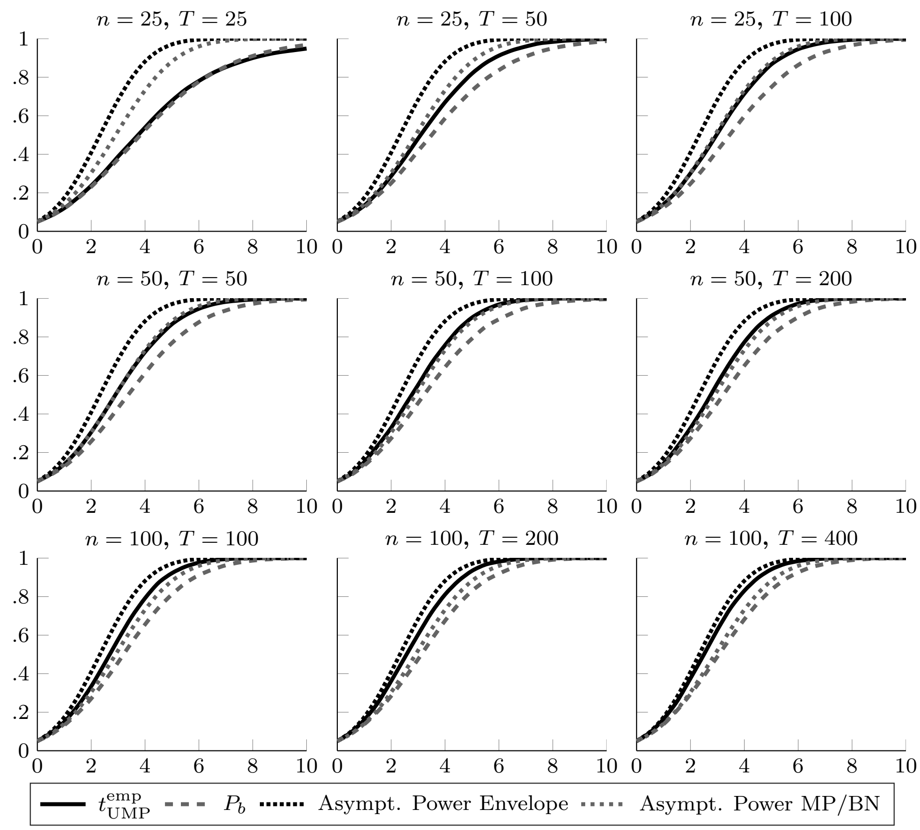

Figure B.3: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with overdifferenced i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. The factor is stationary. Based on 100 000 replications.

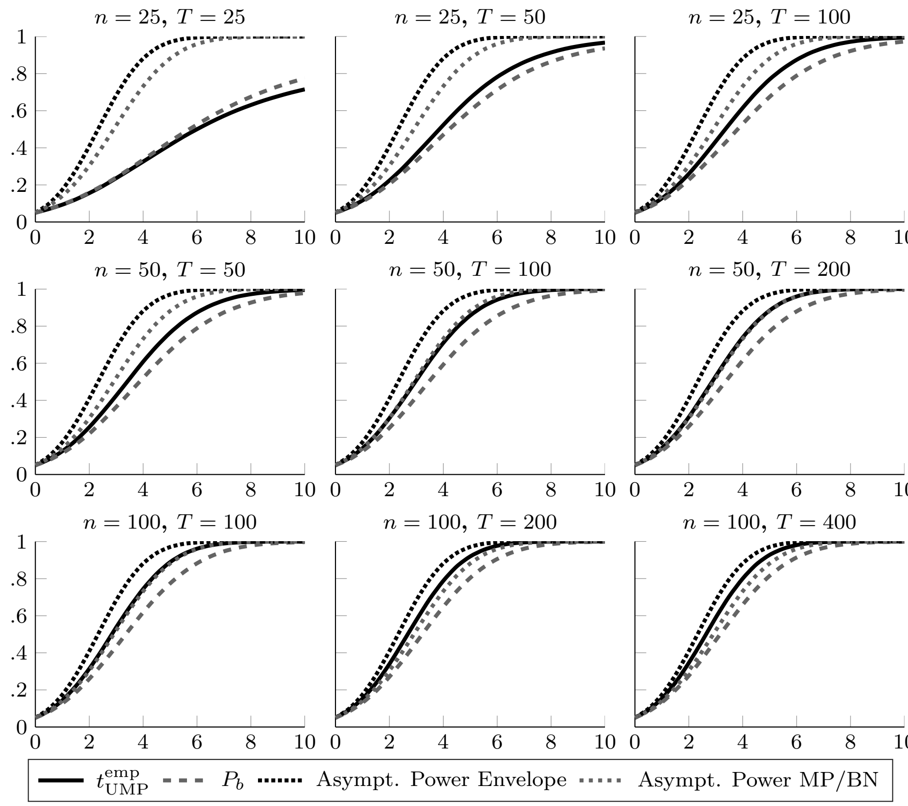

Figure B.4: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Dependence based on three factors. Based on 100 000 replications.

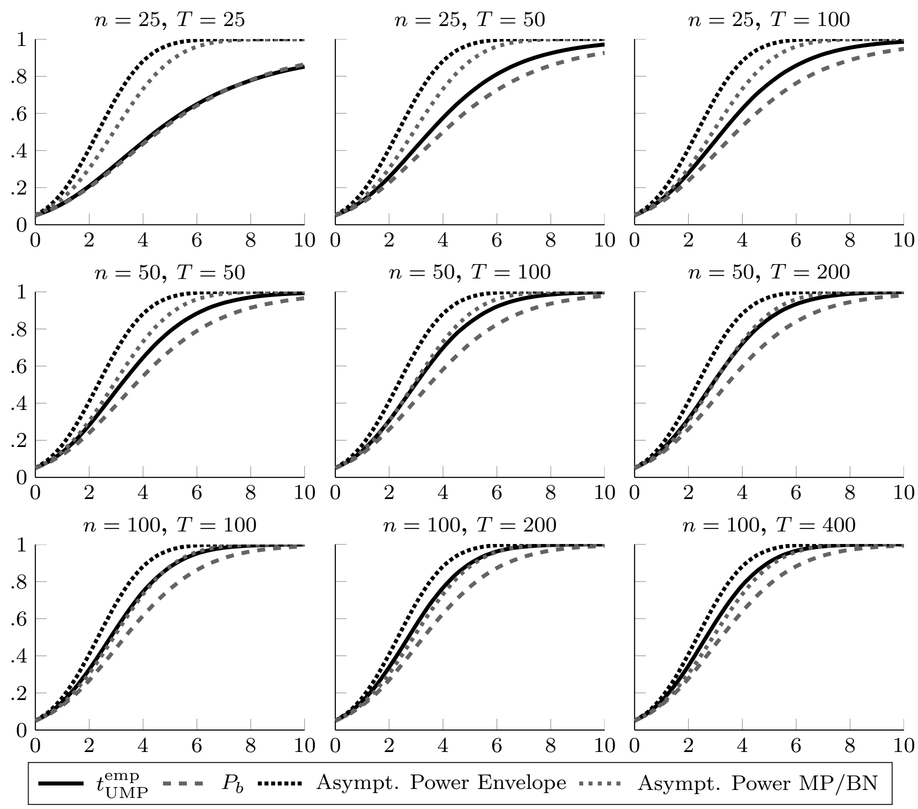

Figure B.5: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Alternatives drawn from a Uniform(0.2,1.8) distribution. Based on 100 000 replications.

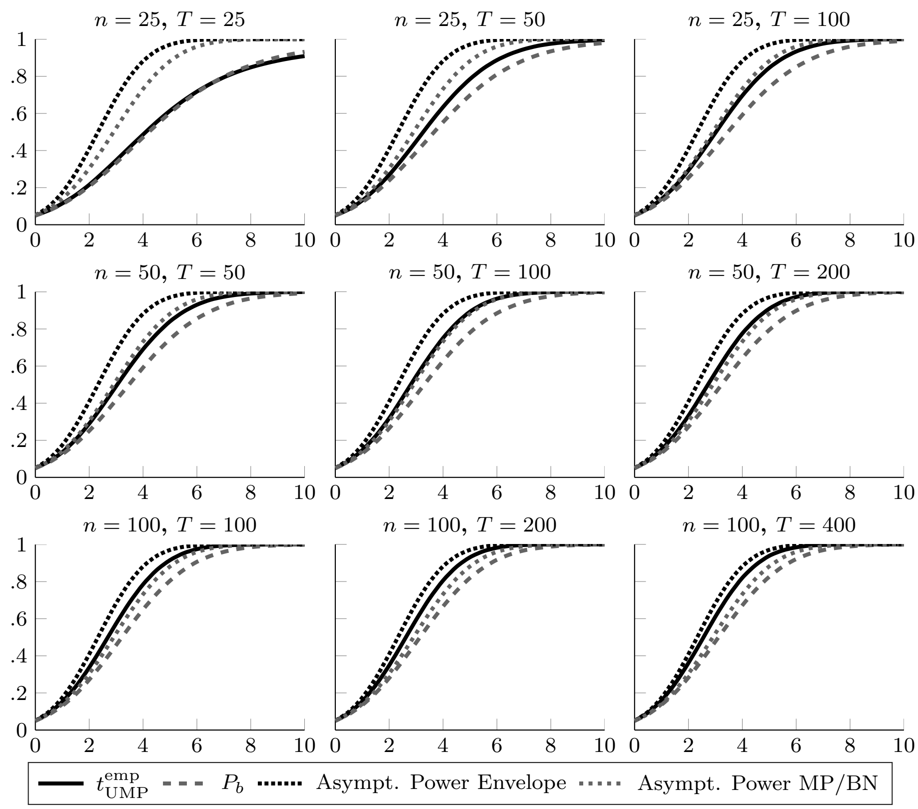

Figure B.6: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Innovations drawn from a t_5 distribution. Note that the power envelopes refer to the Gaussian experiment. Based on 100 000 replications.

n	T	$\sqrt{\omega^4/\phi^4}$	i.i.d. t_{UMP}	i.i.d. $t_{\text{UMP}}^{\text{emp}}$	i.i.d. P_b	AR(1) t_{UMP}	AR(1) $t_{\text{UMP}}^{\text{emp}}$	AR(1) P_b	MA(1) t_{UMP}	MA(1) $t_{\text{UMP}}^{\text{emp}}$	MA(1) P_b
25	25	0.6	0.6	2.8	3.1	1.8	4.5	4.2	2.2	7.0	5.6
25	50	0.6	1.4	4.7	4.0	1.7	4.9	3.6	3.1	8.9	6.2
25	100	0.6	1.8	5.5	4.6	2.3	6.1	4.1	3.9	10.1	6.7
50	50	0.6	2.0	4.3	3.7	2.5	4.5	3.5	5.3	9.9	6.6
50	100	0.6	2.6	5.1	4.2	2.9	5.2	3.7	6.1	11.0	7.0
50	200	0.6	2.9	5.5	4.6	3.4	5.9	4.1	5.3	9.2	6.1
100	100	0.6	3.2	5.0	4.2	3.3	4.9	3.8	9.1	13.1	8.2
100	200	0.6	3.6	5.3	4.5	3.7	5.3	4.1	7.0	10.0	6.6
100	400	0.6	3.6	5.3	4.5	4.3	6.1	4.5	4.9	7.1	5.1
25	25	0.8	0.9	3.1	3.5	1.8	4.3	4.7	2.4	6.7	6.4
25	50	0.8	1.8	5.1	4.6	1.7	4.4	4.0	3.1	8.3	7.2
25	100	0.8	2.3	5.8	5.2	2.2	5.3	4.6	3.9	9.3	7.8
50	50	0.8	2.4	4.6	4.2	2.4	4.2	4.2	5.1	9.3	8.3
50	100	0.8	3.0	5.4	4.8	2.6	4.6	4.3	5.9	10.1	8.5
50	200	0.8	3.3	5.7	5.2	3.1	5.2	4.7	5.0	8.4	7.1
100	100	0.8	3.5	5.1	4.6	3.1	4.4	4.4	8.7	12.3	10.4
100	200	0.8	3.8	5.5	5.0	3.3	4.7	4.5	6.6	9.2	7.9
100	400	0.8	3.9	5.5	5.1	3.9	5.5	5.0	4.7	6.6	5.9
25	25	1.0	1.0	3.3	3.9	1.9	4.3	5.4	2.4	6.5	7.2
25	50	1.0	2.0	5.2	5.1	1.7	4.2	4.5	3.2	8.1	8.2
25	100	1.0	2.6	6.0	5.8	2.1	5.0	5.1	3.9	8.9	8.8
50	50	1.0	2.5	4.7	4.6	2.4	4.0	5.0	5.2	9.2	10.1
50	100	1.0	3.1	5.4	5.2	2.6	4.4	4.8	5.8	9.8	10.0
50	200	1.0	3.4	5.7	5.6	3.0	5.0	5.1	4.9	8.2	8.1
100	100	1.0	3.6	5.2	4.9	3.0	4.2	4.9	8.6	12.1	12.6
100	200	1.0	3.9	5.5	5.3	3.2	4.6	4.9	6.5	9.0	9.1
100	400	1.0	4.0	5.6	5.5	3.8	5.3	5.5	4.6	6.4	6.4
Mean abs. dev. from 5%			2.3	0.6	0.6	2.3	0.5	0.6	1.4	4.1	2.7

Table B.1: Sizes (in percent) of nominal 5% level tests with no heterogeneity in the alternatives. Based on 1 000 000 replications. Andrews Bandwidth.

n	T	$\sqrt{\omega^4/\phi^4}$	i.i.d. t_{UMP}	i.i.d. $t_{\text{UMP}}^{\text{emp}}$	i.i.d. P_b	AR(1) t_{UMP}	AR(1) $t_{\text{UMP}}^{\text{emp}}$	AR(1) P_b	MA(1) t_{UMP}	MA(1) $t_{\text{UMP}}^{\text{emp}}$	MA(1) P_b
25	25	0.6	0.8	3.9	5.9	4.6	10.5	9.5	3.9	10.8	9.6
25	50	0.6	1.4	5.7	6.6	3.1	8.2	6.4	4.2	12.0	9.4
25	100	0.6	1.8	6.5	7.1	3.5	9.3	6.5	5.1	13.7	9.9
50	50	0.6	1.7	4.4	4.7	4.8	8.2	5.6	6.8	12.9	8.4
50	100	0.6	2.1	5.1	5.1	4.3	8.0	4.8	7.4	14.0	8.4
50	200	0.6	2.4	5.5	5.4	4.6	8.5	5.0	6.4	11.9	7.3
100	100	0.6	2.9	5.0	4.6	5.4	7.8	4.7	11.3	16.6	9.3
100	200	0.6	3.1	5.2	4.8	5.0	7.4	4.5	8.5	12.5	7.4
100	400	0.6	3.3	5.3	5.0	5.7	8.3	4.9	6.0	8.9	5.7
25	25	0.8	1.0	3.7	5.2	4.9	9.8	9.6	4.1	10.0	9.5
25	50	0.8	1.9	5.7	6.0	2.8	6.7	6.0	4.0	10.1	9.0
25	100	0.8	2.5	6.6	6.6	2.9	7.0	6.0	4.7	11.1	9.5
50	50	0.8	2.4	5.0	5.0	4.5	7.1	6.5	6.7	11.4	9.9
50	100	0.8	3.0	5.6	5.5	3.6	6.2	5.3	6.8	11.7	9.6
50	200	0.8	3.3	6.0	5.8	3.7	6.3	5.3	5.7	9.7	8.1
100	100	0.8	3.6	5.4	5.0	4.6	6.3	5.7	10.2	14.2	11.6
100	200	0.8	3.8	5.6	5.3	4.0	5.6	5.0	7.4	10.4	8.6
100	400	0.8	3.9	5.6	5.4	4.4	6.2	5.4	5.2	7.3	6.4
25	25	1.0	1.2	4.0	5.2	5.1	9.6	10.2	4.4	9.8	10.1
25	50	1.0	2.4	6.0	6.1	2.8	6.2	6.3	4.1	9.6	9.5
25	100	1.0	3.1	7.0	6.8	2.8	6.2	6.1	4.8	10.4	10.1
50	50	1.0	2.9	5.3	5.4	4.5	6.8	7.7	6.6	10.9	11.5
50	100	1.0	3.4	5.9	5.7	3.4	5.6	5.8	6.7	10.9	10.9
50	200	1.0	3.8	6.2	6.1	3.4	5.5	5.6	5.6	9.0	8.9
100	100	1.0	3.9	5.6	5.3	4.4	5.9	6.6	9.9	13.6	13.9
100	200	1.0	4.1	5.7	5.5	3.7	5.1	5.4	7.2	9.9	9.9
100	400	1.0	4.2	5.8	5.7	4.1	5.6	5.7	5.0	6.8	6.8
Mean abs. dev. from 5%			2.3	0.8	0.6	1.0	2.2	1.2	1.7	6.1	4.2

Table B.2: Sizes (in percent) of nominal 5% level tests with no heterogeneity in the alternatives. Based on 1 000 000 replications. Andrews Bandwidth, three factors.

n	T	$\sqrt{\omega^4/\phi^4}$	i.i.d. t_{UMP}	i.i.d. $t_{\text{UMP}}^{\text{emp}}$	i.i.d. P_b	AR(1) t_{UMP}	AR(1) $t_{\text{UMP}}^{\text{emp}}$	AR(1) P_b	MA(1) t_{UMP}	MA(1) $t_{\text{UMP}}^{\text{emp}}$	MA(1) P_b
25	25	0.6	0.7	2.9	3.3	2.0	4.9	4.6	2.3	7.2	5.9
25	50	0.6	1.4	4.7	4.2	1.8	5.0	3.7	3.2	9.1	6.4
25	100	0.6	1.8	5.5	4.7	2.3	6.1	4.2	3.9	10.1	6.8
50	50	0.6	2.0	4.3	3.7	2.6	4.6	3.6	5.3	10.0	6.8
50	100	0.6	2.6	5.1	4.3	2.9	5.3	3.8	6.1	10.9	7.0
50	200	0.6	2.9	5.4	4.5	3.4	5.9	4.1	5.3	9.2	6.1
100	100	0.6	3.2	5.0	4.2	3.3	4.9	3.8	9.1	13.2	8.2
100	200	0.6	3.6	5.3	4.4	3.6	5.3	4.0	6.9	9.9	6.7
100	400	0.6	3.7	5.4	4.6	4.4	6.2	4.5	4.9	7.1	5.2
25	25	0.8	0.9	3.1	3.5	2.0	4.5	4.9	2.4	6.8	6.5
25	50	0.8	1.8	5.0	4.6	1.7	4.5	4.1	3.1	8.3	7.2
25	100	0.8	2.3	5.9	5.2	2.2	5.3	4.6	3.9	9.3	7.7
50	50	0.8	2.3	4.6	4.2	2.4	4.2	4.3	5.2	9.4	8.3
50	100	0.8	3.0	5.4	4.8	2.6	4.7	4.3	5.9	10.1	8.5
50	200	0.8	3.3	5.7	5.2	3.0	5.2	4.7	5.0	8.4	7.2
100	100	0.8	3.5	5.2	4.7	3.1	4.4	4.4	8.7	12.4	10.4
100	200	0.8	3.8	5.5	5.0	3.3	4.7	4.5	6.6	9.3	7.9
100	400	0.8	3.9	5.5	5.1	3.9	5.5	5.0	4.7	6.5	5.9
25	25	1.0	1.0	3.3	3.8	2.0	4.4	5.6	2.5	6.7	7.3
25	50	1.0	2.0	5.2	5.1	1.7	4.2	4.5	3.3	8.1	8.2
25	100	1.0	2.6	6.0	5.8	2.2	5.1	5.1	3.9	9.0	8.9
50	50	1.0	2.5	4.7	4.6	2.4	4.1	5.0	5.1	9.1	10.0
50	100	1.0	3.1	5.4	5.2	2.6	4.4	4.8	5.8	9.9	10.0
50	200	1.0	3.5	5.8	5.6	3.0	5.0	5.2	4.9	8.1	8.1
100	100	1.0	3.6	5.3	4.9	3.0	4.3	5.0	8.6	12.1	12.6
100	200	1.0	3.9	5.5	5.2	3.2	4.6	4.9	6.4	9.0	9.0
100	400	1.0	4.1	5.6	5.5	3.8	5.3	5.4	4.6	6.3	6.4
Mean abs. dev. from 5%			2.3	0.6	0.6	2.2	0.5	0.6	1.4	4.1	2.8

Table B.3: Sizes (in percent) of nominal 5% level tests with no heterogeneity in the alternatives. Based on 1 000 000 replications. Andrews Bandwidth, t-distribution with five degrees of freedom.

B.2. Finite-Sample Results with the Newey & West (1994) Bandwidth

Tables B.4 to B.6 are analogous to Tables B.1 to B.3. And Figures B.7 to B.15 are analogous to Figures 1 to 3 and B.1 to B.6. In general, the sizes for the MA case are slightly better controlled with the Newey & West (1994) bandwidth, at the expense of slightly lower power for small sample sizes.

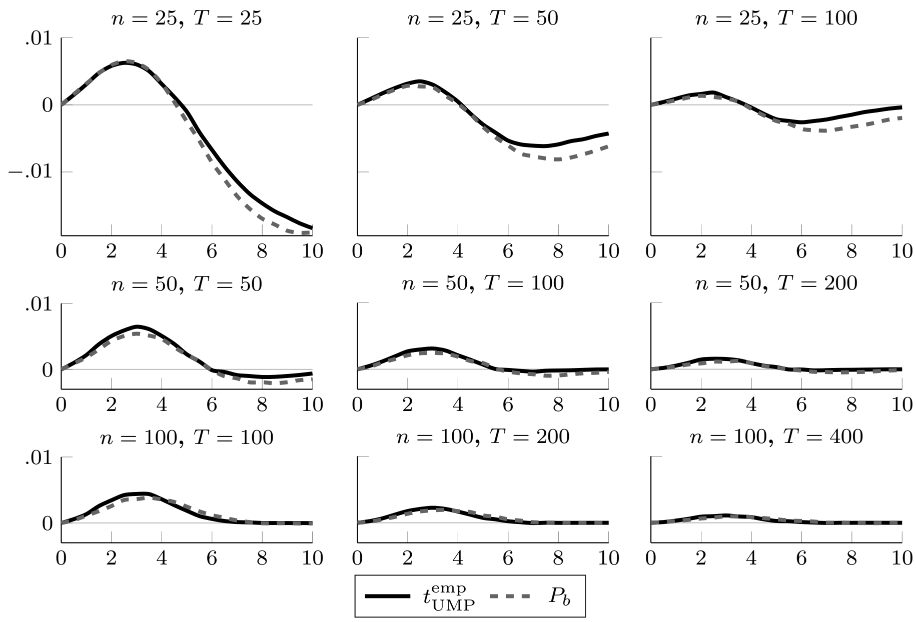

Figure B.7: Difference between powers in the MP vs the PANIC framework as a function of $-h$ with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Based on 1 000 000 replications.

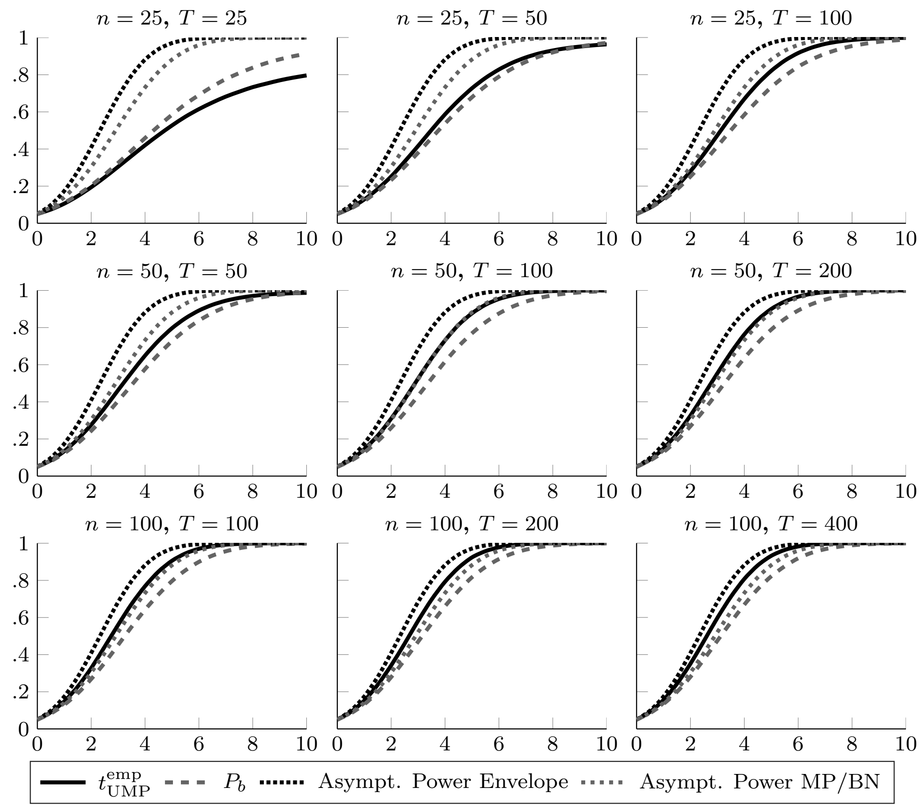

Figure B.8: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Based on 100 000 replications.

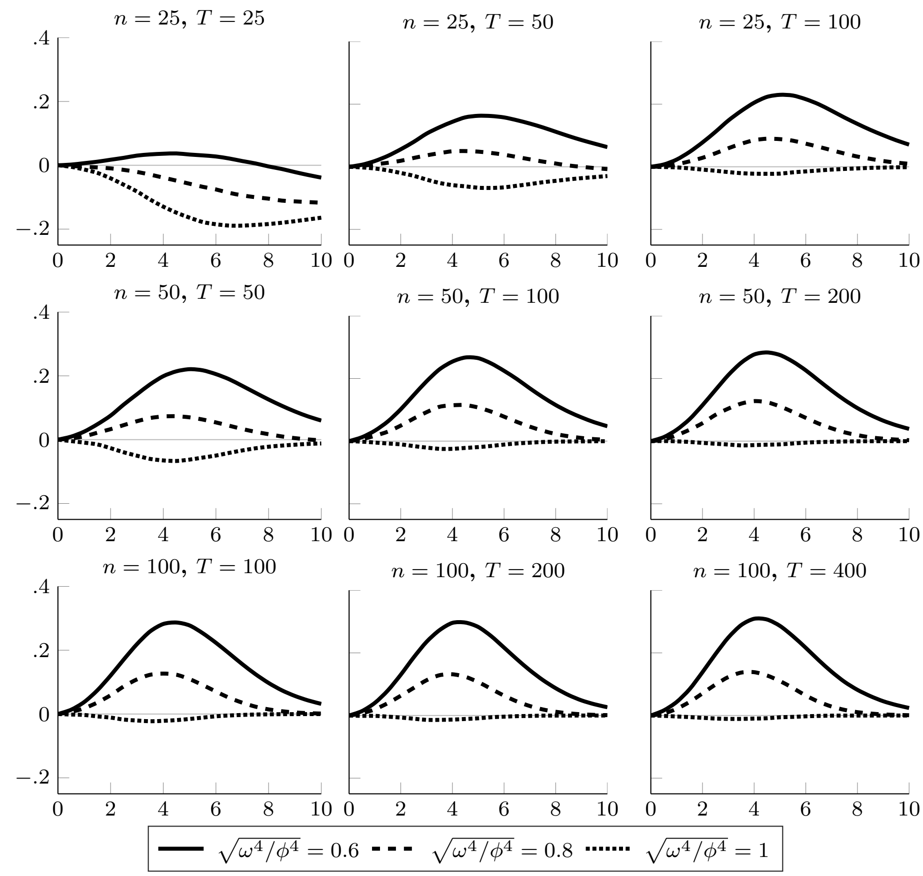

Figure B.9: (Size-corrected) power gains from using $t_{\text{UMP}}^{\text{emp}}$ over P_b for varying values of $\sqrt{\omega^4/\phi^4}$ and sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts. Based on 100 000 replications.

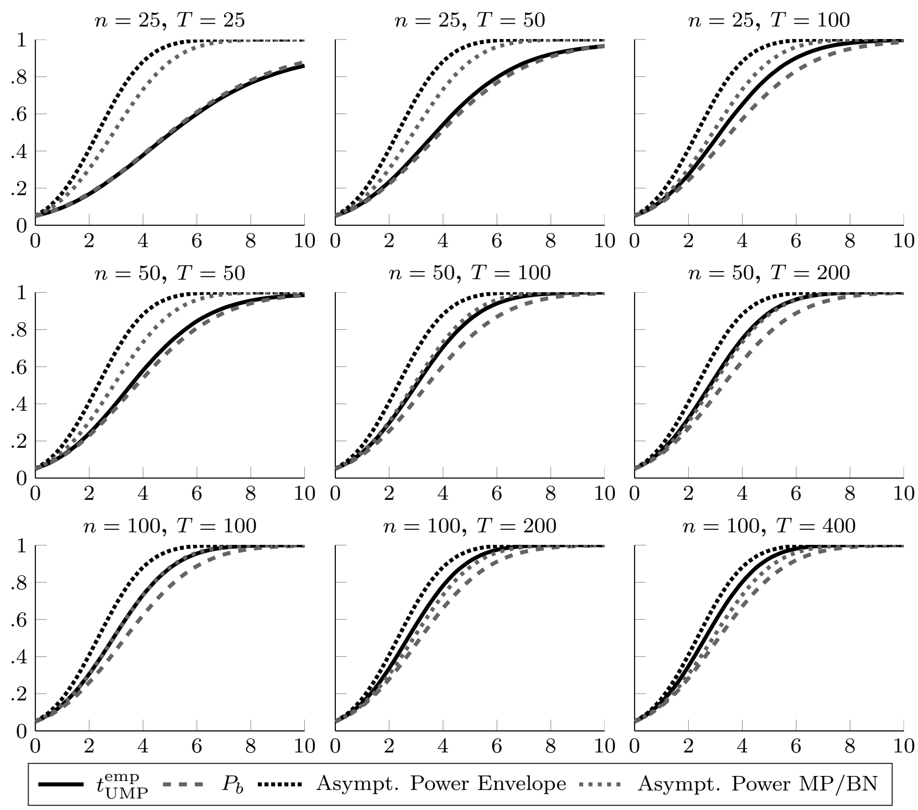

Figure B.10: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with MA factor innovations and MA idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Based on 100 000 replications.

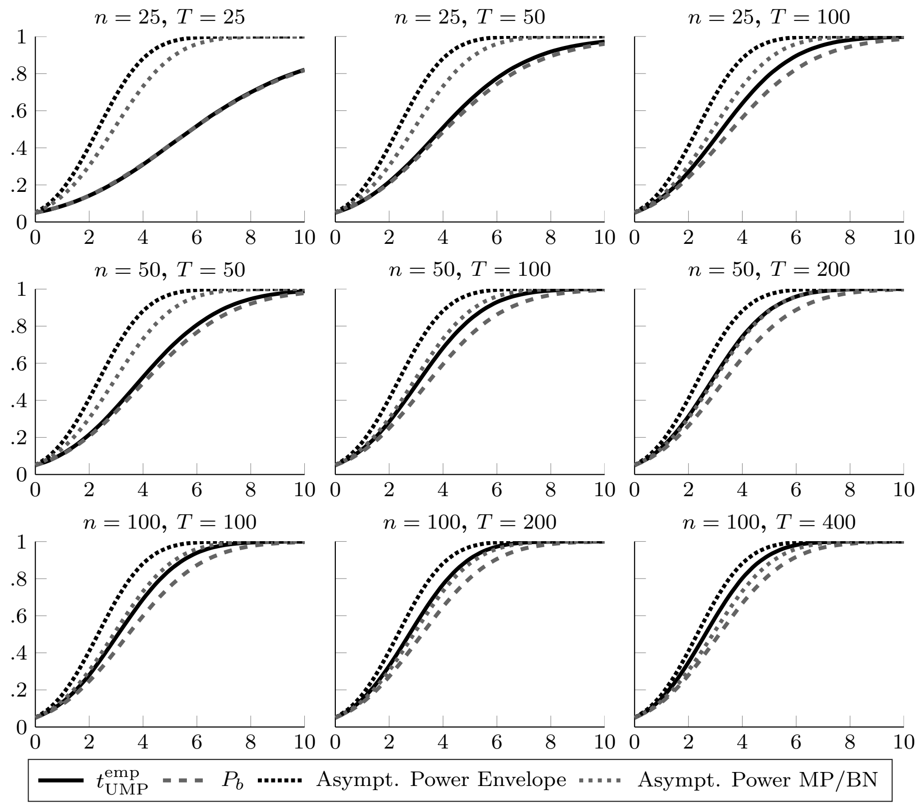

Figure B.11: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with AR factor innovations and AR idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Based on 100 000 replications.

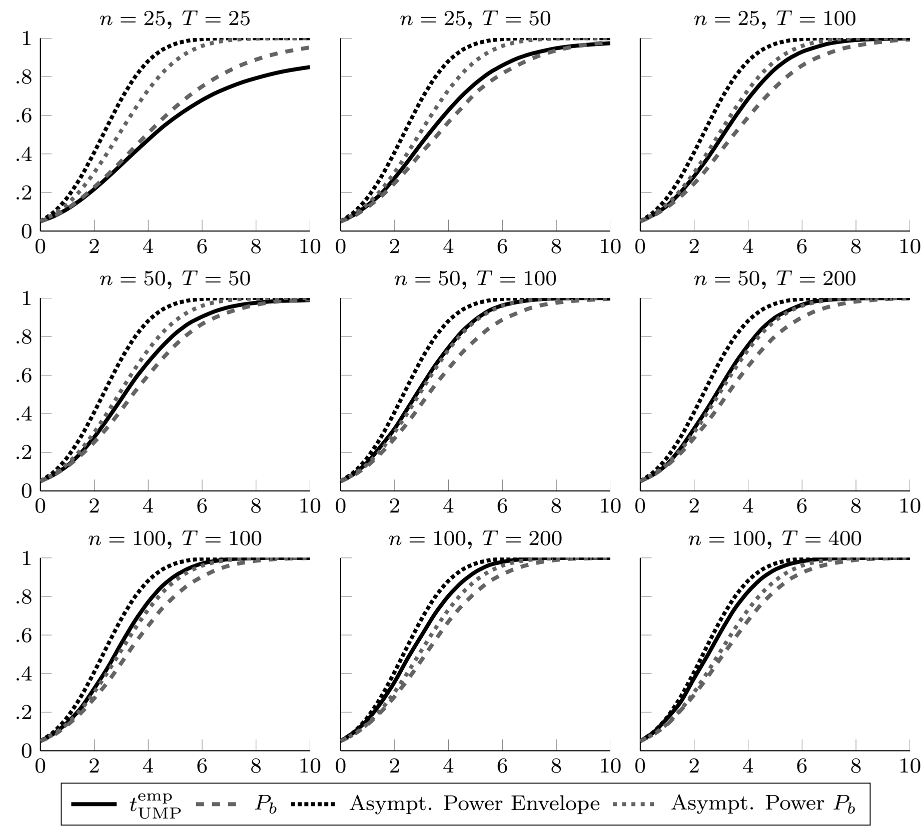

Figure B.12: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with overdifferenced i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. The factor is stationary. Based on 100 000 replications.

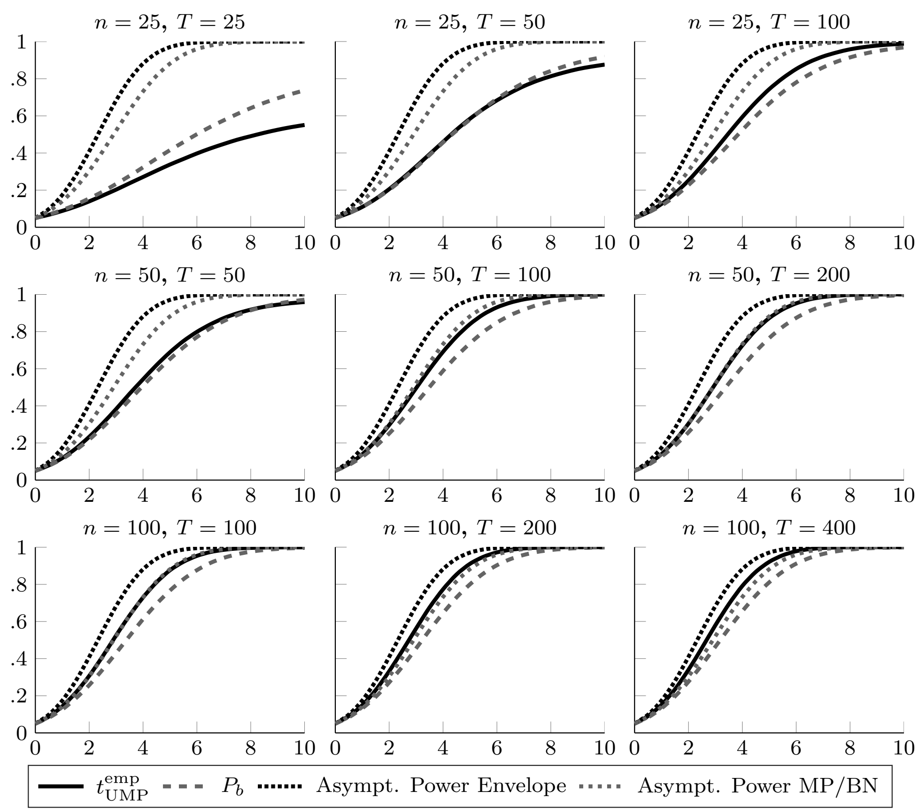

Figure B.13: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Dependence based on three factors. Based on 100 000 replications.

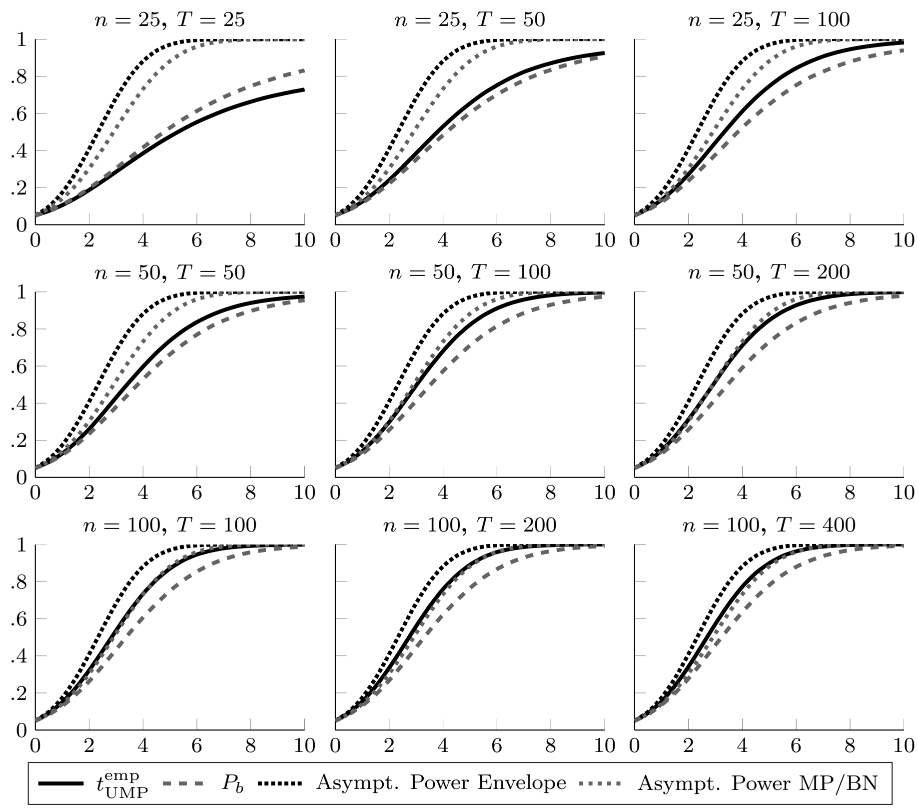

Figure B.14: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Alternatives drawn from a Uniform(0.2,1.8) distribution. Based on 100 000 replications.

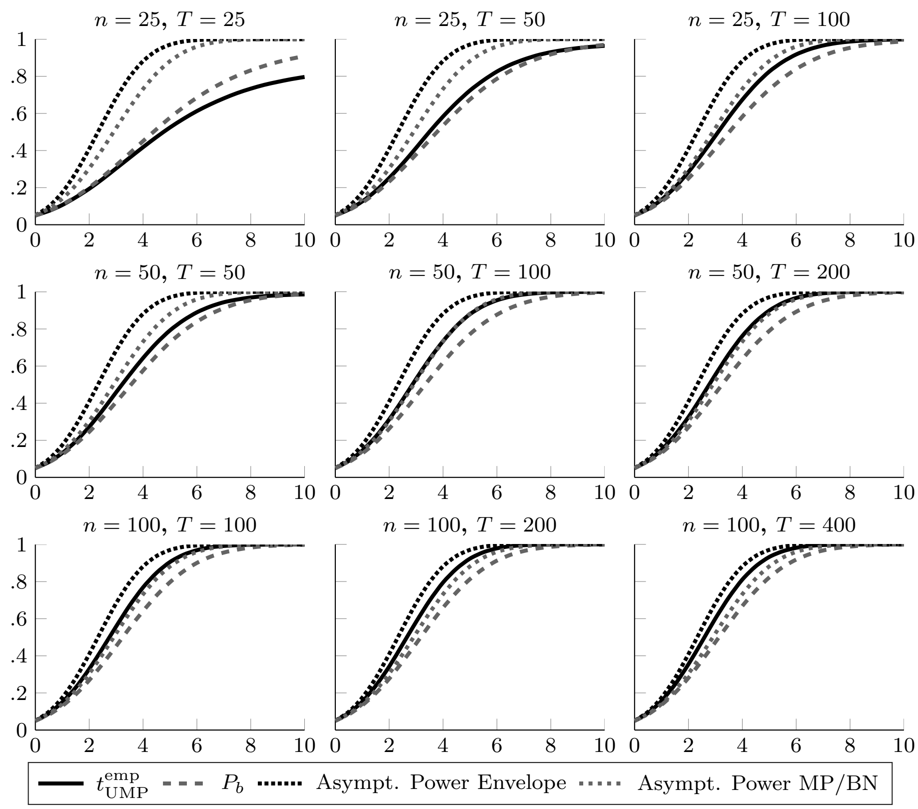

Figure B.15: Size-corrected power of unit-root tests as a function of $-h$ for varying sample sizes in the PANIC framework with i.i.d. factor innovations and i.i.d. idiosyncratic parts and $\sqrt{\omega^4/\phi^4} = 0.8$. Innovations drawn from a t_5 distribution. Note that the power envelopes refer to the Gaussian experiment. Based on 100 000 replications.

n	T	$\sqrt{\omega^4/\phi^4}$	i.i.d. t_{UMP}	i.i.d. $t_{\text{UMP}}^{\text{emp}}$	i.i.d. P_b	AR(1) t_{UMP}	AR(1) $t_{\text{UMP}}^{\text{emp}}$	AR(1) P_b	MA(1) t_{UMP}	MA(1) $t_{\text{UMP}}^{\text{emp}}$	MA(1) P_b
25	25	0.6	0.3	1.2	1.5	1.3	3.4	3.6	1.0	3.3	3.6
25	50	0.6	0.6	2.5	2.3	1.4	4.2	3.1	1.3	4.1	3.3
25	100	0.6	1.3	4.2	3.6	2.3	6.0	4.0	2.2	6.3	4.4
50	50	0.6	0.9	2.1	1.9	2.1	3.9	3.0	1.9	3.8	3.1
50	100	0.6	1.9	3.8	3.1	2.9	5.3	3.6	3.1	6.0	4.2
50	200	0.6	2.4	4.6	3.7	3.4	6.0	3.9	2.8	5.1	3.6
100	100	0.6	2.3	3.7	2.8	3.4	5.1	3.6	4.1	6.1	4.3
100	200	0.6	2.9	4.4	3.5	3.8	5.5	3.8	3.2	4.7	3.4
100	400	0.6	3.2	4.8	3.9	4.2	6.0	4.1	3.1	4.6	3.5
25	25	0.8	0.4	1.3	1.7	1.4	3.2	4.1	1.1	3.2	4.1
25	50	0.8	0.9	2.8	2.6	1.4	3.7	3.4	1.4	3.9	3.7
25	100	0.8	1.7	4.6	4.0	2.1	5.3	4.4	2.3	5.9	5.0
50	50	0.8	1.2	2.4	2.1	2.0	3.6	3.6	1.9	3.7	3.7
50	100	0.8	2.2	4.2	3.4	2.6	4.7	4.1	3.1	5.6	4.8
50	200	0.8	2.8	4.9	4.2	3.1	5.3	4.4	2.7	4.7	4.0
100	100	0.8	2.6	3.9	3.0	3.2	4.6	4.2	4.0	5.8	5.1
100	200	0.8	3.2	4.6	3.8	3.5	4.9	4.2	3.0	4.4	3.7
100	400	0.8	3.5	5.0	4.3	3.9	5.4	4.6	3.0	4.3	3.8
25	25	1.0	0.5	1.5	1.9	1.4	3.3	4.8	1.1	3.2	4.5
25	50	1.0	1.1	3.0	2.9	1.4	3.6	3.9	1.4	3.9	4.2
25	100	1.0	2.0	4.8	4.5	2.1	5.0	4.9	2.4	5.7	5.6
50	50	1.0	1.3	2.5	2.2	2.0	3.5	4.2	2.0	3.6	4.4
50	100	1.0	2.4	4.2	3.7	2.6	4.5	4.6	3.1	5.5	5.4
50	200	1.0	2.9	5.0	4.4	3.0	5.0	4.8	2.8	4.7	4.4
100	100	1.0	2.7	4.0	3.1	3.1	4.4	4.7	3.9	5.7	5.7
100	200	1.0	3.3	4.8	3.9	3.4	4.8	4.5	3.0	4.3	3.9
100	400	1.0	3.7	5.1	4.5	3.8	5.3	4.9	3.0	4.2	3.9
Mean abs. dev. from 5%			3.0	1.3	1.8	2.4	0.7	0.9	2.5	0.9	0.9

Table B.4: Sizes (in percent) of nominal 5% level tests with no heterogeneity in the alternatives. Based on 1 000 000 replications. Newey Bandwidth.

			i.i.d.			AR(1)			MA(1)		
n	T	$\sqrt{\omega^4/\phi^4}$	t_{UMP}	$t_{\text{UMP}}^{\text{emp}}$	P_b	t_{UMP}	$t_{\text{UMP}}^{\text{emp}}$	P_b	t_{UMP}	$t_{\text{UMP}}^{\text{emp}}$	P_b
25	25	0.6	0.5	1.5	3.3	3.0	7.2	7.9	1.6	5.0	6.3
25	50	0.6	0.6	2.6	4.2	2.4	6.7	5.6	1.6	5.6	5.5
25	100	0.6	1.1	4.7	5.7	3.4	9.1	6.2	2.9	8.7	6.9
50	50	0.6	0.6	1.7	2.6	3.9	6.8	4.8	2.4	5.0	4.3
50	100	0.6	1.3	3.5	3.8	4.2	7.9	4.6	3.7	7.6	5.1
50	200	0.6	1.8	4.4	4.4	4.6	8.6	4.7	3.3	6.6	4.4
100	100	0.6	1.9	3.4	3.2	5.5	7.9	4.5	5.0	7.8	4.9
100	200	0.6	2.4	4.1	3.8	5.2	7.7	4.2	3.7	5.8	3.8
100	400	0.6	2.8	4.6	4.2	5.7	8.3	4.5	3.7	5.7	3.8
25	25	0.8	0.5	1.5	2.8	3.2	6.9	8.1	1.8	4.6	6.2
25	50	0.8	0.8	2.8	3.6	2.2	5.5	5.2	1.6	4.7	5.0
25	100	0.8	1.8	5.1	5.2	2.8	6.9	5.8	2.8	7.0	6.3
50	50	0.8	1.0	2.3	2.6	3.7	5.9	5.6	2.5	4.7	4.9
50	100	0.8	2.1	4.2	4.0	3.6	6.2	5.1	3.6	6.4	5.6
50	200	0.8	2.7	5.0	4.7	3.7	6.4	5.0	3.1	5.5	4.6
100	100	0.8	2.5	3.9	3.3	4.7	6.5	5.4	4.7	6.9	5.8
100	200	0.8	3.1	4.7	4.0	4.1	5.9	4.7	3.3	4.9	4.0
100	400	0.8	3.5	5.0	4.5	4.3	6.1	4.9	3.3	4.7	4.0
25	25	1.0	0.7	1.7	2.7	3.4	6.8	8.7	1.9	4.7	6.6
25	50	1.0	1.1	3.2	3.6	2.2	5.1	5.4	1.8	4.6	5.2
25	100	1.0	2.3	5.5	5.3	2.7	6.2	5.9	2.9	6.7	6.6
50	50	1.0	1.3	2.7	2.7	3.7	5.7	6.6	2.6	4.6	5.6
50	100	1.0	2.6	4.5	4.0	3.4	5.6	5.6	3.5	6.1	6.1
50	200	1.0	3.2	5.4	4.8	3.4	5.6	5.3	3.1	5.1	4.8
100	100	1.0	2.9	4.2	3.4	4.5	6.1	6.3	4.7	6.6	6.6
100	200	1.0	3.4	4.9	4.1	3.8	5.3	5.0	3.3	4.7	4.3
100	400	1.0	3.8	5.3	4.7	4.0	5.5	5.2	3.2	4.4	4.1
Mean abs. dev. from 5%			3.1	1.3	1.2	1.3	1.6	0.8	2.0	1.0	0.8

Table B.5: Sizes (in percent) of nominal 5% level tests with no heterogeneity in the alternatives. Based on 1 000 000 replications. Newey Bandwidth, three factors.

n	T	$\sqrt{\omega^4/\phi^4}$	i.i.d.			AR(1)			MA(1)		
			t_{UMP}	$t_{\text{UMP}}^{\text{emp}}$	P_b	t_{UMP}	$t_{\text{UMP}}^{\text{emp}}$	P_b	t_{UMP}	$t_{\text{UMP}}^{\text{emp}}$	P_b
25	25	0.6	0.3	1.2	1.7	1.5	3.7	4.0	1.1	3.4	3.8
25	50	0.6	0.7	2.5	2.4	1.5	4.3	3.2	1.3	4.2	3.5
25	100	0.6	1.3	4.2	3.6	2.3	6.0	4.0	2.3	6.4	4.5
50	50	0.6	0.9	2.1	1.9	2.2	4.0	3.0	1.9	3.9	3.2
50	100	0.6	1.9	3.9	3.1	2.9	5.3	3.6	3.1	6.0	4.2
50	200	0.6	2.4	4.6	3.7	3.4	6.0	3.9	2.8	5.2	3.6
100	100	0.6	2.3	3.7	2.9	3.5	5.1	3.6	4.1	6.1	4.3
100	200	0.6	2.9	4.4	3.4	3.8	5.5	3.8	3.2	4.7	3.4
100	400	0.6	3.3	4.9	3.9	4.3	6.1	4.1	3.2	4.6	3.5
25	25	0.8	0.4	1.3	1.7	1.5	3.5	4.3	1.1	3.3	4.1
25	50	0.8	0.9	2.8	2.6	1.4	3.8	3.5	1.4	3.9	3.7
25	100	0.8	1.7	4.6	4.0	2.1	5.2	4.4	2.3	5.9	5.0
50	50	0.8	1.1	2.4	2.0	2.1	3.6	3.6	2.0	3.7	3.8
50	100	0.8	2.2	4.2	3.4	2.7	4.7	4.2	3.1	5.6	4.9
50	200	0.8	2.8	4.9	4.1	3.0	5.3	4.4	2.7	4.8	4.1
100	100	0.8	2.6	4.0	3.0	3.2	4.6	4.2	4.0	5.8	5.1
100	200	0.8	3.2	4.7	3.8	3.5	4.9	4.2	3.0	4.4	3.7
100	400	0.8	3.5	5.0	4.3	3.9	5.5	4.6	3.0	4.3	3.7
25	25	1.0	0.5	1.4	1.8	1.5	3.4	4.9	1.2	3.2	4.6
25	50	1.0	1.0	3.0	2.9	1.4	3.6	3.8	1.5	3.9	4.2
25	100	1.0	2.0	4.9	4.4	2.1	5.0	4.9	2.4	5.8	5.6
50	50	1.0	1.3	2.5	2.2	2.1	3.6	4.3	2.0	3.6	4.3
50	100	1.0	2.4	4.3	3.6	2.6	4.5	4.6	3.1	5.5	5.5
50	200	1.0	3.0	5.0	4.5	3.0	5.0	4.8	2.7	4.7	4.4
100	100	1.0	2.7	4.0	3.1	3.1	4.5	4.7	3.9	5.7	5.7
100	200	1.0	3.3	4.7	3.9	3.4	4.8	4.5	2.9	4.2	3.9
100	400	1.0	3.7	5.2	4.6	3.7	5.2	4.9	2.9	4.1	3.8
Mean abs. dev. from 5%			3.0	1.3	1.8	2.4	0.7	0.8	2.5	0.9	0.9

Table B.6: Sizes (in percent) of nominal 5% level tests with no heterogeneity in the alternatives. Based on 1 000 000 replications. Newey Bandwidth, t-distribution with five degrees of freedom.

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