Appendix A Proof in section 5

A.1 Proof of Proposition 5.1

Proof. (i). To compare σ_S and $(n-k)\sigma_{k+1} + \sum_{i=1}^{k} \sigma_i$, we first rewrite σ_S^2 as

$$\begin{aligned}\sigma_S^2 &= Var[S] = Var\left[\sum_{i=1}^{k} C_i + \sum_{i=k+1}^{n} C_i\right] \\ &= Var\left[\sum_{i=1}^{k} C_i\right] + Var\left[\sum_{i=k+1}^{n} C_i\right] + 2Cov\left[\sum_{i=1}^{k} C_i, \sum_{i=k+1}^{n} C_i\right]. \end{aligned} \tag{A1}$$

In the following, we shall augment each of the three terms in (A1). Specifically,

$$Var\left[\sum_{i=1}^{k} C_i\right] \leq \left(\sum_{i=1}^{k} \sigma_i\right)^2, \tag{A2}$$

and

$$\begin{aligned} Var\left[\sum_{i=k+1}^{n} C_i\right] &= \sum_{i=k+1}^{n} \sigma_i^2 + 2 \sum_{k+1\leq i<j\leq n} \rho_{ij}\sigma_i\sigma_j \\ &\leq \sum_{i=k+1}^{n} \sigma_i^2 + 2 \sum_{k+1\leq i<j\leq n} \rho_u\sigma_i\sigma_j \\ &= (1-\rho_u) \sum_{i=k+1}^{n} \sigma_i^2 + \rho_u \left(\sum_{i=k+1}^{n} \sigma_i\right)^2 \\ &\leq (1-\rho_u)(n-k-1)\sigma_n^2 + \rho_u((n-k)^2-1)\sigma_n^2 + \sigma_{k+1}^2 \\ &= (n-k-1)((1-\rho_u) + \rho_u(n-k+1))\sigma_n^2 + \sigma_{k+1}^2 \\ &< (n-k-1)(n-k+1)\sigma_{k+1}^2 + \sigma_{k+1}^2 = (n-k)^2\sigma_{k+1}^2, \end{aligned} \tag{A3}$$

where the last inequality follows from the condition (5.2).

Furthermore, note that (5.2) implies that $\sigma_{k+1}^2 \geq \rho_u\sigma_n^2$ and thus $\sigma_{k+1} \geq \rho_u\sigma_n$ due to $\sigma_{k+1} < \sigma_n$. It follows that

$$\begin{aligned} 2\,Cov\left[\sum_{i=1}^{k} C_i, \sum_{i=k+1}^{n} C_i\right] &= 2 \sum_{\substack{1\leq i\leq k \\ k+1\leq j\leq k}} \rho_{ij}\sigma_i\sigma_j \\ \leq 2 \sum_{\substack{1\leq i\leq k \\ k+1\leq j\leq k}} \rho_u\sigma_i\sigma_j &= 2\rho_u \left(\sum_{i=1}^{k} \sigma_i\right) \times \left(\sum_{i=k+1}^{n} \sigma_i\right) \end{aligned}$$

$$\leq 2\rho_u \times (n-k)\sigma_n \times \left(\sum_{i=1}^{k}\sigma_i\right) \leq 2(n-k)\sigma_{k+1} \times \left(\sum_{i=1}^{k}\sigma_i\right). \tag{A4}$$

Plugging (A2), (A3), and (A4) into (A1), we get

$$\sigma_S^2 < \left(\sum_{i=1}^{k}\sigma_i\right)^2 + (n-k)^2\sigma_{k+1}^2 + 2(n-k)\sigma_{k+1} \times \left(\sum_{i=1}^{k}\sigma_i\right) \tag{A5}$$

$$= \left((n-k)\sigma_{k+1} + \sum_{i=1}^{k}\sigma_i\right)^2. \tag{A6}$$

This completes the proof of part (i).

(ii) First suppose (5.3) holds. Following a similar argument as in (i), we decompose σ_S^2 into three components.

$$\sigma_S^2 = Var\left[\sum_{i=1}^{k+1} C_i\right] + Var\left[\sum_{i=k+2}^{n} C_i\right] + 2\,Cov\left[\sum_{i=1}^{k+1} C_i, \sum_{i=k+2}^{n} C_i\right].$$

The three terms can be minified as follows

$$\begin{aligned}
Var\left[\sum_{i=k+2}^{n} C_i\right] &\geq (1-\rho_l)\sum_{i=k+2}^{n}\sigma_i^2 + \rho_l\left(\sum_{i=k+2}^{n}\sigma_i\right)^2 \\
&\geq (1-\rho_l)(n-k-1)\sigma_{k+2}^2 + \rho_l(n-k-1)^2\sigma_{k+2}^2 \\
&\geq \left((n-k-1)^2\rho_l + (n^2-k-1)(1-\rho_l)\right)\sigma_{k+1}^2 \\
&= (n-k-1)^2\sigma_{k+1}^2 + (1-\rho_l)\left((n^2-k-1)-(n-k-1)^2\right)\sigma_{k+1}^2
\end{aligned}$$

where the last inequality follows from condition (5.3),

$$\begin{aligned}
Var\left[\sum_{i=1}^{k+1} C_i\right] &\geq (1-\rho_l)\sum_{i=1}^{k+1}\sigma_i^2 + \rho_l\left(\sum_{i=1}^{k+1}\sigma_i\right)^2 \\
&= \left(\sum_{i=1}^{k+1}\sigma_i\right)^2 - (1-\rho_l)\left(\left(\sum_{i=1}^{k+1}\sigma_i\right)^2 - \sum_{i=1}^{k+1}\sigma_i^2\right) \\
&= \left(\sum_{i=1}^{k+1}\sigma_i\right)^2 - (1-\rho_l)\times 2\sum_{1\leq i<j\leq k+1}\sigma_i\sigma_j \\
&\geq \left(\sum_{i=1}^{k+1}\sigma_i\right)^2 - (1-\rho_l)k(k+1)\sigma_{k+1}^2,
\end{aligned}$$

and

$$
\begin{aligned}
2\,Cov\left[\sum_{i=1}^{k+1} C_i,\ \sum_{i=k+2}^{n} C_i\right] &\geq 2\rho_l(n-k-1)\sigma_{k+2}\left(\sum_{i=1}^{k+1}\sigma_i\right) \\
&\geq 2\rho_l(n-k-1)\sigma_{k+1}\left(\sum_{i=1}^{k+1}\sigma_i\right) \\
&= 2(n-k-1)\sigma_{k+1}\left(\sum_{i=1}^{k+1}\sigma_i\right) - 2(n-k-1)(1-\rho_l)\sigma_{k+1}\left(\sum_{i=1}^{k+1}\sigma_i\right) \\
&\geq 2(n-k-1)\sigma_{k+1}\left(\sum_{i=1}^{k+1}\sigma_i\right) - 2(1-\rho_l)(n-k-1)(k+1)\sigma_{k+1}^2.
\end{aligned}
$$

Combining these three inequalities together yield

$$
\begin{aligned}
\sigma_S^2 \geq\ & (n-k-1)^2\sigma_{k+1}^2 + (1-\rho_l)\left((n^2-k-1)-(n-k-1)^2\right)\sigma_{k+1}^2 \\
&+ \left(\sum_{i=1}^{k+1}\sigma_i\right)^2 - (1-\rho_l)k(k+1)\sigma_{k+1}^2 \\
&+ 2(n-k-1)\sigma_{k+1}\left(\sum_{i=1}^{k+1}\sigma_i\right) - 2(1-\rho_l)(n-k-1)(k+1)\sigma_{k+1}^2 \\
=\ & (n-k-1)^2\sigma_{k+1}^2 + \left(\sum_{i=1}^{k+1}\sigma_i\right)^2 + 2(n-k-1)\sigma_{k+1}\left(\sum_{i=1}^{k+1}\sigma_i\right) \\
=\ & \left((n-k-1)\sigma_{k+1} + \sum_{i=1}^{k+1}\sigma_i\right)^2.
\end{aligned}
$$

This is the desired inequality in (ii).

Now we prove that (5.4) also implies the desired inequality

$$\sigma_S \geq (n-k)\sigma_{k+1} + \sum_{i=1}^{k}\sigma_i.$$

Note that

$$\sigma_S^2 \geq (1-\rho_l)\sum_{i=1}^{n}\sigma_i^2 + \rho_l\left(\sum_{i=1}^{n}\sigma_i\right)^2 \geq \rho_l\left(\sum_{i=1}^{n}\sigma_i\right)^2$$

It suffices to show that

$$\sqrt{\rho_l}\sum_{i=1}^{n}\sigma_i \geq (n-k)\sigma_{k+1} + \sum_{i=1}^{k}\sigma_i$$

which can be rearranged as

$$\sqrt{\rho_l}\sum_{i=k+2}^{n}\sigma_i \geq (n-k-\sqrt{\rho_l})\sigma_{k+1} + (1-\sqrt{\rho_l})\sum_{i=1}^{k}\sigma_i.$$

This can be implied by

$$\sqrt{\rho_l}(n-k-1)\sigma_{k+2} \geq \left(n-(k+1)\sqrt{\rho_l}\right)\sigma_{k+1},$$

which is equivalent to condition (5.4). □

A.2 Proof of Theorem 5.3

In order to prove Theorem 5.3, we first establish two properties of the optimal solution.

Lemma A.1. *Let* $(a_1,\ldots,a_n)$ *be the solution to Problem* (3.10). *If there exists* $k \in \{1,\ldots,n\}$ *such that* $a_k^2 = \frac{\sigma_k^2}{\sigma_S^2}$, *then* $a_i^2 = \frac{\sigma_i^2}{\sigma_S^2}$ *for any* $i = 1,\ldots,k-1$.

Proof. By induction, it suffices to show that $a_{k-1}^2 = \frac{\sigma_{k-1}^2}{\sigma_S^2}$. Assume this does not hold, i.e, $a_{k-1}^2 \neq \frac{\sigma_{k-1}^2}{\sigma_S^2}$. Since $a_{k-1}^2 \leq \frac{\sigma_{k-1}^2}{\sigma_S^2}$ due to the variance reduction constraint, it must hold that $a_{k-1}^2 < \frac{\sigma_{k-1}^2}{\sigma_S^2}$.

Construct sharing ratios $\{\widetilde{a}_1,\ldots,\widetilde{a}_n\}$ with $\widetilde{a}_i = a_i$ for all $i \neq k-1, k$, and

$$\widetilde{a}_{k-1} = a_{k-1} + x, \quad \widetilde{a}_k = a_k - x.$$

We shall prove that there exists x such that the risk sharing strategy $(\widetilde{a}_1,\ldots,\widetilde{a}_n)$ belongs to $\mathcal{A}_{vr}\bigcap\mathcal{A}_{rc}^{\gamma}$ and it results in a total variance smaller than $(a_1,\ldots,a_n)$, which contradicts the fact that $(a_1,\ldots,a_n)$ is the optimal solution and thus proves the desired equality.

Note that $(\widetilde{a}_1,\ldots,\widetilde{a}_n)$ differs from $(a_1,\ldots,a_n)$ only in the positions of $k-1$ and k, it suffices to verify the following conditions:

(i) $\widetilde{a}_{k-1}^2 \leq \frac{\sigma_{k-1}^2}{\sigma_S^2}$ and $\widetilde{a}_k^2 \leq \frac{\sigma_k^2}{\sigma_S^2}$;

(ii) $\frac{a_{k-2}^2}{\sigma_{k-2}^{2\gamma}} \leq \frac{\widetilde{a}_{k-1}^2}{\sigma_{k-1}^{2\gamma}} \leq \frac{\widetilde{a}_k^2}{\sigma_k^{2\gamma}} \leq \frac{a_{k+1}^2}{\sigma_{k+1}^{2\gamma}}$

(iii) $\widetilde{a}_{k-1}^2 + \widetilde{a}_k^2 < a_{k-1}^2 + a_k^2$.

Note that $\widetilde{a}_{k-1}$ is a continuous function of x. Since $\widetilde{a}_{k-1}^2|_{x=0} = a_{k-1}^2 < \frac{\sigma_{k-1}^2}{\sigma_S^2}$, there exists $\delta_1 > 0$ such that

$$\widetilde{a}_{k-1}^2 \leq \frac{\sigma_{k-1}^2}{\sigma_S^2}, \qquad \text{for any } x \in [0, \delta_1). \tag{A7}$$

Note that $a_{k-1}^2 < \frac{\sigma_{k-1}^2}{\sigma_S^2} \leq \frac{\sigma_k^2}{\sigma_S^2} = a_k^2$, we have $a_{k-1} < a_k$. Thus, there exists $\delta_2 \in [0, \delta_1)$ such that for any $x \in [0, \delta_2)$, it holds that

$$a_{k-1} < \widetilde{a}_{k-1} < \widetilde{a}_k < a_k. \tag{A8}$$

The last inequality immediately implies that $\widetilde{a}_k < a_k \leq \frac{\sigma_k^2}{\sigma_S^2}$, which together with (A7) indicates that the strategy $(\widetilde{a}_1, \ldots, \widetilde{a}_n)$ satisfies Condition (i) for any $x \in [0, \delta_2)$.

Noting that $\widetilde{a}_{k-1} + \widetilde{a}_k = a_{k-1} + a_k$, (A8) implies that $(\widetilde{a}_{k-1}, \widetilde{a}_k)$ is strictly majorized by (a_{k-1}, a_k), and thus $\widetilde{a}_{k-1}^2 + \widetilde{a}_k^2 < a_{k-1}^2 + a_k^2$ according to Proposition B.1 of Marshall et al. (2011). Thus, $(\widetilde{a}_1, \ldots, \widetilde{a}_n)$ satisfies Condition (iii).

(A8) also implies that

$$\frac{\widetilde{a}_{k-1}^2}{\sigma_{k-1}^{2\gamma}} \geq \frac{a_{k-1}^2}{\sigma_{k-1}^{2\gamma}} \geq \frac{a_{k-2}^2}{\sigma_{k-2}^{2\gamma}}, \quad \text{and} \quad \frac{\widetilde{a}_k^2}{\sigma_k^{2\gamma}} \leq \frac{a_k^2}{\sigma_k^{2\gamma}} \leq \frac{a_{k+1}^2}{\sigma_{k+1}^{2\gamma}}. \tag{A9}$$

Recalling that $a_{k-1}^2 < \frac{\sigma_{k-1}^2}{\sigma_S^2}$, $a_k^2 = \frac{\sigma_k^2}{\sigma_S^2}$, and $\sigma_{k-1} < \sigma_k$, we have

$$\left.\frac{\widetilde{a}_{k-1}^2}{\sigma_{k-1}^{2\gamma}}\right|_{x=0} = \frac{a_{k-1}^2}{\sigma_{k-1}^{2\gamma}} < \frac{\sigma_{k-1}^{2(1-\gamma)}}{\sigma_S^2} \leq \frac{\sigma_k^{2(1-\gamma)}}{\sigma_S^2} = \frac{a_k^2}{\sigma_k^{2\gamma}} = \left.\frac{\widetilde{a}_k^2}{\sigma_k^{2\gamma}}\right|_{x=0}. \tag{A10}$$

Since $\widetilde{a}_{k-1}^2$ and $\widetilde{a}_k^2$ are both continuous in x, there exists $\delta_3 \in (0, \delta_2)$ such that for any $x \in [0, \delta_3)$, it holds that $\frac{\widetilde{a}_{k-1}^2}{\sigma_{k-1}^{2\gamma}} \leq \frac{\widetilde{a}_k^2}{\sigma_k^{2\gamma}}$. This, together with (A9), implies Condition (ii). ☐

Lemma A.2. *Let $(a_1, \ldots, a_n)$ be the solution to Problem (3.10). If there exists $k \in \{1, \ldots, n\}$ such that $a_k^2 < \frac{\sigma_k^2}{\sigma_S^2}$ for some $1 \leq k \leq n$, then*

$$\frac{a_i^2}{\sigma_i^{2\gamma}} = \frac{a_k^2}{\sigma_k^{2\gamma}}, \qquad i = k+1, \ldots, n.$$

Proof. It can be proved using a similar argument as in the proof of Lemma A.1. ☐

Proof of Theorem 3.10: Let $(a_1, \ldots, a_n)$ be the solution to Problem (3.10). Define $k^* = \sup\{i \in \{1, \ldots, n\} | a_i = \frac{\sigma_i}{\sigma_S}\}$ with the convention of $\sup \emptyset = 0$.

If $k^* = 0$, then $a_i < \frac{\sigma_i}{\sigma_S}$ for all $i = 1, \dots, n$. According to Lemma A.2, we have $\frac{a_i}{\sigma_i^\gamma} = \dots = \frac{a_n}{\sigma_n^\gamma}$, which together with the condition of $a_1 + \dots + a_n = 1$ implies that

$$a_i = \frac{\sigma_i^\gamma}{\sum_{j=1}^n \sigma_j^\gamma}, \qquad i = 1, \dots, n. \tag{A11}$$

If $k^* = k$ for some $k \in \{1, \dots, n\}$, then $a_i = \frac{\sigma_i}{\sigma_S}$ for $i = 1, \dots, k$ and $\frac{a_{k+1}}{\sigma_{k+1}^\gamma} = \dots = \frac{a_n}{\sigma_n^\gamma}$. This, together with the condition of condition of $a_1 + \dots + a_n = 1$ implies that

$$a_i = \frac{\sigma_i^\gamma}{\sum_{j=k+1}^n \sigma_j^\gamma}\Big(1 - \sum_{j=1}^k \frac{\sigma_j}{\sigma_S}\Big), \quad \text{for } i = k+1, \dots, n. \tag{A12}$$

Since k^* must take value in $\{0, 1, \dots, n\}$, the solution to Problem (3.10) must take one of the $n + 1$ forms specified by (A11) and (A12). It is easy to verify that these forms are respectively taken when $(C_1, \dots, C_n)$ falls into $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$. This completes the proof.

A.3 Proof of Proposition 5.6

Proof. According to Proposition 5.4

$$Var[L_i] = \left(\frac{\sigma_i}{\sum\limits_{k=1}^n \sigma_k}\right)^2 \sigma(S)^2, \quad Var[\widehat{L}_i] = \left(\frac{\sigma_i}{\sum\limits_{k=1}^{n+1} \sigma_k}\right)^2 \sigma(\widehat{S})^2.$$

To prove $Var[L_i] \geq Var[\widehat{L}_i]$, it suffices to show that

$$\frac{\sigma(S)^2}{\left(\sum_{k=1}^n \sigma_k\right)^2} \geq \frac{\sigma(\widehat{S})^2}{\left(\sum_{k=1}^{n+1} \sigma_k\right)^2}. \tag{A13}$$

Note that

$$\sigma\left(\widehat{S}\right)^2 = \sigma\left(S\right)^2 + \sigma_{n+1}^2 + 2\sum_{1 \leq i \leq n} \rho_{i,n+1}\sigma_i\sigma_{n+1},$$

$$\left(\sum_{k=1}^{n+1} \sigma_k\right)^2 = \left(\sum_{k=1}^{n} \sigma_k\right)^2 + \sigma_{n+1}^2 + 2\sum_{1 \leq i \leq n} \sigma_i\sigma_{n+1}.$$

It is easy to verify that

$$\frac{a}{b} \geq \frac{c}{d} \Longleftrightarrow \frac{a}{b} \geq \frac{c-a}{d-b}$$

for any $d > b > 0$. Applying this result, we have (A13) is equivalent to

$$\frac{\sigma(S)^2}{\left(\sum_{i=1}^n \sigma_i\right)^2} \geq \frac{\sigma_{n+1}^2 + 2\sigma_{n+1}\sum_{i=1}^n \rho_{i,n+1}\sigma_i}{\sigma_{n+1}^2 + 2\sigma_{n+1}\sum_{i=1}^n \sigma_i}. \tag{A14}$$

Further note that

$$\begin{aligned}\sigma_S^2 &\geq (1-\rho_l)\sum_{i=1}^n \sigma_i^2 + \rho_l\left(\sum_{i=1}^n \sigma_i\right)^2 \\ &\geq (1-\rho_l)\times\frac{1}{n}\left(\sum_{i=1}^n \sigma_i\right)^2 + \rho_l\left(\sum_{i=1}^n \sigma_i\right)^2 = \left(\frac{1+(n-1)\rho_l}{n}\right)\left(\sum_{i=1}^n \sigma_i\right)^2,\end{aligned}$$

and

$$\sigma_{n+1}^2 + 2\sigma_{n+1}\sum_{i=1}^n \rho_{i,n+1}\sigma_i \leq \sigma_{n+1}^2 + 2\widehat{\rho}_h\sigma_{n+1}\sum_{i=1}^n \sigma_i.$$

Therefore, a sufficient condition for (A14) is

$$\begin{aligned}&\frac{\left(\frac{1+(n-1)\rho_l}{n}\right)\left(\sum_{i=1}^n \sigma_i\right)^2}{\left(\sum_{i=1}^n \sigma_i\right)^2} \geq \frac{\sigma_{n+1}^2 + 2\widehat{\rho}_h\sigma_{n+1}\sum_{i=1}^n \sigma_i}{\sigma_{n+1}^2 + 2\sigma_{n+1}\sum_{i=1}^n \sigma_i} \\ \Longleftrightarrow\ &\left(\frac{1+(n-1)\rho_l}{n}\right) \geq \frac{\sigma_{n+1} + 2\widehat{\rho}_h\sum_{i=1}^n \sigma_i}{\sigma_{n+1} + 2\sum_{i=1}^n \sigma_i},\end{aligned}$$

which is easily verified to be equivalent to condition (5.14). □

Appendix B Proof in Section 6

B.1 Proof of Proposition 6.1

Proof. Suppose $(C_1, \ldots, C_n) \in \mathcal{U}_k$, i.e,

$$(n-k+1)\sigma_k + \sum_{i=1}^{k-1}\sigma_i \leq \sigma_S < (n-k)\sigma_{k+1} + \sum_{i=1}^{k}\sigma_i. \tag{B1}$$

According to Theorem 3.1, the solution to Problem (2.3) admits the form specified by (3.1), thus Problem (2.3) reduces to

$$\min_{(a_1,\ldots,a_n)\in\mathcal{A}_{vr}} \quad \sum_{i=1}^n a_i^2.$$

Note that

$$\begin{aligned}\sum_{i=1}^{n} a_i^2 &= \sum_{i=1}^{k} a_i^2 + \sum_{i=k+1}^{n} a_i^2 \\ &\geq \sum_{i=1}^{k} a_i^2 + \frac{1}{n-k}\left(\sum_{i=k+1}^{n} a_i\right)^2 \qquad \text{(B2)} \\ &= \sum_{i=1}^{k} a_i^2 + \frac{1}{n-k}\left(1-\sum_{i=1}^{k} a_i\right)^2 \triangleq g(a_1,\ldots,g_k),\end{aligned}$$

where (B2) follows from Cauchy-Schwarz inequality with equality holds if any only if $a_{k+1} = \cdots = a_n$.

Recall that $a_i \leq \frac{\sigma_i}{\sigma_S}$ for $i = 1,\ldots,n$ due to the variance reduction constraint $(a_1,\ldots,a_n) \in \mathcal{A}_{vr}$. For any $j = 1,\ldots,k$ and $a_j < \frac{\sigma_j}{\sigma_S}$, we have

$$\begin{aligned}\frac{\partial}{\partial a_j} g(a_1,\ldots,a_k) &= 2a_j + \frac{2}{n-k}\left(\sum_{i=1}^{k} a_i - 1\right) \\ &< \frac{2\sigma_j}{\sigma_S} + \frac{2}{n-k}\left(\sum_{i=1}^{k} \frac{\sigma_i}{\sigma_S} - 1\right) \\ &= \frac{2}{(n-k)\sigma_S}\left((n-k)\sigma_j + \sum_{i=1}^{k} \sigma_i - \sigma_S\right) \leq 0\end{aligned}$$

where the last inequality is due to (B1). This implies $g(a_1,\ldots,g_k)$ decreases in a_j on $[0, \frac{\sigma_j}{\sigma_S})$ for all $j = 1,\ldots,k$. Thus, $g(a_i,\ldots,a_k)$ attains its minimum over $\mathcal{A}_{vr}$ at $a_1 = \frac{\sigma_1}{\sigma_S},\ldots,a_k = \frac{\sigma_k}{\sigma_S}$. Following (B2), $\sum_{i=1}^{n} a_i^2$ attains its minimum over $\mathcal{A}_{vr}$ at

$$a_1^* = \frac{\sigma_1}{\sigma_S},\ldots,a_k^* = \frac{\sigma_k}{\sigma_S}, \quad a_{k+1}^* = \cdots = a_n^* = \frac{\sigma_S - \sum_{i=1}^{k}\sigma_i}{(n-k)\sigma_S}. \qquad \text{(B3)}$$

This solution is the same as the solution to Problem (2.5), as specified by (5.5) with $\gamma = 0$. □

B.2 Proof of Proposition 6.2

Proof. According to Theorem 3.1, Problem (2.3) reduces to

$$\min_{(a_1,\ldots,a_n)\in\mathcal{A}_{rc}^1} \sum_{i=1}^{n} a_i^2.$$

Rewrite the objective function as $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n-1} a_i^2 + (1-\sum_{i=1}^{n-1} a_i)^2$ and denote it as $g(a_1, \dots, a_{n-1})$. Note that for any $j = 1, \dots, k-1$,

$$\frac{\partial}{\partial a_j} g(a_1, \dots, g_{n-1}) = 2a_j - 2\left(1 - \sum_{i=1}^{n-1} a_i\right) = 2(a_j - a_n).$$

For any $(a_1, \dots, a_n) \in \mathcal{A}_{rc}^1$, we have $\frac{a_1}{\sigma_1} \leq \cdots \leq \frac{a_n}{\sigma_n}$. Since it is assumed that $\sigma_1 < \dots < \sigma_n$, then $a_j < a_n$ for any $j = 1, \dots, n-1$, which implies that $g(a_1, \dots, a_{n-1})$ is decreasing in each a_j on $\mathcal{A}_{rc}^1$. Furthermore, combining the constraints of $\frac{a_1}{\sigma_1} \leq \cdots \leq \frac{a_n}{\sigma_n}$ and $a_1 + \cdots + a_n = 1$, we can solve that $a_j \leq \frac{\sigma_j}{\sum_{i=1}^{n} \sigma_i}$ for any $j = 1, \dots, n-1$. Therefore, the optimal sharing ratios that minimize the objective function $g(a_1, \dots, a_{n-1})$ over $\mathcal{A}_{rc}^1$ are

$$a_1^* = \frac{\sigma_1}{\sum_{i=1}^{n} \sigma_i}, \dots, a_{n-1}^* = \frac{\sigma_{n-1}}{\sum_{i=1}^{n} \sigma_i},$$

which implies that $a_n^* = \frac{\sigma_n}{\sum_{i=1}^{n} \sigma_i}$. This solution is the same as the solution to Problem (2.5), as specified by (5.5) with $\gamma = 1$. □

B.3 Proof of Proposition 6.3

Proof. Suppose $(C_1, \dots, C_n) \in \mathcal{U}_k$, i.e,

$$\sum_{i=1}^{k-1} \sigma_i + \sigma_k^{1-\gamma} \times \sum_{i=k}^{n} {\sigma_i}^{\gamma} \leq \sigma_S < \sum_{i=1}^{k} \sigma_i + \sigma_{k+1}^{1-\gamma} \times \sum_{i=k+1}^{n} \sigma_i^{\gamma}. \tag{B4}$$

According to Theorem 3.1, Problem (2.3) reduces to

$$\min_{(a_1, \dots, a_n) \in \mathcal{A}_{vr}} \quad \sum_{i=1}^{n} \frac{a_i^2}{\sigma_i^{\gamma}}.$$

Note that

$$\begin{aligned} \sum_{i=1}^{n} \frac{a_i^2}{\sigma_i^{\gamma}} &= \sum s_{i=1}^{k} \frac{a_i^2}{\sigma_i^{\gamma}} + \sum_{i=k+1}^{n} \frac{a_i^2}{\sigma_i^{\gamma}} \\ &\geq \sum_{i=1}^{k} \frac{a_i^2}{\sigma_i^{\gamma}} + \frac{\left(\sum\limits_{i=k+1}^{n} a_i\right)^2}{\sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}} \\ &= \sum_{i=1}^{k} \frac{a_i^2}{\sigma_i^{\gamma}} + \frac{\left(1 - \sum\limits_{i=1}^{k} a_i\right)^2}{\sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}} \triangleq g(a_1, \dots, a_k), \end{aligned}$$

where the inequality follows from Cauchy-Schwarz inequality with the equality holds if and only if

$$\frac{a_{k+1}}{\sigma_{k+1}^{\gamma}} = \cdots = \frac{a_n}{\sigma_n^{\gamma}}. \tag{B5}$$

Further note that, for any $j = 1, \dots, k$ and $a_j < \frac{\sigma_j}{\sigma_S}$, it holds that

$$\begin{aligned} \frac{\partial}{\partial a_j} g(a_1, \dots, a_k) &= 2\frac{a_j}{\sigma_j^{\gamma}} + \frac{2}{\sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}} \left(\sum_{i=1}^{k} a_i - 1 \right) \\ &< 2\frac{\sigma_j^{1-\gamma}}{\sigma_S} + \frac{2}{\sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}} \left(\sum_{i=1}^{k} \frac{\sigma_i}{\sigma_S} - 1 \right) \\ &= \frac{2}{\sigma_S \sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}} \left(\sigma_j^{1-\gamma} \sum_{i=k+1}^{n} \sigma_i^{\gamma} + \sum_{i=1}^{k} \sigma_i - \sigma_S \right) \leq 0 \end{aligned}$$

where the last inequality is due to (B4). This implies $g(a_1, \dots, g_k)$ decreases in a_j on $[0, \frac{\sigma_j}{\sigma_S})$ for all $j = 1, \dots, k$. Thus, $g(a_i, \dots, a_k)$ attains its minimum over $\mathcal{A}_{vr}$ at $a_1 = \frac{\sigma_1}{\sigma_S}, \dots, a_k = \frac{\sigma_k}{\sigma_S}$. Following (B5), $\sum_{i=1}^{n} a_i^2$ attains its minimum over $\mathcal{A}_{vr}$ at

$$a_1^* = \frac{\sigma_1}{\sigma_S}, \dots, a_k^* = \frac{\sigma_k}{\sigma_S}, a_{k+1}^* = \frac{\sigma_{k+1}^{\gamma} \left(\sigma_S - \sum\limits_{i=1}^{k} \sigma_i \right)}{\sigma_S \sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}}, \dots, a_n^* = \frac{\sigma_n^{\gamma} \left(\sigma_S - \sum\limits_{i=1}^{k} \sigma_i \right)}{\sigma_S \sum\limits_{i=k+1}^{n} \sigma_i^{\gamma}}.$$

This is the same as the solution to Problem (2.4) as specified by (5.5). □