Online Appendix to "Faster Uniform Convergence Rates for Deconvolution Estimators From Repeated Measurements"

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S.1 Supplement to Table 1

The constants κ_s for $s = 1, \ldots, 5$ in Table 1 are defined as

$$\kappa_1 := \frac{2\beta_\epsilon - 2\rho_\epsilon + 2 + c^{\rho_\epsilon}(-2\beta_\epsilon + 3/2)}{\rho_\epsilon(c^{\rho_\epsilon} + 2)},$$

$$\kappa_2 := \frac{\beta_x - \rho_x + 1 + c^{\rho_x}(-\beta_x + \beta_\epsilon + 3/2)}{\rho_x(c^{\rho_x} + 1)},$$

$$\kappa_3 := \frac{(\mu_x + \mu_\epsilon)(-\rho + \beta_x + 1) - c^\rho \mu_\epsilon(\beta_x + \beta_\epsilon - 3/2)}{\rho[\mu_x + (c^\rho + 1)\mu_\epsilon]},$$

$$\kappa_4 := \frac{(\beta_x - \rho + 1)[\mu_x \vee (3\mu_x - \mu_\epsilon)] + c^\rho \mu_\epsilon[\mathbb{I}\{2\mu_x < \mu_\epsilon\}(1 - \beta_\epsilon) + \mathbb{I}\{2\mu_x > \mu_\epsilon\}(\beta_x - 3\beta_\epsilon + 3/2) + \zeta]}{\rho[\mu_x + c^\rho \mu_\epsilon + 0 \vee (2\mu_x - \mu_\epsilon)]},$$

and

$$\kappa_5 := \frac{2\mu_\epsilon(-\rho + \beta_\epsilon + 1) + c^\rho \mu_x(-2\beta_x + 3/2)}{\rho(2\mu_\epsilon + c^\rho \mu_x)},$$

respectively, where

$$\zeta := \mathbb{I}\{2\mu_x = \mu_\epsilon\}[(1 - \beta_\epsilon) \vee (\beta_x - 3\beta_\epsilon + 3/2)].$$

Besides, let

$$\begin{aligned}\zeta' := &\ \mathbb{I}\{2\mu_x < \mu_\epsilon\}(\beta_x + \beta_\epsilon - \rho) + \mathbb{I}\{2\mu_x > \mu_\epsilon\}(3\beta_\epsilon - \rho - 1/2) \\ &+ \mathbb{I}\{2\mu_x = \mu_\epsilon\}[-\rho + \beta_x + 1 - (1 - \beta_\epsilon) \vee (\beta_x - 3\beta_\epsilon + 3/2)],\end{aligned}$$

then the optimal choices of h_n corresponding to Table 1 can be summarized as follows.

Table S1: Chosen Bandwidths Corresponding to Table 1

		h_n
Case 1	$\hat{f}_X$	$n^{-1/(4\beta_x+2\beta_\epsilon+1)}$
	$\tilde{f}_X$	$n^{-1/[(6\beta_\epsilon+1)\vee(2\beta_x+2\beta_\epsilon)]}$
	$\hat{f}_\epsilon$	$n^{-1/(4\beta_x+2\beta_\epsilon+1)}$
	$\tilde{f}_\epsilon$	$n^{-1/(6\beta_\epsilon+1)}$
Case 2	$\hat{f}_X$	$\left[\frac{\mu_\epsilon}{2}\log n+\frac{\mu_\epsilon(\beta_\epsilon-2\beta_x-1/2)}{\rho_\epsilon}\log\log n\right]^{-1/\rho_\epsilon}$
	$\tilde{f}_X$	$\left[\frac{\mu_\epsilon}{6}\log n+\frac{\mu_\epsilon(3\beta_\epsilon-1/2)}{3\rho_\epsilon}\log\log n\right]^{-1/\rho_\epsilon}$
	$\hat{f}_\epsilon$	$\left[\frac{\mu_\epsilon}{2c^{\rho_\epsilon}}\log n-\frac{\mu_\epsilon(2\beta_x+\rho_\epsilon-\beta_\epsilon+1/2)}{\rho_\epsilon c^{\rho_\epsilon}}\log\log n\right]^{-1/\rho_\epsilon}$
	$\tilde{f}_\epsilon$	$\left[\frac{\mu_\epsilon}{2(c^{\rho_\epsilon}+2)}\log n+\frac{\mu_\epsilon(3\beta_\epsilon-\rho_\epsilon-1/2)}{\rho_\epsilon(c^{\rho_\epsilon}+2)}\log\log n\right]^{-1/\rho_\epsilon}$
Case 3	$\hat{f}_X$	$\left[\frac{\mu_x}{2(c^{\rho_x}+1)}\log n+\frac{\mu_x(2\beta_x-\rho_x-\beta_\epsilon-1/2)}{\rho_x(c^{\rho_x}+1)}\log\log n\right]^{-1/\rho_x}$
	$\tilde{f}_X$	$\left[\frac{\mu_x}{2c^{\rho_x}}\log n+\frac{\mu_x(\beta_x-\rho_x-\beta_\epsilon)}{\rho_x c^{\rho_x}}\log\log n\right]^{-1/\rho_x}$
	$\hat{f}_\epsilon$	$\left[\frac{\mu_x}{4}\log n+\frac{\mu_x(2\beta_x-\beta_\epsilon-1/2)}{2\rho_x}\log\log n\right]^{-1/\rho_x}$
	$\tilde{f}_\epsilon$	$n^{-1/(6\beta_\epsilon+1)}$
Case 4	$\hat{f}_X$	$\left[\frac{\mu_x\mu_\epsilon}{2[\mu_x+(c^\rho+1)\mu_\epsilon]}\log n+\frac{\mu_x\mu_\epsilon(2\beta_x-\rho+\beta_\epsilon-1/2)}{\rho[\mu_x+(c^\rho+1)\mu_\epsilon]}\log\log n\right]^{-1/\rho}$
	$\tilde{f}_X$	$\left[\frac{\mu_x\mu_\epsilon}{2[\mu_x+c^\rho\mu_\epsilon+0\vee(2\mu_x-\mu_\epsilon)]}\log n+\frac{\mu_x\mu_\epsilon\zeta'}{\rho[\mu_x+c^\rho\mu_\epsilon+0\vee(2\mu_x-\mu_\epsilon)]}\log\log n\right]^{-1/\rho}$
	$\hat{f}_\epsilon$	$\left[\frac{\mu_x\mu_\epsilon}{2(2\mu_\epsilon+c^\rho\mu_x)}\log n+\frac{\mu_x\mu_\epsilon(2\beta_x-\rho+\beta_\epsilon-1/2)}{\rho(2\mu_\epsilon+c^\rho\mu_x)}\log\log n\right]^{-1/\rho}$
	$\tilde{f}_\epsilon$	$\left[\frac{\mu_\epsilon}{2(c^{\rho_\epsilon}+2)}\log n+\frac{\mu_\epsilon(3\beta_\epsilon-\rho_\epsilon-1/2)}{\rho_\epsilon(c^{\rho_\epsilon}+2)}\log\log n\right]^{-1/\rho_\epsilon}$

S.2 Supplement to Table 2

The constants $\eta_1,\eta_2,\eta_3,\eta_4$ in Table 2 are defined as

$$\eta_1 := \frac{\beta_\epsilon-\rho_\epsilon+1+c^{\rho_\epsilon}(-\beta_\epsilon+\gamma_\epsilon+3/2)}{\rho_\epsilon(c^{\rho_\epsilon}+1)},$$

$$\eta_2 := \frac{\mu_x(-\rho+\beta_x+1)+c^\rho\mu_\epsilon(\gamma_x-\beta_\epsilon+3/2)}{\rho(\mu_x+c^\rho\mu_\epsilon)},$$

$$\eta_3 := \frac{(\beta_x-\rho+1)[\mu_x\vee(2\mu_x-\mu_\epsilon)]+c^\rho\mu_\epsilon[\mathbb{I}\{\mu_x<\mu_\epsilon\}(1-\beta_\epsilon)+\mathbb{I}\{\mu_x>\mu_\epsilon\}(\beta_x-2\beta_\epsilon+\gamma_\epsilon+3/2)+\xi]}{\rho[\mu_x+c^\rho\mu_\epsilon+0\vee(\mu_x-\mu_\epsilon)]},$$

and

$$\eta_4 := \frac{\mu_\epsilon(-\rho+\beta_\epsilon+1)+c^\rho\mu_x(\gamma_x-\beta_x+3/2)}{\rho(\mu_\epsilon+c^\rho\mu_x)},$$

respectively, where

$$\xi := \mathbb{I}\{\mu_x=\mu_\epsilon\}[(1-\beta_\epsilon)\vee(\beta_x-2\beta_\epsilon+\gamma_\epsilon+3/2)].$$

Besides, let

$$\begin{aligned}\xi' := {} & \mathbb{I}\{\mu_x<\mu_\epsilon\}(\beta_x+\beta_\epsilon-\rho)+\mathbb{I}\{\mu_x>\mu_\epsilon\}(2\beta_\epsilon-\rho-\gamma_\epsilon-1/2)\\ & +\mathbb{I}\{\mu_x=\mu_\epsilon\}[-\rho+\beta_x+1-(1-\beta_\epsilon)\vee(\beta_x-2\beta_\epsilon+\gamma_\epsilon+3/2)],\end{aligned}$$

then the optimal choices of h_n corresponding to Table 2 can be summarized as follows.

Table S2: Chosen Bandwidths Corresponding to Table 2

		h_n
Case 1	$\hat{f}_X$	$n^{-1/(2\beta_x+2\beta_\epsilon+1)}$
	$\tilde{f}_X$	$n^{-1/[(4\beta_\epsilon+1)\vee(2\beta_x+2\beta_\epsilon)]}$
	$\hat{f}_\epsilon$	$n^{-1/(2\beta_x+2\beta_\epsilon+1)}$
	$\tilde{f}_\epsilon$	$n^{-1/(4\beta_\epsilon+1)}$
Case 2	$\hat{f}_X$	$\left[\frac{\mu_\epsilon}{2}\log n+\frac{\mu_\epsilon(\beta_\epsilon-\beta_x-1/2)}{\rho_\epsilon}\log\log n\right]^{-1/\rho_\epsilon}$
	$\tilde{f}_X$	$\left[\frac{\mu_\epsilon}{4}\log n+\frac{\mu_\epsilon(2\beta_\epsilon-\gamma_\epsilon-1/2)}{2\rho_\epsilon}\log\log n\right]^{-1/\rho_\epsilon}$
	$\hat{f}_\epsilon$	$\left[\frac{\mu_\epsilon}{2c^{\rho_\epsilon}}\log n-\frac{\mu_\epsilon(\beta_x+\rho_\epsilon-\beta_\epsilon+1/2)}{\rho_\epsilon c^{\rho_\epsilon}}\log\log n\right]^{-1/\rho_\epsilon}$
	$\tilde{f}_\epsilon$	$\left[\frac{\mu_\epsilon}{2(c^{\rho_\epsilon}+1)}\log n+\frac{\mu_\epsilon(2\beta_\epsilon-\gamma_\epsilon-\rho_\epsilon-1/2)}{\rho_\epsilon(c^{\rho_\epsilon}+1)}\log\log n\right]^{-1/\rho_\epsilon}$
Case 3	$\hat{f}_X$	$\left[\frac{\mu_x}{2c^{\rho_x}}\log n+\frac{\mu_x(\beta_x-\rho_x-\beta_\epsilon-\gamma_x-1/2)}{\rho_x c^{\rho_x}}\log\log n\right]^{-1/\rho_x}$
	$\tilde{f}_X$	$\left[\frac{\mu_x}{2c^{\rho_x}}\log n+\frac{\mu_x(\beta_x-\rho_x-\beta_\epsilon)}{\rho_x c^{\rho_x}}\log\log n\right]^{-1/\rho_x}$
	$\hat{f}_\epsilon$	$\left[\frac{\mu_x}{2}\log n+\frac{\mu_x(\beta_x-\gamma_x-\beta_\epsilon-1/2)}{\rho_x}\log\log n\right]^{-1/\rho_x}$
	$\tilde{f}_\epsilon$	$n^{-1/(4\beta_\epsilon+1)}$
Case 4	$\hat{f}_X$	$\left[\frac{\mu_x\mu_\epsilon}{2(\mu_x+c^\rho\mu_\epsilon)}\log n+\frac{\mu_x\mu_\epsilon(\beta_x-\rho+\beta_\epsilon-\gamma_x-1/2)}{\rho(\mu_x+c^\rho\mu_\epsilon)}\log\log n\right]^{-1/\rho}$
	$\tilde{f}_X$	$\left[\frac{\mu_x\mu_\epsilon}{2[\mu_x+c^\rho\mu_\epsilon+0\vee(\mu_x-\mu_\epsilon)]}\log n+\frac{\mu_x\mu_\epsilon\xi'}{\rho[\mu_x+c^\rho\mu_\epsilon+0\vee(\mu_x-\mu_\epsilon)]}\log\log n\right]^{-1/\rho}$
	$\hat{f}_\epsilon$	$\left[\frac{\mu_x\mu_\epsilon}{2(\mu_\epsilon+c^\rho\mu_x)}\log n+\frac{\mu_x\mu_\epsilon(\beta_x-\rho+\beta_\epsilon-\gamma_x-1/2)}{\rho(\mu_\epsilon+c^\rho\mu_x)}\log\log n\right]^{-1/\rho}$
	$\tilde{f}_\epsilon$	$\left[\frac{\mu_\epsilon}{2(1+c^{\rho_\epsilon})}\log n+\frac{\mu_\epsilon(-\rho_\epsilon+2\beta_\epsilon-\gamma_\epsilon-1/2)}{\rho_\epsilon(1+c^{\rho_\epsilon})}\log\log n\right]^{-1/\rho_\epsilon}$

S.3 Proof of Theorem 2

Recall that $a(s), b(s), \hat{a}(s), \hat{b}(s)$, $\Delta_j(t)$ for $j = 1, \ldots, 6$, $\Delta_2'(t)$ and $\Delta_2''(t)$ are defined in Section 4 and Step 1 and 2 in the proof of Theorem 1, $c(s), d(s), \tilde{c}(s), \tilde{d}(s)$, $\tilde{\Delta}_j(t)$ for $j = 1, \ldots, 7$, $\tilde{\Delta}_2'(t)$ and $\tilde{\Delta}_2''(t)$ are defined in Step 3 in the proof of Theorem 1.

Step 1:

Similar to the proof of Lemma 3, one can show that

$$\sup_{|t|\le h_n^{-1}} |\Delta_1(t)| = O_P(n^{-1/2} h_n^{-\beta_x+\beta_\epsilon-1} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)),$$

$$\sup_{|t|\le h_n^{-1}} |\Delta_2(t)| = O_P(n^{-1/2} h_n^{-2\beta_x+\beta_\epsilon-1} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)),$$

$$\sup_{|t|\le h_n^{-1}} |\Delta_3(t)| = O_P(n^{-1} h_n^{-2\beta_x+2\beta_\epsilon-1} \exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)),$$

so $\sup_{|t|\le h_n^{-1}} \mathbb{I}\{|\Delta(t)| > 1\} = 0$ with probability approaching 1.

Moreover, similar to the proof of Theorem 1, we have

$$\sup_{x\in\mathbb{R}} \left|\hat{f}_X(x) - f_X(x)\right| \lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)|\, dt + O(h_n^{\beta_x-1}),$$

and it suffices to show

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta(t)| dt = O_P\left(n^{-1/2} h_n^{-\beta_x+\beta_\epsilon-3/2} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right). \tag{S.3.1}$$

Now note that

$$\begin{aligned}\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\Delta_1(t)| dt &\lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\mathbb{E}\left|Y_1 \int_0^t e^{\mathrm{i}sY_2} \frac{1}{b(s)} ds\right|^2\right)^{1/2} dt \\ &\lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\int_0^{|t|} \left|\frac{1}{b(s)}\right|^2 ds\right)^{1/2} dt \\ &= O\left(n^{-1/2} h_n^{\beta_\epsilon-3/2} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),\end{aligned}$$

$$\begin{aligned}\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\Delta_2'(t)| dt &\lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\int_0^{|t|} \left|\frac{a(s)}{b(s)^2}\right|^2 ds\right)^{1/2} dt \\ &= O\left(n^{-1/2} h_n^{-\beta_x+\beta_\epsilon-3/2} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),\end{aligned}$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta_2''(t)|dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \sup_{|s|\leq|t|} \frac{1}{|b(s)|} dt \cdot \int_0^{h_n^{-1}} \left|\hat{b}(s) - b(s)\right|^2 ds \cdot \sup_{|s|\leq h_n^{-1}} \left|\frac{a(s)}{b(s)}\right| \left|\frac{1}{\hat{b}(s)}\right|$$
$$= O_P\left(n^{-1} h_n^{-2\beta_x+2\beta_\epsilon-2} \exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

and

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta_3(t)|dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \sup_{|s|\leq|t|} \frac{1}{|b(s)|} dt \cdot \int_0^{h_n^{-1}} |\hat{a}(s) - a(s)||\hat{b}(s) - b(s)| ds \cdot \sup_{|s|\leq h_n^{-1}} \left|\frac{1}{\hat{b}(s)}\right|$$
$$= O_P\left(n^{-1} h_n^{-\beta_x+2\beta_\epsilon-2} \exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

from which the desired result (S.3.1) follows.

Step 2:

Similar to the proof of Theorem 1 and (A.26) in the proof of Theorem 5, one can show that

$$\sup_{\varepsilon\in\mathbb{R}} \left|\hat{f}_\epsilon(\varepsilon) - f_\epsilon(\varepsilon)\right| \lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)|\, dt + O(\varsigma_{h,q}^\epsilon)$$
$$= \int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_4(t) + \Delta_5(t) + \Delta_6(t)|\, dt + O(\varsigma_{h,q}^\epsilon).$$

Thus the desired result follows from

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_4(t)|\, dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_{Y_1}(t) - \varphi_{Y_1}(t)| dt \cdot \sup_{|t|\leq h_n^{-1}} \frac{1}{|\varphi_X(t)|} = O_P\left(n^{-1/2} h_n^{-\beta_x-1}\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_5(t)|\, dt \leq 2\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_\epsilon(t)||\Delta(t)| dt + \int_{-h_n^{-1}}^{h_n^{-1}} \frac{|\varphi_\epsilon(t)|}{|\varphi_X(t)|} |\hat{\varphi}_X(t) - \varphi_X(t)|\, dt \cdot \sup_{|t|\leq h_n^{-1}} \mathbb{I}\{|\Delta(t)| > 1\}$$
$$= O_P\left(n^{-1/2} h_n^{-2\beta_x-3/2}\right),$$

and

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_6(t)|\, dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| dt \cdot \frac{\sup_{|t|\leq h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)|}{\inf_{|t|\leq h_n^{-1}} |\varphi_X(t)|}$$
$$= o_P\left(\int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| dt\right),$$

where the last equation holds because, similar to the proof of Lemma 4, it can be shown that

$$\sup_{|t|\leq h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)| = O_P\left(n^{-1/2} h_n^{-\beta_x+\beta_\epsilon-1} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right).$$

Step 3:

Similar to previous proofs, it holds that

$$\sup_{x\in\mathbb{R}}\left|\tilde{f}_X(x)-f_X(x)\right|\lesssim\int_{-h_n^{-1}}^{h_n^{-1}}|\tilde{\varphi}_X(t)-\varphi_X(t)|\,dt+O(h_n^{\beta_x-1})$$
$$=\int_{-h_n^{-1}}^{h_n^{-1}}|\tilde{\Delta}_5(t)+\tilde{\Delta}_6(t)+\tilde{\Delta}_7(t)|dt+O(h_n^{\beta_x-1}).$$

First, it can be shown that

$$\int_{-h_n^{-1}}^{h_n^{-1}}\left|\tilde{\Delta}_5(t)\right|dt\le\int_{-h_n^{-1}}^{h_n^{-1}}|\hat{\varphi}_{Y_1}(t)-\varphi_{Y_1}(t)|dt\cdot\sup_{|t|\le h_n^{-1}}\frac{1}{|\varphi_\epsilon(t)|}=O_P\left(n^{-1/2}h_n^{\beta_\epsilon-1}\exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right).$$

Second, we have

$$\int_{-h_n^{-1}}^{h_n^{-1}}|\varphi_X(t)|\cdot\mathbb{E}|\tilde{\Delta}_1(t)|dt=O\left(n^{-1/2}h_n^{\beta_x+2\beta_\epsilon-3/2}\exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}}|\varphi_X(t)|\cdot\mathbb{E}|\tilde{\Delta}_2'(t)|dt=O\left(n^{-1/2}h_n^{\beta_x+3\beta_\epsilon-3/2}\exp(3h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}}|\varphi_X(t)||\tilde{\Delta}_2''(t)|dt=O_P\left(n^{-1}h_n^{\beta_x+5\beta_\epsilon-2}\exp(5h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}}|\varphi_X(t)||\tilde{\Delta}_3(t)|dt=O_P\left(n^{-1}h_n^{\beta_x+4\beta_\epsilon-2}\exp(4h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}}|\varphi_X(t)||\tilde{\Delta}_4(t)|dt\lesssim\sup_{|t|\le h_n^{-1}}|\tilde{\Delta}_4(t)|=O_P(n^{-1/2}h_n^{-1}),$$

which further imply that

$$\int_{-h_n^{-1}}^{h_n^{-1}}\left|\tilde{\Delta}_6(t)\right|dt\le 2\int_{-h_n^{-1}}^{h_n^{-1}}|\varphi_X(t)||\tilde{\Delta}(t)|dt+\int_{-h_n^{-1}}^{h_n^{-1}}\frac{|\varphi_X(t)|}{|\varphi_\epsilon(t)|}|\tilde{\varphi}_\epsilon(t)-\varphi_\epsilon(t)|\,dt\cdot\sup_{|t|\le h_n^{-1}}\mathbb{I}\{|\tilde{\Delta}(t)|>1\}$$
$$=O_P\left(n^{-1/2}h_n^{\beta_x+3\beta_\epsilon-3/2}\exp(3h_n^{-\rho_\epsilon}/\mu_\epsilon)\right).$$

Third, similar to Lemma 4,

$$\sup_{|t|\le h_n^{-1}}|\tilde{\varphi}_\epsilon(t)-\varphi_\epsilon(t)|=O_P\left(n^{-1/2}h_n^{2\beta_\epsilon-1}\exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

so we have

$$\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_7(t)\right| dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)| dt \cdot \frac{\sup_{|t|\leq h_n^{-1}} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)|}{\inf_{|t|\leq h_n^{-1}} |\varphi_\epsilon(t)|}$$
$$= o_P\left(\int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)| dt\right),$$

and this completes the proof of Step 3.

S.4 Proof of Theorem 3

Step 1:
Similar to the proof of Lemma 3, one can show that

$$\sup_{|t|\leq h_n^{-1}} |\Delta_1(t)| = O_P(n^{-1/2} h_n^{\beta_x - \beta_\epsilon - 1} \exp(h_n^{-\rho_x}/\mu_x)),$$

$$\sup_{|t|\leq h_n^{-1}} |\Delta_2(t)| = O_P(n^{-1/2} h_n^{2\beta_x - \beta_\epsilon - 1} \exp(2h_n^{-\rho_x}/\mu_x)),$$

$$\sup_{|t|\leq h_n^{-1}} |\Delta_3(t)| = O_P(n^{-1} h_n^{2\beta_x - 2\beta_\epsilon - 1} \exp(2h_n^{-\rho_x}/\mu_x)),$$

so $\sup_{|t|\leq h_n^{-1}} \mathbb{I}\{|\Delta(t)| > 1\} = 0$ with probability approaching 1.

Moreover, similar to the proof of Theorem 1 and (A.26) in the proof of Theorem 5, we have

$$\sup_{x\in\mathbb{R}} \left|\hat{f}_X(x) - f_X(x)\right| \lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)|\, dt + O(\varsigma_{h,q}^x),$$

and it suffices to show

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta(t)| dt = O_P\left(n^{-1/2} h_n^{\beta_x - \beta_\epsilon - 3/2} \exp(h_n^{-\rho_x}/\mu_x)\right). \tag{S.4.1}$$

Now note that

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\Delta_1(t)| dt \lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\int_0^{|t|} \left|\frac{1}{b(s)}\right|^2 ds\right)^{1/2} dt$$
$$= O\left(n^{-1/2} h_n^{-\beta_\epsilon - 3/2}\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\Delta_2'(t)| dt \lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\int_0^{|t|} \left| \frac{a(s)}{b(s)^2} \right|^2 ds \right)^{1/2} dt$$
$$= O\left(n^{-1/2} h_n^{\beta_x - \beta_\epsilon - 3/2} \exp(h_n^{-\rho_x}/\mu_x) \right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta_2''(t)| dt \le \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \sup_{|s| \le |t|} \frac{1}{|b(s)|} dt \cdot \int_0^{h_n^{-1}} \left| \hat{b}(s) - b(s) \right|^2 ds \cdot \sup_{|s| \le h_n^{-1}} \left| \frac{a(s)}{b(s)} \right| \left| \frac{1}{\hat{b}(s)} \right|$$
$$= O_P\left(n^{-1} h_n^{2\beta_x - 2\beta_\epsilon - 2} \exp(2 h_n^{-\rho_x}/\mu_x) \right),$$

and

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta_3(t)| dt \le \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \sup_{|s| \le |t|} \frac{1}{|b(s)|} dt \cdot \int_0^{h_n^{-1}} |\hat{a}(s) - a(s)||\hat{b}(s) - b(s)| ds \cdot \sup_{|s| \le h_n^{-1}} \left| \frac{1}{\hat{b}(s)} \right|$$
$$= O_P\left(n^{-1} h_n^{\beta_x - 2\beta_\epsilon - 2} \exp(h_n^{-\rho_x}/\mu_x) \right),$$

from which the desired result (S.4.1) follows.

Step 2:
Similar to the proof of Theorem 1, one can show that

$$\sup_{\varepsilon \in \mathbb{R}} \left| \hat{f}_\epsilon(\varepsilon) - f_\epsilon(\varepsilon) \right| \lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| \, dt + O(h_n^{\beta_\epsilon - 1})$$
$$= \int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_4(t) + \Delta_5(t) + \Delta_6(t)| \, dt + O(h_n^{\beta_\epsilon - 1}).$$

Thus the desired result follows from

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_4(t)| \, dt \le \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_{Y_1}(t) - \varphi_{Y_1}(t)| dt \cdot \sup_{|t| \le h_n^{-1}} \frac{1}{|\varphi_X(t)|} = O_P\left(n^{-1/2} h_n^{\beta_x - 1} \exp(h_n^{-\rho_x}/\mu_x) \right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_5(t)| \, dt \le 2 \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_\epsilon(t)||\Delta(t)| dt + \int_{-h_n^{-1}}^{h_n^{-1}} \frac{|\varphi_\epsilon(t)|}{|\varphi_X(t)|} |\hat{\varphi}_X(t) - \varphi_X(t)| \, dt \cdot \sup_{|t| \le h_n^{-1}} \mathbb{I}\{|\Delta(t)| > 1\}$$
$$= O_P\left(n^{-1/2} h_n^{2\beta_x - 3/2} \exp(2 h_n^{-\rho_x}/\mu_x) \right),$$

and

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_6(t)| \, dt \le \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| dt \cdot \frac{\sup_{|t| \le h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)|}{\inf_{|t| \le h_n^{-1}} |\varphi_X(t)|}$$
$$= o_P\left(\int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| dt \right),$$

where the last equation holds because, similar to the proof of Lemma 4, it can be shown that

$$\sup_{|t|\leq h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)| = O_P\left(n^{-1/2} h_n^{\beta_x - \beta_\epsilon - 1} \exp(h_n^{-\rho_x}/\mu_x)\right).$$

Step 3:
Similar to previous proofs, it holds that

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|\tilde{f}_X(x) - f_X(x)\right| &\lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)|\, dt + O(\varsigma_{h,q}^x) \\ &= \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\Delta}_5(t) + \tilde{\Delta}_6(t) + \tilde{\Delta}_7(t)| dt + O(\varsigma_{h,q}^x).\end{aligned}$$

First, it can be shown that

$$\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_5(t)\right| dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_{Y_1}(t) - \varphi_{Y_1}(t)| dt \cdot \sup_{|t|\leq h_n^{-1}} \frac{1}{|\varphi_\epsilon(t)|} = O_P\left(n^{-1/2} h_n^{-\beta_\epsilon - 1}\right).$$

Second, we have

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\tilde{\Delta}_1(t)| dt = O\left(n^{-1/2}\right), \quad \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\tilde{\Delta}_2'(t)| dt = O\left(n^{-1/2}\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}_2''(t)| dt = O_P\left(n^{-1} h_n^{-3\beta_\epsilon - 1}\right), \quad \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}_3(t)| dt = O_P\left(n^{-1} h_n^{-2\beta_\epsilon - 1}\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}_4(t)| dt = O_P(n^{-1/2} h_n^{-1}),$$

which further imply that

$$\begin{aligned}\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_6(t)\right| dt &\leq 2\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}(t)| dt + \int_{-h_n^{-1}}^{h_n^{-1}} \frac{|\varphi_X(t)|}{|\varphi_\epsilon(t)|} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)|\, dt \cdot \sup_{|t|\leq h_n^{-1}} \mathbb{I}\{|\tilde{\Delta}(t)| > 1\} \\ &= O_P\left(n^{-1/2} h_n^{-1}\right).\end{aligned}$$

Third, similar to Lemma 4,

$$\sup_{|t|\leq h_n^{-1}} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| = O_P\left(n^{-1/2} h_n^{-2\beta_\epsilon - 1}\right),$$

so we have

$$\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_7(t)\right| dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)| dt \cdot \frac{\sup_{|t|\leq h_n^{-1}} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)|}{\inf_{|t|\leq h_n^{-1}} |\varphi_\epsilon(t)|}$$
$$= o_P\left(\int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)| dt\right),$$

and this completes the proof of Step 3.

S.5 Proof of Theorem 4

Step 1:

Similar to the proof of Lemma 3, one can show that

$$\sup_{|t|\leq h_n^{-1}} |\Delta_1(t)| = O_P(n^{-1/2} h_n^{\beta_x+\beta_\epsilon-1} \exp(h_n^{-\rho_x}/\mu_x + h_n^{-\rho_\epsilon}/\mu_\epsilon)),$$

$$\sup_{|t|\leq h_n^{-1}} |\Delta_2(t)| = O_P(n^{-1/2} h_n^{2\beta_x+\beta_\epsilon-1} \exp(2h_n^{-\rho_x}/\mu_x + h_n^{-\rho_\epsilon}/\mu_\epsilon)),$$

$$\sup_{|t|\leq h_n^{-1}} |\Delta_3(t)| = O_P(n^{-1} h_n^{2\beta_x+2\beta_\epsilon-1} \exp(2h_n^{-\rho_x}/\mu_x + 2h_n^{-\rho_\epsilon}/\mu_\epsilon)),$$

so $\sup_{|t|\leq h_n^{-1}} \mathbb{I}\{|\Delta(t)| > 1\} = 0$ with probability approaching 1.

Moreover, similar to the proof of Theorem 1 and (A.26) in the proof of Theorem 5, we have

$$\sup_{x\in\mathbb{R}} \left|\hat{f}_X(x) - f_X(x)\right| \lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)|\, dt + O(\varsigma_{h,q}^x),$$

and it suffices to show

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta(t)| dt = O_P\left(n^{-1/2} h_n^{\beta_x+\beta_\epsilon-3/2} \exp(h_n^{-\rho_x}/\mu_x + h_n^{-\rho_\epsilon}/\mu_\epsilon)\right). \tag{S.5.1}$$

Now note that

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\Delta_1(t)| dt \lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\int_0^{|t|} \left|\frac{1}{b(s)}\right|^2 ds\right)^{1/2} dt$$
$$= O\left(n^{-1/2} h_n^{\beta_\epsilon-3/2} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\Delta_2'(t)| dt \lesssim n^{-1/2} \cdot \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \left(\int_0^{|t|} \left| \frac{a(s)}{b(s)^2} \right|^2 ds \right)^{1/2} dt$$
$$= O\left(n^{-1/2} h_n^{\beta_x + \beta_\epsilon - 3/2} \exp(h_n^{-\rho_x}/\mu_x + h_n^{-\rho_\epsilon}/\mu_\epsilon) \right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta_2''(t)| dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \sup_{|s|\leq|t|} \frac{1}{|b(s)|} dt \cdot \int_0^{h_n^{-1}} \left| \hat{b}(s) - b(s) \right|^2 ds \cdot \sup_{|s|\leq h_n^{-1}} \left| \frac{a(s)}{b(s)} \right| \left| \frac{1}{\hat{b}(s)} \right|$$
$$= O_P\left(n^{-1} h_n^{2\beta_x + 2\beta_\epsilon - 2} \exp(2h_n^{-\rho_x}/\mu_x + 2h_n^{-\rho_\epsilon}/\mu_\epsilon) \right),$$

and

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\Delta_3(t)| dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \sup_{|s|\leq|t|} \frac{1}{|b(s)|} dt \cdot \int_0^{h_n^{-1}} |\hat{a}(s) - a(s)||\hat{b}(s) - b(s)| ds \cdot \sup_{|s|\leq h_n^{-1}} \left| \frac{1}{\hat{b}(s)} \right|$$
$$= O_P\left(n^{-1} h_n^{\beta_x + 2\beta_\epsilon - 2} \exp(h_n^{-\rho_x}/\mu_x + 2h_n^{-\rho_\epsilon}/\mu_\epsilon) \right),$$

from which the desired result (S.5.1) follows.

Step 2:
Similar to the proof of Theorem 1 and (A.26) in the proof of Theorem 5, one can show that

$$\sup_{\varepsilon\in\mathbb{R}} \left| \hat{f}_\epsilon(\varepsilon) - f_\epsilon(\varepsilon) \right| \lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| \, dt + O(\varsigma_{h,q}^\epsilon)$$
$$= \int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_4(t) + \Delta_5(t) + \Delta_6(t)| \, dt + O(\varsigma_{h,q}^\epsilon).$$

Thus the desired result follows from

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_4(t)| \, dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_{Y_1}(t) - \varphi_{Y_1}(t)| dt \cdot \sup_{|t|\leq h_n^{-1}} \frac{1}{|\varphi_X(t)|} = O_P\left(n^{-1/2} h_n^{\beta_x - 1} \exp(h_n^{-\rho_x}/\mu_x) \right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_5(t)| \, dt \leq 2 \int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_\epsilon(t)||\Delta(t)| dt + \int_{-h_n^{-1}}^{h_n^{-1}} \frac{|\varphi_\epsilon(t)|}{|\varphi_X(t)|} |\hat{\varphi}_X(t) - \varphi_X(t)| \, dt \cdot \sup_{|t|\leq h_n^{-1}} \mathbb{I}\{|\Delta(t)| > 1\}$$
$$= O_P\left(n^{-1/2} h_n^{2\beta_x - 3/2} \exp(2h_n^{-\rho_x}/\mu_x) \right),$$

and

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\Delta_6(t)| \, dt \leq \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| dt \cdot \frac{\sup_{|t|\leq h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)|}{\inf_{|t|\leq h_n^{-1}} |\varphi_X(t)|}$$
$$= o_P\left(\int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| dt \right),$$

where the last equation holds because, similar to the proof of Lemma 4, it can be shown that

$$\sup_{|t|\le h_n^{-1}} |\hat{\varphi}_X(t) - \varphi_X(t)| = O_P\left(n^{-1/2} h_n^{\beta_x+\beta_\epsilon-1} \exp(h_n^{-\rho_x}/\mu_x + h_n^{-\rho_\epsilon}/\mu_\epsilon)\right).$$

Step 3:

Similar to previous proofs, it holds that

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|\tilde{f}_X(x) - f_X(x)\right| &\lesssim \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)|\, dt + O(\varsigma_{h,q}^x) \\ &= \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\Delta}_5(t) + \tilde{\Delta}_6(t) + \tilde{\Delta}_7(t)| dt + O(\varsigma_{h,q}^x).\end{aligned}$$

First, it can be shown that

$$\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_5(t)\right| dt \le \int_{-h_n^{-1}}^{h_n^{-1}} |\hat{\varphi}_{Y_1}(t) - \varphi_{Y_1}(t)| dt \cdot \sup_{|t|\le h_n^{-1}} \frac{1}{|\varphi_\epsilon(t)|} = O_P\left(n^{-1/2} h_n^{\beta_\epsilon-1} \exp(h_n^{-\rho_\epsilon}/\mu_\epsilon)\right).$$

Second, we have

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\tilde{\Delta}_1(t)| dt = O\left(n^{-1/2}[1 \vee (h_n^{-\beta_x+2\beta_\epsilon-3/2} \exp(-h_n^{-\rho_x}/\mu_x + 2h_n^{-\rho_\epsilon}/\mu_\epsilon))]\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)| \cdot \mathbb{E}|\tilde{\Delta}_2'(t)| dt = O\left(n^{-1/2}[1 \vee (h_n^{-\beta_x+3\beta_\epsilon-3/2} \exp(-h_n^{-\rho_x}/\mu_x + 3h_n^{-\rho_\epsilon}/\mu_\epsilon))]\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}_2''(t)| dt = O_P\left(n^{-1} h_n^{3\beta_\epsilon-1} \exp(3h_n^{-\rho_\epsilon}/\mu_\epsilon)[1 \vee (h_n^{-\beta_x+2\beta_\epsilon-1} \exp(-h_n^{-\rho_x}/\mu_x + 2h_n^{-\rho_\epsilon}/\mu_\epsilon))]\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}_3(t)| dt = O_P\left(n^{-1} h_n^{2\beta_\epsilon-1} \exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)[1 \vee (h_n^{-\beta_x+2\beta_\epsilon-1} \exp(-h_n^{-\rho_x}/\mu_x + 2h_n^{-\rho_\epsilon}/\mu_\epsilon))]\right),$$

$$\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}_4(t)| dt = O_P(n^{-1/2} h_n^{-1}),$$

which further imply that

$$\begin{aligned}\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_6(t)\right| dt &\le 2\int_{-h_n^{-1}}^{h_n^{-1}} |\varphi_X(t)||\tilde{\Delta}(t)| dt + \int_{-h_n^{-1}}^{h_n^{-1}} \frac{|\varphi_X(t)|}{|\varphi_\epsilon(t)|} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)|\, dt \cdot \sup_{|t|\le h_n^{-1}} \mathbb{I}\{|\tilde{\Delta}(t)| > 1\} \\ &= O_P\left(n^{-1/2}[h_n^{-1} \vee (h_n^{-\beta_x+3\beta_\epsilon-3/2} \exp(-h_n^{-\rho_x}/\mu_x + 3h_n^{-\rho_\epsilon}/\mu_\epsilon))]\right).\end{aligned}$$

Third, similar to Lemma 4,

$$\sup_{|t|\le h_n^{-1}} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)| = O_P\left(n^{-1/2} h_n^{2\beta_\epsilon - 1} \exp(2h_n^{-\rho_\epsilon}/\mu_\epsilon)\right),$$

so we have

$$\begin{aligned}
\int_{-h_n^{-1}}^{h_n^{-1}} \left|\tilde{\Delta}_7(t)\right| dt &\le \int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)| dt \cdot \frac{\sup_{|t|\le h_n^{-1}} |\tilde{\varphi}_\epsilon(t) - \varphi_\epsilon(t)|}{\inf_{|t|\le h_n^{-1}} |\varphi_\epsilon(t)|} \\
&= o_P\left(\int_{-h_n^{-1}}^{h_n^{-1}} |\tilde{\varphi}_X(t) - \varphi_X(t)| dt\right),
\end{aligned}$$

and this completes the proof of Step 3.