Appendix

September 4, 2024

1 Further context

There is one crucial distinction that must be noted between the approach that I take in this letter and the closest approach that I build on in the literature, that of Eggers and Nowacki (2024). Eggers and Nowacki (2024) consider the probability of being pivotal in causing a candidate to be dropped in some round whether or not that changes the ultimate election winner. I take a narrower definition of pivotality: I focus exclusively on the probability of changing the election's overall winner.

Another important substantive caveat to acknowledge is that the precise probability of being pivotal may or may not ultimately play a role in voter decision-making. The equation that I will derive is particularly complex (which I attribute to the inherent complexity of the electoral system), and Bartholdi III and Orlin (1991, p. 341) have already shown that identifying a ballot that would elect a certain candidate in IRV is NP-complete, so identifying a strategic vote "is not practical, even though theoretically possible". However, the pivotal probability expression has substantive and methodological relevance beyond its implications for voter behaviour. For example, if ties tend to be more common in IRV, then voters may find themselves acting pivotally more often, especially in small elections, whether they plan to or not. Small mistakes in election administration, or discrepancies between original and audited vote counts, could also more frequently show different winners if election results tend to be closer.

It is also important to mention an active area of inquiry related to pivotal probability in IRV, which is the minimum number of ballots that would need to be changed in order to change the election winner, called the "margin of victory" (Atsusaka et al., 2024, Blom et al., 2016, Magrino et al., 2011). Pivotal probability is related to the margin of victory, since, all else equal, a smaller margin of victory implies a larger pivotal probability. However, identifying the margin of victory is not sufficient to know the pivotal probability, since the margin of victory concerns the *fewest* number of ballots that could upset an election result (Magrino et al., 2011, §3.3), whereas a correct expression for pivotal probability will require modeling the probability of *any* way that a given ballot could change the election result. This work on the margin of victory has also often focused on efficient estimation and bounding, aimed at the practical problem of how many ballots to sample in an election audit.

Finally, I should clearly define IRV. In an IRV contest with κ candidates, voters are allowed to rank some number of those candidates. The election administrator counts the number of times that each candidate was ranked first, and the candidate with the fewest first-place votes is eliminated. For each ballot, any remaining candidate that was ranked immediately after the eliminated candidate has one vote added to their vote total. Again, whichever candidate has the fewest votes is eliminated. This procedure is repeated until $\kappa - 1$ candidates have been eliminated. The remaining candidate wins.

2 The counterfactual

The primary counterfactual state in the paper is abstention. So, throughout the paper, all probabilities and utilities should be understood as being implicitly compared to abstention. How does this affect the derivation?

The paper seeks an expression for the utility of casting a ballot β as a function of the number of votes for each candidate. Let $u(\beta)$ be the utility of ballot β, and $\mathbb{P}_{\text{pivotal}}(\beta)$ be the probability that β is pivotal, and $u_{\text{pivotal}}(\beta)$ be the utility obtained by the voter conditional on β changing the election winner. We will express each of these variables in terms of candidates' vote totals, and arrive at an expression of the form:

$$u(\beta) = \mathbb{P}_{\text{pivotal}}(\beta) \cdot u_{\text{pivotal}}(\beta)$$

The correct interpretation of this equation is the utility obtained from ballot β *as opposed to not casting any ballot at all*. This interpretation omits any cost T associated with the act of voting, but this is not a serious limitation, and a reader who wishes to include that cost could instead compute:

$$u(\beta) = \mathbb{P}_{\text{pivotal}}(\beta) \cdot u_{\text{pivotal}}(\beta) - T$$

What modifications would be necessary to consider a different counterfactual vote choice, other than abstention? If the voter is considering the effects of casting ballot β as opposed to some other ballot β', then they should instead calculate the utility $u(\beta' \to \beta)$ of switching from β' to β as:

$$u(\beta' \to \beta) = \mathbb{P}_{\text{pivotal}}(\beta) \cdot u_{\text{pivotal}}(\beta) - \mathbb{P}_{\text{pivotal}}(\beta') \cdot u_{\text{pivotal}}(\beta')$$

It must also be noted that, in addition to now needing to calculate $\mathbb{P}_{\text{pivotal}}(\beta')$, the fact that the voter would otherwise have cast β' also requires an adjustment to $\mathbb{P}_{\text{pivotal}}(\beta)$: the voter must now account for the fact that their own vote has been taken away from various candidates by removing β' from the pool of ballots cast and replacing it with β. β' would have contributed one vote to a candidate immediately, to another candidate if the first candidate were eliminated, to a different candidate if the first two were eliminated, and so on. When calculating $\mathbb{P}_{\text{pivotal}}(\beta)$ with β' as the counterfactual of interest, the voter must subtract a vote from whichever candidate would have received a vote from β' in every hypothetical scenario. This is standard in formal treatments of strategic voting under SMDP, e.g. that of Mebane et al. (2019), but these kinds of conditional cascades make it more complicated in IRV.

I take abstention to be the primary counterfactual only for the sake of simplifying the exposition of my derivation (and because the problem of *which* β' to choose as the counterfactual state is immense, given how many possible ballots can be cast). When I discuss (for example) how much more likely a ballot is to be pivotal when candidate A is ranked in position i, I do not consider the effects of ranking candidate A *as opposed to some other candidate*, which would require subtracting 1 vote from the appropriate other candidate's vote total. Instead, I only consider the effect of adding one vote to whichever candidate β is currently counted for, compared to if the voter contributed 0 votes to any candidate. A reader who wishes to consider the utility of ballot β as opposed to some β' only needs to subtract 1 vote from whichever candidate β' would have been counted towards in each hypothetical scenario.

3 Simplifying assumptions

I take the following two assumptions, entirely for the sake of communication, since without these assumptions the equations would be too explosively complex to write. These assumptions have the following attributes: a) they are at least as reasonable as any alternative assumptions, b) they are necessary for exposition, and c) their limitations do not in any way seriously limit the article's results.

For any candidates $A, B, C,$ and D among the κ candidates in the election, with v_a^b denoting the vote total of candidate a in round b, and using $E_1 \perp E_2 \perp E_3$ to mean that events $E_1, E_2,$ and E_3 are pairwise independent,

Assumption 1: $\left[v_A^r \geq v_B^r\right] \perp \left[v_A^r \geq v_C^r\right] \perp \left[v_C^r \geq v_D^r\right]$
Assumption 2: $\left[v_A^r \geq v_B^r\right] \perp \left[v_A^{r'} \geq v_B^{r'}\right] \perp \left[v_A^{r'} \geq v_C^{r'}\right] \perp \left[v_C^{r'} \geq v_D^{r'}\right]$

where $A \neq C$, $A \neq D$, $B \neq C$, and $B \neq D$.

Assumption 1 says that any two pairwise comparisons of vote totals are independent, even conditional on any set of candidates having been dropped and any relative ordering of other vote totals. Assumption 2 says that the vote totals of two candidates in some round is unrelated to the vote totals of those or other candidates in another round. Their usefulness to the paper is as follows: Assumption 1 allows us to multiply the probability of two relative vote totals being observed *within* a round and argue that it represents the joint probability of those orderings, while Assumption 2 allows us to obtain a joint probability by multiplying relative vote totals *across* rounds.

Assumption 1 is a feature of the Poisson voting games framework that I will rely on to generate specific probabilities, and has been discussed and used extensively in that context (Mebane et al., 2019, Myerson, 1998, Vasselai, 2022). Assumption 2 is more challenging, because it is not a familiar assumption, and it is not obvious why we should accept it. However, I argue that it is at least as reasonable as any alternative, and doing away with it would introduce enormous complication without adding any insight.

Clearly, knowing (for example) the round 1 vote totals of two candidates does give us information about what their round 2 vote totals will be. But what information? To state how likely a candidate is to win the election given that they were leading by a certain margin in round 1 votes, we would need to know the probability that a certain round 1 vote margin will translate into a certain round 2 vote margin, the probability that some round 2 vote margin will turn into some round 3 vote margin, and so on. That probability presumably varies dramatically from election to election, based on arbitrarily complicated features: voters' preferences, campaign effects, coalition dynamics, and so on. Rather than limiting our attention to one such probability, I take the simplest assumption instead: that the vote totals are independent across rounds.

This is *not* a fundamental methodological limitation; it just makes exposition possible. A reader who wishes to do away with either assumption can simply pick their preferred joint probability distribution and substitute it into the derivation in the place of multiplication.[1] However, that would impose an assumption that is no more realistic and at least as strong as Assumption 2, while also severely complicating the derivation.

Two structural points of the derivation should also be mentioned. First, I set aside ties between more than two candidates, which are extremely unlikely in large-population elections and would make the derivations infeasibly more complicated. In SMDP such ties are known to be negligible in large electorates, except in cases of extremely low turnout (Vasselai, 2022).

A word must also be said about tie-breakers. An alternative path to the one described in Equation 6 in the article, that S_1 must have fewer votes than S_2 in order to lose to it, is that of course instead S_1 could have had *the same number of votes* as S_2, and then *lost the tie-breaker*. I omit this only for the sake of exposition, because it is completely infeasible to write out all the comparisons with equality included. Numerical computations of the pivotal probability would have to include every tie-breaking event that could produce a tied vote total between c and S_1. Separately, note that casting a vote which breaks a tie between c and S_1 is not necessarily pivotal. It is only pivotal in the event that c would have lost the tie-breaker against S_1. This tie-breaking probability is feasible to represent, so we include a factor of $\frac{1}{2}$ to represent the event that c would lose a fair tie-breaker without the pivotal vote boosting them into first place. I assume a fair tie-breaker.

4 Indirect pivotality from switching sequences

There are two crucial observations. First, c is not the winner at the end of the sequence A'; otherwise, this would be a directly pivotal event (or not pivotal, if $A_{-1} = c$). Second, adding a vote to c cannot change the sequence in which candidates would be dropped *before* c. This is because I am only considering the pure effects of adding a vote to c, as if the elector's only other option were abstention (as explained in §2 of this Appendix), and not the act of

[1] If a reader wishes to do away with Assumption 1, they cannot obtain the joint probability of two vote totals having certain relative sizes within the same round by simply multiplying the probability that each ordering is realized. Likewise, without Assumption 2, they cannot obtain the joint probability of observing that some vote total is larger than another in round 1, and that also some vote total is larger than another in round 2, by simply multiplying those probabilities. Instead, they must propose exactly what the probability is that the first pair has some relative size *and* the second pair has some relative size. So, wherever the probabilities of relative orderings are multiplied either within rounds or across rounds, they need only introduce a term that gives the probability of these two events occurring. This will produce results identical to the probability that I derive in this paper, up to multiplication of each term by some real number.

replacing a candidate with c.

Monotonicity failure, when $A_{-1} = c$ and $A'_{-1} \neq c$, occurs because of this kind of replacement, which increases the vote total of c while simultaneously taking that vote away from another candidate (Ornstein and Norman, 2014, Smith, 1973). Also note that indirect pivotality is distinct from a violation of the participation criterion (the "no-show paradox"), in which a voter can cause a more-preferred candidate to win by abstaining rather than voting. There are two important distinctions: first, indirect pivotality can arise from a voter just changing the order of the ballot they were already going to cast rather than switching from abstention to voting, and second, it does not necessarily cause the victory of a more-preferred candidate.

Indirect pivotality is however related to the strategic defection scenarios that Eggers and Nowacki (2024) identify in IRV, which they note is also related to Cox's (1997) notion of a "turkey" who strategically supports a spoiler in a runoff system.

Now, we can phrase these two observations as constraints on A and A', where y represents the index of c in A.

Constraint 1: A and A' are the same up to the pivotal event: for some y, $A_{1:y-1} = A'_{1:y-1}$

Constraint 2: The winner in A is not the winner in A': $A_{-1} \neq A'_{-1}$

Together these two constraints imply a third: the last element in A cannot be the candidate in A_y, that is, the winner is some candidate other than c. Another simplifying fact is that, for a given pair A and A' (and recalling that I do not consider ties between more than two candidates), whichever candidate c tied with is the candidate that is dropped in that round instead of c, so the tie must have been between c and whichever candidate is listed in A'_y.

The probability that ranking c in position y will be pivotal is the probability that, for any pair of sequences A, A' satisfying Conditions 1 and 2, the vote for c causes the candidates to be dropped in the sequence specified by A' rather than A. I seek that probability by first considering just one list A.

There must be a tie between c and one of the candidates which remain in the contest by the time that c is dropped. Because I assume that there is a negligible probability of more than two candidates being tied, these events are mutually exclusive by assumption. This reasoning leads to the equation in the text.

5 Example pivotal probability calculation

Using the framework for numerically specific pivot probabilities introduced in Appendix §8, I now perform a full example computation of pivotal probability in IRV. I will use different numbers from the example in the main text because with such large numbers, exact pivotal probabilities become almost inexpressibly small. So, consider instead the following contrived example. Imagine that the candidates in an IRV election are Al Gross, Santa Claus,[2] Mark

[2]A candidate of this name did actually contest Alaska's house seat.

Begich, Mary Peltola, and Sarah Palin, and for simplicity consider ballots of length 3.[3] Suppose that the following ballot types are cast, with the following frequency:

Number	Ballot
2	[Gross, Palin, X]
2	[Claus, Gross, Palin]
3	[Begich, Gross, Palin]
6	[Palin, Peltola, X]
12	[Peltola, X, X]

where "X" represents any legal vote. We will take the perspective of an elector who expects these ballots to be cast, and is deciding whether to abstain or cast a ballot.

To perform the full pivotal probability calculations, the voter cannot just consider one way of being directly pivotal and one way of being indirectly pivotal, but the combinations of drop orders in a 5-candidate election is too large for us to explicitly write out the entire procedure for calculating it. So instead, just compute the probability of the drop sequences that we have identified as examples. We will also only fully work through the first ballot locations.

Direct pivotality: In this example, $\text{Sym}(\text{C} \setminus i)$ is the group of all permutations of the set {Gross, Claus, Begich, Palin}. We will focus on the drop order provoked in this example, namely $S = [\text{Gross}, \text{Claus}, \text{Begich}, \text{Palin}]$, but a voter who computes direct pivotality would need to consider every other permutation as well. Denote by $p_{\text{direct}}(S)$ the probability that the ballot under consideration will be directly pivotal through the particular chain S under consideration. Then,

[3]This is only in the interest of exposition; similar examples with ballots of length 4 or 5 are straightforward to construct.

$$
\begin{aligned}
p_{\text{direct}}(S) = \mathbb{P}\Big(\mu^1_{A_1}|A_{1:0} < \mu^1_{A_2}|A_{1:0}\Big) \cdot \mathbb{P}\Big(\mu^1_{A_1}|A_{1:0} < \mu^1_{A_3}|A_{1:0}\Big) \cdot \mathbb{P}\Big(\mu^1_{A_1}|A_{1:0} < \mu^1_{A_4}|A_{1:0}\Big)\cdot \\
\mathbb{P}\Big(\mu^1_{A_1}|A_{1:0} < \mu^1_{A_5}|A_{1:0}\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_2}|A_{1:0} + \mu^2_{A_2}|A_{1:1}\big] < \big[\mu^1_{A_3}|A_{1:0} + \mu^2_{A_3}|A_{1:1}\big]\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_2}|A_{1:0} + \mu^2_{A_2}|A_{1:1}\big] < \big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1}\big]\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_2}|A_{1:0} + \mu^2_{A_2}|A_{1:1}\big] < \big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1}\big]\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_3}|A_{1:0} + \mu^2_{A_3}|A_{1:1} + \mu^3_{A_3}|A_{1:2}\big] < \big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1} + \mu^3_{A_4}|A_{1:2}\big]\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_3}|A_{1:0} + \mu^2_{A_3}|A_{1:1} + \mu^3_{A_3}|A_{1:2}\big] < \big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1} + \mu^3_{A_5}|A_{1:2}\big]\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1} + \mu^3_{A_4}|A_{1:2} + \mu^4_{A_4}|A_{1:3}\big] = \\
\big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1} + \mu^3_{A_5}|A_{1:2} + \mu^4_{A_5}|A_{1:3}\big]\Big)\cdot \\
\mathbb{P}\Big(\big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1} + \mu^3_{A_4}|A_{1:2} + \mu^4_{A_4}|A_{1:3}\big] = 1+ \\
\big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1} + \mu^3_{A_5}|A_{1:2} + \mu^4_{A_5}|A_{1:3}\big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
p_{\text{direct}}(S) = \mathbb{P}\Big(\mu^1_{\text{Gross}} < \mu^1_{\text{Claus}}\Big) \cdot \mathbb{P}\Big(\mu^1_{\text{Gross}} < \mu^1_{\text{Begich}}\Big) \cdot \mathbb{P}\Big(\mu^1_{\text{Gross}} < \mu^1_{\text{Palin}}\Big) \cdot \mathbb{P}\Big(\mu^1_{\text{Gross}} < \mu^1_{\text{Peltola}}\Big) \cdot \\
\mathbb{P}\Big(\big[\mu^1_{\text{Claus}} + \mu^2_{\text{Claus}}|\{\text{Gross}\}\big] < \big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\{\text{Gross}\}\big]\Big) \cdot \\
\mathbb{P}\Big(\big[\mu^1_{\text{Claus}} + \mu^2_{\text{Claus}}|\{\text{Gross}\}\big] < \big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\{\text{Gross}\}\big]\Big) \cdot \\
\mathbb{P}\Big(\big[\mu^1_{\text{Claus}} + \mu^2_{\text{Claus}}|\{\text{Gross}\}\big] < \big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\{\text{Gross}\}\big]\Big) \cdot \\
\mathbb{P}\Big(\big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\{\text{Gross}\} + \mu^3_{\text{Begich}}\{\text{Gross}, \text{Claus}\}\big] < \\
\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\{\text{Gross}\} + \mu^3_{\text{Palin}}|\{\text{Gross}, \text{Claus}\}\big]\Big) \\
\mathbb{P}\Big(\big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\{\text{Gross}\} + \mu^3_{\text{Begich}}\{\text{Gross}, \text{Claus}\}\big] < \\
\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\{\text{Gross}\} + \mu^3_{\text{Peltola}}|\{\text{Gross}, \text{Claus}\}\big]\Big) \\
\mathbb{P}\Big(\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\{\text{Gross}\} + \mu^3_{\text{Palin}}|\{\text{Gross}, \text{Claus}\} + \mu^4_{\text{Palin}}|\{\text{Gross}, \text{Claus}, \text{Begich}\}\big] = \\
\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\{\text{Gross}\} + \mu^3_{\text{Peltola}}|\{\text{Gross}, \text{Claus}\} + \mu^4_{\text{Peltola}}|\{\text{Gross}, \text{Claus}, \text{Begich}\}\big]\Big) \cdot \\
\mathbb{P}\Big(\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\{\text{Gross}\} + \mu^3_{\text{Palin}}|\{\text{Gross}, \text{Claus}\} + \mu^4_{\text{Palin}}|\{\text{Gross}, \text{Claus}, \text{Begich}\}\big] = 1 + \\
\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\{\text{Gross}\} + \mu^3_{\text{Peltola}}|\{\text{Gross}, \text{Claus}\} + \mu^4_{\text{Peltola}}|\{\text{Gross}, \text{Claus}, \text{Begich}\}\big]\Big) \cdot
\end{aligned}
$$

Now we can substitute in the actual numbers, and represent the Skellam CDF by $\mathcal{C} \equiv \sum_{w=0}^{\infty} \mathcal{S}(w, a, b)$, which is the probability that $a > b$ when a, b are random variables following the Poisson distribution. Then,

$$
\begin{aligned}
p_{\text{direct}}(S) = {} & \mathcal{C}(2,2) \cdot \mathcal{C}(3,2) \cdot \mathcal{C}(6,2) \cdot \mathcal{C}(12,2) \cdot \\
& \mathcal{C}(3+0, 2+0) \cdot \mathcal{C}(6+2, 2+0) \cdot \mathcal{C}(12+0, 2+0) \cdot \\
& \mathcal{C}(6+2+2, 3+0+0) \cdot \\
& \mathcal{C}(12+0+0, 3+0+0) \cdot \\
& (\mathcal{S}(0, 6+2+2+0, 12+0+0+0) + \\
& \frac{1}{2} \cdot \mathcal{S}(0, 6+2+2+0, 1+12+0+0+0))
\end{aligned}
$$

$p_{\text{direct}}(S) \approx 0.023$

Indirect pivotality: We will calculate just the indirect pivotal probability of moving from the default drop sequence $A = [\text{Gross}, \text{Claus}, \text{Begich}, \text{Peltola}, \text{Palin}]$ to the alternative drop sequence $A' = [\text{Claus}, \text{Begich}, \text{Palin}, \text{Gross}, \text{Peltola}]$. The indirectly pivotal event is boosting Gross above Claus, so we will only consider the probability of a last-place tie between those two candidates. Denote by $p_{\neg d}(E)$ indirect pivotal probability through this specific event. Then,

$$
\begin{aligned}
p_{\neg d}(E) =& \mathbb{P}\Big(\mu^1_{A_2}|A_{1:0} > \mu^1_{A_1}|A_{1:0}\Big) \cdot \mathbb{P}\Big(\mu^1_{A_3}|A_{1:0} > \mu^1_{A_1}|A_{1:0}\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{A_3}|A_{1:0} + \mu^2_{A_3}|A_{1:1}\big] > \big[\mu^1_{A_2}|A_{1:0} + \mu^2_{A_2}|A_{1:1}\big]\Big) \cdot \\
& \mathbb{P}\Big(\mu^1_{A_4}|A_{1:0} > \mu^1_{A_1}|A_{1:0}\Big) \cdot \mathbb{P}\Big(\big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1}\big] > \big[\mu^1_{A_2}|A_{1:0} + \mu^2_{A_2}|A_{1:1}\big]\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1} + \mu^3_{A_4}|A_{1:2}\big] > \big[\mu^1_{A_3}|A_{1:0} + \mu^2_{A_3}|A_{1:1} + \mu^3_{A_3}|A_{1:2}\big]\Big) \cdot \\
& \mathbb{P}\Big(\mu^1_{A_5}|A_{1:0} > \mu^1_{A_1}|A_{1:0}\Big) \cdot \mathbb{P}\Big(\big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1}\big] > \big[\mu^1_{A_2}|A_{1:0} + \mu^2_{A_2}|A_{1:1}\big]\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1} + \mu^3_{A_5}|A_{1:2}\big] > \big[\mu^1_{A_3}|A_{1:0} + \mu^2_{A_3}|A_{1:1} + \mu^3_{A_3}|A_{1:2}\big]\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{A_5}|A_{1:0} + \mu^2_{A_5}|A_{1:1} + \mu^3_{A_5}|A_{1:2} + \mu^4_{A_5}|A_{1:3}\big] > \\
& \big[\mu^1_{A_4}|A_{1:0} + \mu^2_{A_4}|A_{1:1} + \mu^3_{A_4}|A_{1:2} + \mu^4_{A_4}|A_{1:3}\big]\Big) \cdot \\
& \mathbb{P}\Big(\mu^1_{G_2}|G_{1:0} > \mu^1_{G_1}|G_{1:0}\Big) \cdot \mathbb{P}\Big(\mu^1_{G_3}|G_{1:0} > \mu^1_{G_1}|G_{1:0}\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{G_3}|G_{1:0} + \mu^2_{G_3}|G_{1:1}\big] > \big[\mu^1_{G_2}|G_{1:0} + \mu^2_{G_2}|G_{1:1}\big]\Big) \cdot \\
& \mathbb{P}\Big(\mu^1_{G_4}|G_{1:0} > \mu^1_{G_1}|G_{1:0}\Big) \cdot \mathbb{P}\Big(\big[\mu^1_{G_4}|G_{1:0} + \mu^2_{G_4}|G_{1:1}\big] > \big[\mu^1_{G_2}|G_{1:0} + \mu^2_{G_2}|G_{1:1}\big]\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{G_4}|G_{1:0} + \mu^2_{G_4}|G_{1:1} + \mu^3_{G_4}|G_{1:2}\big] > \big[\mu^1_{G_3}|G_{1:0} + \mu^2_{G_3}|G_{1:1} + \mu^3_{G_3}|G_{1:2}\big]\Big) \cdot \\
& \mathbb{P}\Big(\mu^1_{G_5}|G_{1:0} > \mu^1_{G_1}|G_{1:0}\Big) \cdot \mathbb{P}\Big(\big[\mu^1_{G_5}|G_{1:0} + \mu^2_{G_5}|G_{1:1}\big] > \big[\mu^1_{G_2}|G_{1:0} + \mu^2_{G_2}|G_{1:1}\big]\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{G_5}|G_{1:0} + \mu^2_{G_5}|G_{1:1} + \mu^3_{G_5}|G_{1:2}\big] > \big[\mu^1_{G_3}|G_{1:0} + \mu^2_{G_3}|G_{1:1} + \mu^3_{G_3}|G_{1:2}\big]\Big) \cdot \\
& \mathbb{P}\Big(\big[\mu^1_{G_5}|G_{1:0} + \mu^2_{G_5}|G_{1:1} + \mu^3_{G_5}|G_{1:2} + \mu^4_{G_5}|G_{1:3}\big] > \\
& \big[\mu^1_{G_4}|G_{1:0} + \mu^2_{G_4}|G_{1:1} + \mu^3_{G_4}|G_{1:2} + \mu^4_{G_4}|G_{1:3}\big]\Big) \cdot \\
& \mathbb{P}\Big(\mu^1_{\text{Gross}} = \mu^1_{\text{Claus}}\Big) + \frac{1}{2} \cdot \mathbb{P}\Big(\mu^1_{\text{Gross}} = \mu^1_{\text{Claus}} - 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
p_{\neg d}(E) =& \mathbb{P}\Big(\mu^1_{\text{Gross}} = \mu^1_{\text{Claus}}\Big) + \frac{1}{2}\cdot\mathbb{P}\Big(\mu^1_{\text{Gross}} = \mu^1_{\text{Claus}} - 1\Big)\cdot\mathbb{P}\Big(\mu^1_{\text{Claus}} > \mu^1_{\text{Gross}}\Big)\cdot\mathbb{P}\Big(\mu^1_{\text{Begich}} > \mu^1_{\text{Gross}}\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\text{Gross}\big] > \big[\mu^1_{\text{Claus}} + \mu^2_{\text{Claus}}|\text{Gross}\big]\Big)\cdot \\
& \mathbb{P}\Big(\mu^1_{\text{Palin}} > \mu^1_{\text{Gross}}\Big)\cdot\mathbb{P}\Big(\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\text{Gross}\big] > \big[\mu^1_{\text{Claus}} + \mu^2_{\text{Claus}}|\text{Gross}\big]\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\text{Gross} + \mu^3_{\text{Palin}}|\{\text{Gross}, \text{Claus}\}\big] > \\
& \big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\text{Gross} + \mu^3_{\text{Begich}}|\{\text{Gross}, \text{Claus}\}\big]\Big)\cdot \\
& \mathbb{P}\Big(\mu^1_{\text{Peltola}} > \mu^1_{\text{Gross}}\Big)\cdot\mathbb{P}\Big(\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\text{Gross}\big] > \big[\mu^1_{\text{Claus}} + \mu^2_{\text{Claus}}|\text{Gross}\big]\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\text{Gross} + \mu^3_{\text{Peltola}}|\{\text{Gross}, \text{Claus}\}\big] > \\
& \big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\text{Gross} + \mu^3_{\text{Begich}}|\{\text{Gross}, \text{Claus}\}\big]\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\text{Gross} + \mu^3_{\text{Palin}}|\{\text{Gross}, \text{Claus}\} + \mu^4_{\text{Palin}}|\{\text{Gross}, \text{Claus}, \text{Begich}\}\big] > \\
& \big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\text{Gross} + \mu^3_{\text{Peltola}}|\{\text{Gross}, \text{Claus}\} + \mu^4_{\text{Peltola}}|\{\text{Gross}, \text{Claus}, \text{Begich}\}\big]\Big)\cdot \\
& \mathbb{P}\Big(\mu^1_{\text{Begich}} > \mu^1_{\text{Claus}}\Big)\cdot\mathbb{P}\Big(\mu^1_{\text{Palin}} > \mu^1_{\text{Claus}}\Big)\cdot\mathbb{P}\Big(\big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\text{Claus}\big] > \big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\text{Claus}\big]\Big)\cdot \\
& \mathbb{P}\Big(\mu^1_{\text{Gross}} > \mu^1_{\text{Claus}}\Big)\cdot\mathbb{P}\Big(\big[\mu^1_{\text{Gross}} + \mu^2_{\text{Gross}}|\text{Claus}\big] > \big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\text{Claus}\big]\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Gross}} + \mu^2_{\text{Gross}}|\text{Claus} + \mu^3_{\text{Gross}}|\{\text{Claus}, \text{Begich}\}\big] > \\
& \big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\text{Claus} + \mu^3_{\text{Palin}}|\{\text{Claus}, \text{Begich}\}\big]\Big)\cdot \\
& \mathbb{P}\Big(\mu^1_{\text{Peltola}} > \mu^1_{\text{Claus}}\Big)\cdot\mathbb{P}\Big(\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\text{Claus}\big] > \big[\mu^1_{\text{Begich}} + \mu^2_{\text{Begich}}|\text{Claus}\big]\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\text{Claus} + \mu^3_{\text{Peltola}}|\{\text{Claus}, \text{Begich}\}\big] > \\
& \big[\mu^1_{\text{Palin}} + \mu^2_{\text{Palin}}|\text{Claus} + \mu^3_{\text{Palin}}|\{\text{Claus}, \text{Begich}\}\big]\Big)\cdot \\
& \mathbb{P}\Big(\big[\mu^1_{\text{Peltola}} + \mu^2_{\text{Peltola}}|\text{Claus} + \mu^3_{\text{Peltola}}|\{\text{Claus}, \text{Begich}\} + \mu^4_{\text{Peltola}}|\{\text{Claus}, \text{Begich}, \text{Palin}\}\big] > \\
& \big[\mu^1_{\text{Gross}} + \mu^2_{\text{Gross}}|\text{Claus} + \mu^3_{\text{Gross}}|\{\text{Claus}, \text{Begich}\} + \mu^4_{\text{Gross}}|\{\text{Claus}, \text{Begich}, \text{Palin}\}\big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
p_{\neg d}(E) = {} & \mathcal{C}(2,2) \cdot \mathcal{C}(3,2) \cdot \mathcal{C}(3+0,2+0) \cdot \mathcal{C}(6,2) \cdot \mathcal{C}(6+2,2+0) \cdot \mathcal{C}(6+2+2,3+0+0) \cdot \\
& \mathcal{C}(12,2) \cdot \mathcal{C}(12+0,2+0) \cdot \mathcal{C}(12+0+0,3+0+0) \cdot \\
& \mathcal{C}(6+2+2+3,12+0+0+0) \cdot \mathcal{C}(3,2) \cdot \mathcal{C}(6,2) \cdot \mathcal{C}(6+0,3+0) \cdot \mathcal{C}(2,2) \cdot \\
& \mathcal{C}(2+2,3+0) \cdot \mathcal{C}(2+2+3,6+0+0) \cdot \mathcal{C}(12,2) \cdot \mathcal{C}(12+0,3+0) \cdot \\
& \mathcal{C}(12+0+0,6+0+0) \cdot \mathcal{C}(12+0+0+6,2+2+3+0) \cdot \\
& (\mathcal{S}(0,2,2) + \frac{1}{2} \cdot \mathcal{S}(0,2,2-1))
\end{aligned}
$$

$$
p_{\neg d}(E) \approx 0.003
$$

The probability of this particular example of indirect pivotality was found to be an order of magnitude smaller than the probability of the direct pivotality example, consistent with the fact that indirect pivotality requires a much more specific situation to occur.

6 Pseudocode

Here I provide pseudocode for computing direct and indirect pivotality. I leave counting the votes non-explicit, but each vote total should represent the vote total relevant to the comparison at hand. I also continue to not explicitly represent the probability of ties arising within drop sequences, though these must be computed. The following algorithms are run on the set of all possible ballots, which we have assumed to be $\mathcal{P}_L(\mathrm{Sym}(C))$, that is, the set of all length-L subsets of all of the permutations of the set of candidates C.

Algorithm 1 Direct pivotality

```
for voter in voters do
    for β in allBallots do
        ballotDirPivot ← 0
        for i in [1 : L] do
            for S in Sym(C \ {S_i}) do
                A ← S ∪ {i}
                if β[1 : i] in A then
                    for each ℓ in A_{1:κ−2} do
                        candDropProbs ← 1
                        for h in A_{ℓ:κ} do
                            chainProb ← Σ_{w=1}^{∞} 𝒮(w, v_ℓ, v_h)
                            candDropProbs ← candDropProbs · chainProb
                        end for
                    end for
                    breakTieProb ← 𝒮(0, v_i, v_{S_{−1}})
                    makeTieProb ← 𝒮(−1, v_i, v_{S_{−1}})
                    pivotProb ← (candDropProbs)(1/2 · breakTieProb + 1/2 · makeTieProb)
                    ballotDirPivot ← ballotDirPivot + pivotProb
                end if
            end for
        end for
        allBallotPivots[β] ← ballotDirPivot
    end for
end for
```

Algorithm 2 Indirect pivotality

```
for voter in voters do
    for $\beta$ in allBallots do
        ballotIndirPivot $\leftarrow 0$
        for $A$ in Sym($C$) do
            $\mathbf{A} \leftarrow$ set of all permutations of $A$ satisfying the two requirements
            baseChainProb $\leftarrow 1$
            if $\beta[1:i]$ in $A_{1:y}$ then
                for each $\ell$ in $A_{2:}$ do
                    for $h$ in $A_{1:\ell-1}$ do
                        baseChainProb $\leftarrow$ baseChainProb $\cdot \sum_{w=1}^{\infty}(w, v_\ell, v_h)$
                    end for
                    for $A'$ in $\mathbf{A}$ do
                        $G \leftarrow A'_{y:\kappa}$
                        altProb $\leftarrow 1$
                        for each $d$ in $G$ do
                            for $d$ in $[1 : length(G)]$ do
                                for $h$ in $[1 : d-1]$ do
                                    altProb $\leftarrow$ altProb $\cdot \sum_{w=1}^{\infty}(w, v_{G_d}, v_{G_h})$
                                end for
                            end for
                        end for
                        $t \leftarrow A'_y$
                        breakProb $\leftarrow \mathcal{S}(0, v_i, v_t)$
                        makeProb $\leftarrow \mathcal{S}(0, v_i, v_t - 1)$
                        pivotProb $\leftarrow$ chainProb $\cdot$ altProb $\cdot$ ($\frac{1}{2}\cdot$ breakProb + $\frac{1}{2}\cdot$ makeProb)
                        ballotIndirPivot $\leftarrow$ ballotIndirPivot + pivotProb
                    end for
                end for
            end if
        end for
    end for
end for
```

The pivotal probability of each contest can be multiplied by the utility the voter would obtain from that result, and then the ballot with the largest expected utility selected, with the caveat that a tie-breaking rule might also be necessary.

7 Equal utility ballots

There are several ways that voters may find the same expected utility for multiple different ballots. I will informally outline three ways.

First, if a voter has a non-strict preference ordering, then they will expect the same utility from any pair of ballots which are equal up to the relative ordering of any set of candidates to which they attach equal sincere utility.

Second, if voters have non-strict preference orderings, then the utilities they expect from casting two ballots may still be equal. For example, in a 4-way race between the set of candidates $\{A, B, C\}$, the ballots $\beta_1 = [A, B, D]$ and $\beta_2 = [A, C, D]$ have the same expected utility if and only if $p_{B,C} \cdot u(B, C) + 2p_{B,D} \cdot u(B, D) = -p_{C,B} \cdot u(C, B) + p_{D,C} \cdot u(D, C)$, that is if the two ways that the second position can be pivotal balance each other out, where $p_{x,y}$ denotes the probability of being pivotal in deciding the election for candidate x over candidate y, and $u(x, y)$ represents the utility obtained from the victory of x minus the utility obtained from the victory of y. Because we can freely set the sincere utility obtained from the victory of every candidate, it is straightforward to produce a numerical example that satisfies this equality.

Third, suppose that one candidate is expected to not receive any votes. We assign zero probability to the event that such a candidate's vote total will exceed the vote total of a candidate who is expected to get a positive number of votes. Then any two ballots will have equal expected utility if they are equal up to the location of the candidate who is expected to receive zero votes.

For this reason, a complete specification of optimal vote choice in IRV requires some means of breaking ties. One option is to pick a framework for assigning probabilities to pivotal events which guarantees that ties will not arise (though it is not obvious what such a framework would be), and to assume that all voters have strict preference orderings. Barring that, a tie-breaking rule must assign a unique expected utility to any two non-equal ballots, and a reasonable second condition would be that a voter should break ties in proportion to their sincere utilities.

8 Modeling the probabilities

There are many ways to model expected vote totals, but I motivated Assumption 1 (and to a lesser extent Assumption 2) as being especially well-supported by one prominent framework: Poisson voting games (Myerson, 1998). For that reason I proceed by suggesting how to compute probabilities in that framework, by extending a derivation of pivotal probabilities in single-vote elections by Mebane et al. (2019). Importantly, however, we have already seen the full pivotal probability equation before ever making the probabilities numerically specific. So, all of the preceding work can be immediately adapted to any other framework for computing probabilities in voting games, such as the Dirichlet beliefs in an iterated polling framework that have been used in previous work on strategic voting in IRV (Eggers and Nowacki, 2024).

Following Mebane et al. (2019), if the number of voters is drawn from a Poisson distribution, then the number of voters with a preference ordering following each possible preference ordering will also follow a Poisson distribution with known parameter, and their difference follows the Skellam distribution. So one way of computing the expected utility of a ballot is as follows:

$$
\begin{aligned}
u(\beta) = \sum_{i=1}^{L} \Bigg(\sum_{\substack{S \in \\ \text{Sym}(C \backslash i) \\ \beta_{1:i-1} \subset S}} & \Bigg\{ \Bigg[\prod_{\ell=1}^{\kappa-2} \prod_{r=\ell+1}^{\kappa} \sum_{w=0}^{\infty} \mathcal{S}\Bigg(w, \sum_{q=0}^{\ell-1} \mu_{A_r}^{q+1} | A_{1:q}, \sum_{q=0}^{\ell-1} \mu_{A_\ell}^{q+1} | A_{1:q} \Bigg) \Bigg] \\
& \Bigg[\frac{1}{2} \cdot \mathcal{S}\Bigg(0, \sum_{q=0}^{\kappa-2} \mu_{A_{\kappa-1}}^{q+1} | A_{1:q}, \sum_{q=0}^{\kappa-2} \mu_{i}^{q+1} | A_{1:q} \Bigg) + \\
& \frac{1}{2} \cdot \mathcal{S}\Bigg(1, \sum_{q=0}^{\kappa-2} \mu_{A_{\kappa-1}}^{q+1} | A_{1:q}, \sum_{q=0}^{\kappa-2} \mu_{i}^{q+1} | A_{1:q} \Bigg) \Bigg] \Bigg\} \cdot \Bigg[u(i) - u(S_{-1}) \Bigg] + \\
& \sum_{\substack{A \in \text{Sym}(C) \\ \beta_{1:i} \subset A_{1:y}}} \Bigg[\prod_{\ell=2}^{\kappa} \Bigg[\prod_{h=1}^{\ell-1} \sum_{w=0}^{\infty} \mathcal{S}\Bigg(w, \sum_{q=0}^{h-1} \mu_{A_\ell}^{q+1} | A_{1:q}, \ \sum_{q=0}^{h-1} \mu_{A_h}^{q+1} | A_{1:q} \Bigg) \Bigg] \\
& \cdot \sum_{A' \in \mathbf{A}} \Bigg\{ \prod_{d=1}^{|G|} \Bigg[\prod_{h=1}^{d-1} \sum_{w=0}^{\infty} \mathcal{S}\Bigg(w, \sum_{q=0}^{y+d} \mu_{G_d}^{q+1} | A_{1:q}, \ \sum_{q=0}^{y+d} \mu_{G_h}^{q+1} | A_{1:q} \Bigg) \Bigg] \\
& \cdot \Bigg[\frac{1}{2} \cdot \mathcal{S}\Bigg(0, \sum_{q=0}^{y} \mu_{c}^{q+1} | A_{1:q}, \ \sum_{q=0}^{y} \mu_{t}^{q+1} | A_{1:q} \Bigg) \\
& + \frac{1}{2} \cdot \mathcal{S}\Bigg(1, \sum_{q=0}^{y} \mu_{c}^{q+1} | A_{1:q}, \ \sum_{q=0}^{y} \mu_{t}^{q+1} | A_{1:q} \Bigg) \Bigg] \Bigg\} \Bigg] \\
& \cdot \Bigg[u(A'_{-1}) - u(A_{-1}) \Bigg] \Bigg)
\end{aligned}
$$

In the article I implement the IRV pivotal probability algorithm in Python and simulate the magnitude of pivotal probabilities for identical election setups in IRV and SMDP. Now that we can estimate pivotal probabilities in both systems, we can assess the claim that there is more incentive to vote strategically in one system than in the other.[4]

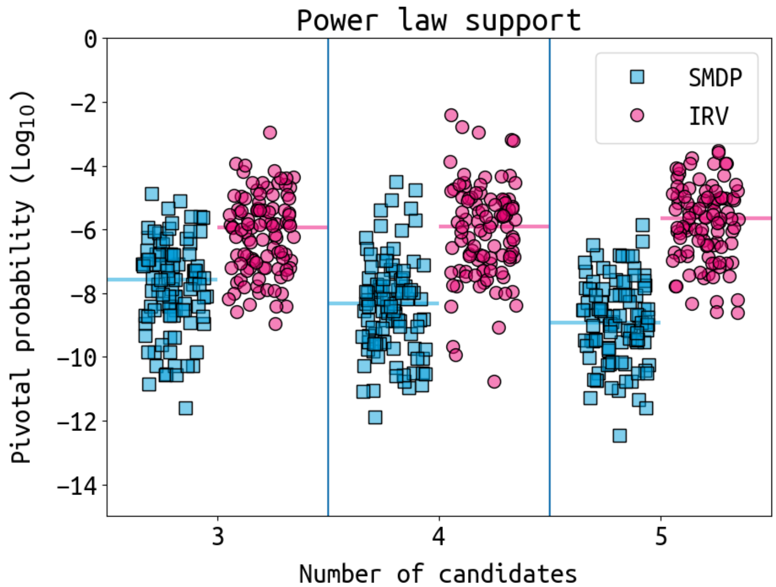
9 Simulated pivotal probabilities

In Appendix §8 I suggest a method for modeling the probability of the pivotal events identified in this paper. Here I show results from 100 runs of a model, where each run consists of one IRV contest and one SMDP contest with identical parameters, re-run for elections with 3, 4, and 5 candidates. Preferences are drawn in two ways. One is a uniform distribution, where every candidate has similar numbers of people most-preferring them, second-most-preferring them, third-most-preferring them, and so on. The other is a power law distribution, where half of the population holds one preference order, and the other half has a uniformly random preference order. These preference types are opposite ends of a spectrum: the uniform case

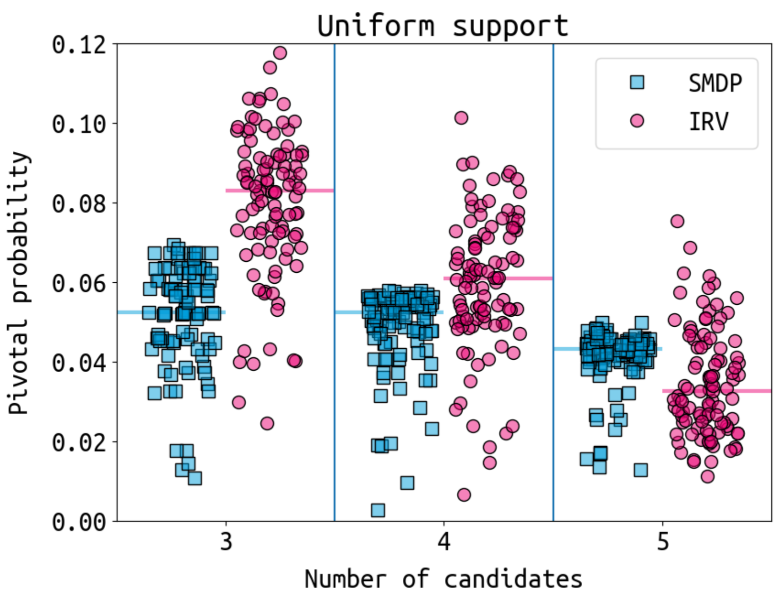
[4] To facilitate direct comparison, I simulate SMDP pivotal probabilities using the derivation from Mebane et al. (2019) which I extended to cover IRV.

is an extremely competitive election, in which every candidate has similar levels of support, while the power law case is a model of an election where there is a clear front-runner. In both IRV and SMDP, pivotal probabilities should be larger in competitive elections than in uncompetitive elections, but it is not obvious in advance how the different preference distributions should increase or decrease pivotal probabilities more in one system than the other.

Figure 1 shows the total pivotal probability for two stylized preference distributions in either IRV and SMDP: one where support for 3, 4, or 5 candidates is roughly uniform, and another where it follows a power law. I conduct 100 runs, and within each of the 100 runs, the order of each voter's ballot is determined randomly, in a way that ensures that support for the candidates follows the specified distribution. The total pivotal probability on the y-axis is the sum of the pivotal probability of every possible ballot. Each dot represents the sum of the pivotal probabilities of all ballots in one run of the simulation. Note that the figures have different y-axis scales so that the range of both is clearly visible; this means that they should not be directly compared to one another without taking the y-axis scales into account.



(a) Pivotal probabilities with power law support



(b) Pivotal probabilities with uniform support

Figure 1: The comparative pivotal probabilities of IRV and SMDP with 3, 4, and 5 candidates, when 1,000 voters can rank every candidate. The y-value sums the pivotal probability of *all* ballots. 100 runs are shown with different random number seeds. For each run, one contest is held under IRV and an identical contest under SMDP, for each number of candidates, and with preferences drawn from either a power law or a uniform distribution.

There are differences, including IRV having a *slightly* higher pivotal probability overall and IRV exhibiting slightly lower variance. When support for candidates resembles a power law distribution, IRV has very slightly higher pivotal probabilities than SMDP (averaging about 10^{-6} compared to about 10^{-7} with these particular preference structures). In the case of uniformly distributed support, the chances to be pivotal in IRV may be slightly higher than in SMDP when there are only 3 candidates, but when there are 4 or 5 candidates neither is clearly larger than the other.[5] The results do not support widespread claims that there are either stronger or weaker incentives for voters to cast strategic votes under IRV compared to SMDP.

However, these are stylized figures meant to illustrate how my derivations could be applied in the presence of actual preference data. Instead perhaps the overall most important feature of Figure 1 is that IRV and SMDP have pivotal probabilities that often coincide up to the order of magnitude or even the decimal place. This is particularly true when considering that the simplifying assumptions from Appendix §3 were made for logistical and not empirical reasons, so differences between the SMDP and IRV pivotal probabilities could certainly change if a user were to conjecture specific joint probabilities to replace the independence assumptions, as might the relationship between the number of candidates and the difference between IRV and SMDP pivot probabilities.

10 Translation to existing notation

In this article I have had to develop substantial notation in order to express relative vote totals and candidate drop orders under hypothetical scenarios in arbitrarily many-candidate IRV elections which, to my knowledge, have not previously been written explicitly in symbolic notation. The article that I believe has expressed the *most* similar ideas is that of Eggers and Nowacki (2024). The relaxation of the assumption that the election includes only three candidates, and the consideration of pivotal events beyond the next round, mean that it would not be convenient to directly adapt their notation to our purposes. However, for the sake of comparability, and to aide readers who are familiar with their notation in understanding the new notation that I introduce, in this section I translate some of the events that they consider from their notation into mine.

Eggers and Nowacki (2024) consider certain pivotal probabilities in their Appendix section A.1.2., and particularly Table 1, which contains expressions for particular "pivot events in IRV". To illustrate the connections, and also some differences, between the quantities that I derive and the ones they derive, I will obtain the first equation in their table from one of the expressions in my article.

For illustration, we will consider the pivotal event that "i and j tie after k is eliminated in 1st round". Eggers and Nowacki (2024) call this a "second-round" pivot event, meaning

[5] However, I strongly caution against reading substantively into the appearance that pivotal probabilities in IRV fall more quickly as the number of candidates rises than pivotal probabilities in SMDP, since this result may be attributable to the varying quality of Assumption 2 as more candidates are introduced. Also, note that these are all out-of-equilibrium pivotal probabilities. Observing these pivotal probabilities would cause rational voters to update their expected vote choice, which should change the expected vote totals of each candidate, and in turn alter the pivotal probabilities. This process may or may not converge to a fixed equilibrium. So, these should be understood as the *initial* pivotal probabilities of these election structures.

that they assume a certain result from the first round of a three-candidate contest (in this case, that candidate k was dropped first), and express the probability of one of the other two candidates winning. The expression they give is:

$$\pi_{ij,2} = \Pr\left(v_j + v_{kj} - \frac{1}{2} \in (0, N^{-1}) \cap\ v_k < v_i \cap\ v_k < v_j\right)$$

where N is the total number of voters, and v_i is the share of votes received by candidate i. Can we obtain this expression from the ones in the main body of this paper? First we must consider the assumptions underlying the expression above.

The first structural difference is that Eggers and Nowacki (2024, Appendix p.3) take the perspective of a voter who assumes that the vector of votes $\mathbf{v}$ is a continuous random variable. This means that it is possible for the share of votes to fall in an interval between 0 votes and 1 vote. The second difference is that Eggers and Nowacki (2024) consider shares of votes, whereas I consider the total number of votes received.

Two simple substitutions will adjust for these differences. To address the second difference, instead of representing a single ballot as its share of all votes N^{-1}, we will just consider it as 1 vote and not phrase it in terms of the total number of votes cast. Second, while for Eggers and Nowacki (2024) a single ballot can be pivotal if the relevant vote shares fall in the interval between 0 and $\frac{1}{N}$, since we consider raw numbers of discrete-valued votes for us the difference in votes between the candidate to beat and the candidate being voted for must be exactly either 0 or 1. Making these adjustments to the above equation yields:

$$\pi_{ij,2} = \Pr\left[\left(v_j + v_{kj} - \frac{1}{2} = 0\right) \cup \left(v_j + v_{kj} - \frac{1}{2} = 1\right)\right] \cap \left[v_k < v_i \cap v_k < v_j\right]$$

By Kolmogorov's second and third axioms and the rules of IRV,

$$v_j + v_{kj} + v_i + v_{ki} = 1$$

With this identity, the expression of Eggers and Nowacki (2024) becomes the following:

$$\pi_{ij,2} = \Pr\left[\left(v_j + v_{kj} = v_i + v_{ki}\right) \cup \left(v_j + v_{kj} = 1 + v_i + v_{ki}\right)\right] \cap \left[v_k < v_i \cap v_k < v_j\right]$$

Since the first two events are mutually exclusive, the probability of either one occurring can be obtained by adding the probability of each one occurring. Now we reach the final difference in assumptions between the two derivations. Assumptions 1 and 2 in this Appendix — which, I must again stress, are the most realistic assumptions I could find that would allow me to fit the full equations I derive onto a standard sheet of paper — allow us to multiply the probability of certain events rather than positing the precise chance that they co-occur. Then,

$$\pi_{ij,2} = \Big[\Pr\,\big(v_j + v_{kj} = v_i + v_{ki}\big) + \Pr\,\big(v_j + v_{kj} = 1 + v_i + v_{ki}\big)\Big]\cdot \Pr\,(v_k < v_i)\cdot \Pr\,(v_k < v_j)$$

To avoid a collision with some of the notation that we have developed in the body of this paper, particularly overloading the variable i, let us re-define the candidate labels before we turn to matching the two equations. Let $x \equiv i$, $y \equiv j$, and $z \equiv k$, and, to match the style we chose in the body of the paper, represent probabilities by $\mathbb{P}$ rather than Pr. Then the above equation becomes:

$$\pi_{xy,2} = \Big[\mathbb{P}\big(v_y + v_{zy} = v_x + v_{zx}\big) + \mathbb{P}\big(v_y + v_{zy} = 1 + v_x + v_{zx}\big)\Big]\cdot \mathbb{P}(v_z < v_x)\cdot \mathbb{P}(v_z < v_y)$$

Now that we have rephrased the equation of Eggers and Nowacki (2024) using the same assumptions we have made throughout this paper, can we obtain it from one of the equations that we have derived? Because this situation presumes that k is dropped first, there is no opportunity for indirect pivotality; the relevant equation is the expression for direct pivotal probability, Equation 8 in the main article. Eggers and Nowacki (2024) imagine a three-candidate contest with length 2 ballots. Equation 8 gives the following identity for the direct pivotal chances of any ballot β:

$$\begin{aligned} p_{\text{direct}}(\beta) = \sum_{i=1}^{L}\Bigg(\sum_{\substack{S\in \\ \mathrm{Sym}(C\setminus i) \\ \beta_{1:i-1}\subset S}} \Bigg\{ \Bigg[\prod_{\ell=1}^{\kappa-2} \prod_{r=\ell+1}^{\kappa} \mathbb{P}\Bigg(\sum_{q=0}^{\ell-1} \mu_{A_\ell}^{q+1} | A_{1:q} < \sum_{q=0}^{\ell-1} \mu_{A_r}^{q+1} | A_{1:q} \Bigg) \Bigg] \cdot \\ \Bigg[\frac{1}{2}\cdot \mathbb{P}\Bigg(\sum_{q=0}^{\kappa-2} \mu_{A_{\kappa-1}}^{q+1} | A_{1:q} = \sum_{q=0}^{\kappa-2} \mu_i^{q+1} | A_{1:q} \Bigg) + \\ \frac{1}{2}\cdot \mathbb{P}\Bigg(\sum_{q=0}^{\kappa-2} \mu_{A_{\kappa-1}}^{q+1} | A_{1:q} = 1 + \sum_{q=0}^{\kappa-2} \mu_i^{q+1} | A_{1:q} \Bigg) \Bigg] \Bigg\} \Bigg) \end{aligned}$$

In this scenario, $L = 2$. Eggers and Nowacki (2024) do not consider any particular ballot β, focusing rather on the probability of a tie occurring, which we have noted is the same as the direct pivotal probability of any ballot listing $x \succ y$ or $y \succ x$. Because we are wlog positing a vote for candidate x, the relevant drop sequences are all the permutations of the set $\{y, z\}$, such that all previous entries in the ballot have already been dropped. This restricts our attention to $S = \begin{bmatrix} z & y \end{bmatrix}$ and therefore $A = \begin{bmatrix} z & y & x \end{bmatrix}$. With $\kappa = 3$, we only consider $\ell = 1$.

We will need one more identity to obtain the tie probability from the direct pitoval probability. The direct pivotal chances account for the chance that the ballot is cast for a candidate in an exact tie *who would have won anyways* by winning the tiebreaker, and the probabilty that the ballot is cast for a candidate who is one vote behind but who then loses the tiebreaker. Each of these events has probability $\frac{1}{2}$ of occurring if the tiebreaker is fair. However, the tie probability should not adjust for these events. So, all else equal,

$$\pi_{xy,2} = 2 \cdot p_{\text{direct}}$$

We also should omit the initial sum over all candidates on the ballot, $\sum_{i=1}^{L}$, because this would represent the probability of breaking the tie either for x or for y, but we are seeking only the probability that they are tied. Then we can substitute the numbers above as well as the above identity into the direct pivotal probability to obtain our expression for $\pi_{xy,2}$.

$$\begin{aligned}\pi_{xy,2} =& \Bigg\{ \Bigg[\prod_{r=2}^{3} \mathbb{P}\Bigg(\sum_{q=0}^{0} \mu_{A_1}^{q+1} | A_{1:q} < \sum_{q=0}^{0} \mu_{A_r}^{q+1} | A_{1:q} \Bigg) \Bigg] \cdot \\ & \Bigg[\frac{1}{2} \cdot \mathbb{P}\Bigg(\sum_{q=0}^{1} \mu_{A_2}^{q+1} | A_{1:q} = \sum_{q=0}^{1} \mu_{i}^{q+1} | A_{1:q} \Bigg) + \\ & \frac{1}{2} \cdot \mathbb{P}\Bigg(\sum_{q=0}^{1} \mu_{A_2}^{q+1} | A_{1:q} = 1 + \sum_{q=0}^{\kappa-2} \mu_{i}^{q+1} | A_{1:q} \Bigg) \Bigg] \Bigg\} \cdot 2\end{aligned}$$

$$\begin{aligned}\pi_{xy,2} =& \prod_{r=2}^{3} \Bigg[\mathbb{P}\Bigg(\mu_z^1 < \mu_{A_r}^1 \Bigg) \Bigg] \cdot \\ & \Bigg[\mathbb{P}\Bigg(\sum_{q=0}^{1} \mu_{y}^{q+1} | A_{1:q} = \sum_{q=0}^{1} \mu_{x}^{q+1} | A_{1:q} \Bigg) + \mathbb{P}\Bigg(\sum_{q=0}^{1} \mu_{y}^{q+1} | A_{1:q} = 1 + \sum_{q=0}^{1} \mu_{x}^{q+1} | A_{1:q} \Bigg) \Bigg]\end{aligned}$$

$$\begin{aligned}\pi_{AB,2} = & \mathbb{P}\Bigg(\mu_C^1 < \mu_B^1 \Bigg) \cdot \mathbb{P}\Bigg(\mu_C^1 < \mu_A^1 \Bigg) \cdot \\ & \Bigg[\mathbb{P}\Bigg(\mu_B^1 + \mu_B^2 | C = \mu_A^1 + \mu_A^2 | C \Bigg) + \mathbb{P}\Bigg(\mu_B^1 + \mu_B^2 | C = 1 + \mu_A^1 + \mu_A^2 | C \Bigg) \Bigg]\end{aligned}$$

We developed this notation to represent situations that can arise in IRV elections with very large numbers of candidates. In the notation of Eggers and Nowacki (2024), v is used rather than μ, and cases like $\mu_B^2|C$ is written as v_{CB}, so that the superscripts can also be ommitted. Rephrasing the above expression using those conventions,

$$\pi_{AB,2} = [\mathbb{P}(v_B + v_{CB} = v_A + v_{CA}) + \mathbb{P}(v_B + v_{CB} = 1 + v_A + v_{CA})] \cdot \mathbb{P}(v_C < v_B) \cdot \mathbb{P}(v_C < v_A)$$

This exactly matches our rendition of the expression from Eggers and Nowacki (2024). What, then, should we take from this exercise? Most importantly, it is that the two papers agree when applied to the same situation. My results generalize and build on that previous result, and in no way should they be understood as contradicting or disagreeing with it. And yet, it is not quite the case that the equations that I derive can be written entirely as sums or products of equations already in Eggers and Nowacki (2024). The most direct reason is that three different structural assumptions or scope decisions were made, which we needed to modify in order to obtain one equation from the other:

1. Eggers and Nowacki (2024) consider beliefs over a continuous-valued vote share rather than the raw vote count

2. My assumptions 1 and 2 in this Appendix permit multiplication of certan probabilities, to stand in for the probability of certain joint events with unknowable joint distributions

3. Eggers and Nowacki (2024) restrict their attention to three- or four-candidate contests

4. Eggers and Nowacki (2024) separately state the probability of events occurring *within each round*, so that just changing the loser in a certain round is a pivotal event, whereas I only consider it a pivotal event if it changes the ultimate election winner

None of these features are any better or worse. They are simply fitted to the needs of their authors. The major new feature of my equations is that they work for *any number of candidates* and *any ballot length*. Previously it was not feasible to write out, say, the probability that candidates i and j tie in round five given that candidates a, b, c, and d have all be dropped. My addition is to make any situation like this expressable in closed form.

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