Online Appendix: "Simulating multiple equilibria in rational expectations models with occasionally-binding constraints: An algorithm and a policy application"[1]

This appendix provides further details of the numerical examples solved the main paper, and there are some sanity checks, including examples from Guerrieri and Iacoviello (2015). In addition, we show how our algorithm can be applied to models in which the constraint binds at steady state, the case of *multiple* occasionally-binding constraints, and a model with an upper bound constraint that binds multiple times due to cyclical dynamics. The codes for the simulations are available at the author's GitHub page at: github.com/MCHatcher.

1 General framework

We consider models of the form

$$\begin{aligned} B_{1,t}x_t &= B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \quad \forall t \geq 1 \\ B_{i,t} &= \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}\overline{B}_i + (1 - \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}})\tilde{B}_i \end{aligned} \tag{1}$$

where $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}$ is an indicator, $x^*_{1,t} > \underline{x}_1 \ \forall t > T$, $\underline{x}_1 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ given, e_t is a vector of known shocks with $e_t = 0_{m \times 1} \ \forall t > T$, and the 'shadow value' of the bounded variable is

$$x^*_{1,t} = F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + Ge_t + H \tag{2}$$

where $H \in \mathbb{R}$, F is a $1 \times 3n$ vector with $f_{11} = 0$ and G is a $1 \times m$ vector.

The matrices $B_{i,t}$ are regime dependent. In the reference regime $B_{i,t} = \overline{B}_i$; in the alternative regime $B_{i,t} = \tilde{B}_i$. The indicator variable $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}$ determines which regime is realized at a given t. The assumption that $x^*_{1,t} > \underline{x}_1 \ \forall t > T$ is a terminal condition which states that the bounded variable, $x_{1,t}$, permanently escapes the bound after a finite number of periods T.

1.1 Model solutions

Solutions to the problem in (1) are found by trialling sequences for the indicator variable of the form $(\mathbb{1}_t)_{t=1}^T$ (with $\mathbb{1}_t \in \{0, 1\}$ specified for all t) and $\mathbb{1}_t = 1 \ \forall t > T$; this in turn determines the sequences $\{B_{1,t}, B_{2,t}, B_{3,t}, B_{4,t}, B_{5,t}\}_{t \geq 1}$ which can be used to find a time path $(x_t)_{t \geq 1}$ using the Algorithm presented in the main text. Only time paths consistent with the terminal condition and the occasionally-binding constraint are accepted as solutions.

Any solution(s) to problem (1) found using our Algorithm have the form:

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for all } t > T \end{cases} \tag{3}$$

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where, for $t = 1, ..., T$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t} \tag{4}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}) \tag{5}$$

and $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ has eigenvalues in the unit circle, $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$, $\Psi_{T+1} = \overline{\Psi}$, $\Omega_{T+1} = \overline{\Omega}$, $\Gamma_{T+1} := 0_{n\times m}$ and $e_t = 0_{m\times 1}$ for all $t > T$.

1.2 Finding the M matrix

To compute the matrix $M \in \mathbb{R}^{T\times T}$ of impulse responses of the bounded variable to news shocks at dates $t = 1, ..., T$ (see Holden, 2023), let $v_t := [v_{1,t} \;\; 0_{1\times(n-1)}]'$ be an $n \times 1$ vector of known shocks to the bounded variable, where $v_{1,t} \in \{0, 1\}$ for $t = 1, ..., T$ and $v_{1,t} = 0$ for all $t > T$. Letting $\hat{x}_t := x_t - \overline{x}$ (Assumption 1, main paper), we can solve the following model:

$$\overline{B}_1\hat{x}_t = \overline{B}_2\hat{x}_{t+1} + \overline{B}_3\hat{x}_{t-1} + \overline{B}_4e_t + v_t, \quad \forall t \geq 1 \tag{6}$$

whose solution and M matrix are described in Remark 1.

Remark 1 *The solution to the perfect foresight model in* (6) *is given by*

$$\hat{x}_t = \overline{\Omega}\hat{x}_{t-1} + \hat{\Gamma}\tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{7}$$

where $\tilde{e}_t := \overline{B}_4e_t + v_t$, $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_2(\Psi_{t+1} + \hat{\Gamma}\tilde{e}_{t+1})$ $\forall t \in [1, T]$, *with* $\Psi_t = 0_{n\times 1}$ *for all* $t > T$, $\overline{\Omega}$ *as in* (3), *and the corresponding* M *matrix is*

$$M_{ij} = \frac{\partial \hat{x}_{1,i}}{\partial v_{1,j}} = \hat{x}_{1,i}|_{v_{1,j}=1} - \hat{x}_{1,i}|_{v_{1,j}=0} \quad \text{for } i, j \in \{1, ..., T\}.$$

2 Solution details: Fisherian example

Recall that for all $t \geq 1$ the model consists of a Taylor-type rule with a zero lower bound and the Fisher equation (see Holden, 2023):

$$i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + e_t\} \tag{8}$$

$$i_t = r + E_t\pi_{t+1} \tag{9}$$

where $\phi - \psi > 1$, $\psi > 0$, $\pi_0, e_1 \in \mathbb{R}$, $r > 0$ is a fixed real interest rate, and $e_t = 0$ for all $t > 1$. To simplify presentation, we set $\phi = 2$. The results are not specific to this case.

As discussed in the main paper, there are two solutions to the model (8)–(9): one is away from the bound in all periods, and the other has the constraint binding only in period 1. We now show that our Algorithm finds the same solutions. Letting $x_t := [i_t \;\; \pi_t]'$, the matrices in the reference regime and the alternative regime are given by

$$\overline{B}_1 = \begin{bmatrix} 1 & -\phi \\ 1 & 0 \end{bmatrix}, \;\; \overline{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \;\; \overline{B}_3 = \begin{bmatrix} 0 & -\psi \\ 0 & 0 \end{bmatrix}, \;\; \overline{B}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \;\; \overline{B}_5 = \begin{bmatrix} r \\ r \end{bmatrix}$$

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_5 = \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

Hence, analogous to (1)–(2), the model for all $t \geq 1$ is

$$\begin{aligned} B_{1,t}x_t &= B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \\ B_{j,t} &= \mathbb{1}_{\{i_t^*>0\}}\overline{B}_j + (1 - \mathbb{1}_{\{i_t^*>0\}})\tilde{B}_j, \quad \forall j \in [5] \end{aligned} \tag{10}$$

where $e_t = 0$ for all $t > 1$, and

$$i_t^* = F\begin{bmatrix} x_t' & E_t x_{t+1}' & x_{t-1}' \end{bmatrix}' + Ge_t + H, \quad \text{with } F = \begin{bmatrix} 0 & \phi & 0 & 0 & 0 & -\psi \end{bmatrix}, G = \begin{bmatrix} 1 \end{bmatrix}, H = \begin{bmatrix} r \end{bmatrix}.$$

Let $\mathbb{1}_t$ denote the guess on the indicator. Consider first the solution away from the bound. This solution corresponds to the guess $\mathbb{1}_t = 1$ for all $t \geq 1$, such that $B_{j,t} = \overline{B}_j$ $\forall j \in [5]$ and

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5, \quad \forall t \geq 1. \tag{11}$$

The guessed solution $x_t = [i_t \;\; \pi_t]'$ thus follows the Algorithm with $T = 1$ and $\mathbb{1}_t = 1$ $\forall t$:

$$x_t = \begin{cases} \Omega_1 x_0 + \Gamma_1 e_1 + \Psi_1 & \text{for } t = 1 \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > 1 \end{cases} \tag{12}$$

where $\Omega_1 = \overline{\Omega}$, $\Psi_1 = \overline{\Psi}$, $\Gamma_1 = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_4 = \begin{bmatrix} -\frac{\omega}{\phi-\omega} & -\frac{1}{\phi-\omega} \end{bmatrix}'$, and

$$\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3 = \begin{bmatrix} 0 & \omega^2 \\ 0 & \omega \end{bmatrix}, \quad \overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5) = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

with $\omega = 1 - \sqrt{1-\psi}$.[2]

The guessed solution is verified provided $i_1^* = r + \phi\pi_1 - \psi\pi_0 + e_1 \geq 0$, $i_t^* = r + \phi\pi_t - \psi\pi_{t-1} > 0$ for all $t > 1$, which requires $\pi_0 \geq -\frac{r}{\omega^2}$ for $e_1 = 0$ (see main text).

Now consider the second solution. We guess that the lower bound constraint binds only in period 1, such that $\mathbb{1}_1 = 0$ and $\mathbb{1}_t = 1$ $\forall t > 1$; hence $B_{j,t} = \tilde{B}_j$ for $t = 1$ and $B_{j,t} = \overline{B}_j$ for all $t > 1$, $j \in [5]$. Note that this guess implies that $T = 1$ and $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$ for all $t > 1$. Therefore, the guessed solution is

$$x_t = \begin{cases} \Psi_1 & \text{for } t = 1 \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > 1 \end{cases} \tag{13}$$

where $\Psi_1 = (\tilde{B}_1 - \tilde{B}_2\overline{\Omega})^{-1}(\tilde{B}_2\overline{\Psi} + \tilde{B}_5) = \begin{bmatrix} 0 & -\frac{r}{\omega} \end{bmatrix}'$.

To verify this solution, we require $i_1^* = r + \phi\pi_1 - \psi\pi_0 + e_1 \leq 0$ and $i_t^* = r + \phi\pi_t - \psi\pi_{t-1} > 0$ for all $t > 1$, which again requires $\pi_0 \geq -\frac{r}{\omega^2}$ for $e_1 = 0$. The two solutions are plotted in Figure 1 in the main paper, along with the corresponding shadow nominal rates i_t^*.

[2] We can guess that the left column of $\overline{\Omega}$ is zero (because i_{t-1} does not enter the model), which makes it straightforward to solve the resulting quadratic equation in $\overline{\Omega}$ for the right-hand column. In general, $\overline{\Omega}, \overline{\Psi}$ can be found using numerical algorithms (e.g. Binder and Pesaran, 1997; Sims, 2002; Cho and Moreno, 2011).

3 New Keynesian model

3.1 Baseline model

The baseline model has the form:

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{14}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) \tag{15}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{16}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{17}$$

where $\theta_\pi > 1$, $\beta \in (0,1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0,1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

Let $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t \end{bmatrix}'$ and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i, \quad \text{for } i \in \{2,3,4\}, \quad \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Given $x_{1,t} = i_t$, $i_t = \max\{\underline{i},\ i_t^*\}$ can be written in the form $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ by setting $\underline{x}_1 = \underline{i}$ and $x_{1,t}^* = i_t^*$, or in vector form as in (2) with $F = \begin{bmatrix} 0 & 1 & 0_{1\times 10} \end{bmatrix}$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$. We set the parameters at the values given in the main text, whenever these parameters were not being varied as part of the analysis. Our computed perfect foresight paths correspond to initial conditions $x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}'$, $e_1 = 0.01$, $e_t = 0\ \forall t \geq 2$, unless otherwise stated.

3.2 Computing the M matrix

To compute the M matrix of impulse responses of the bounded variable, we solve the model in (14)–(17) ignoring the bound (i.e. with the max operator removed) and with a 'news shock' $v_{1,t} \in \{0,1\}$ added, such that: $i_t = i_t^* + v_{1,t}$. The resulting model can be written as:

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \tilde{e}_t, \quad \forall t \geq 1 \tag{18}$$

where $\tilde{e}_t := \overline{B}_4 e_t + v_t$ and $v_t := \begin{bmatrix} v_{1,t} & 0 & 0 & 0 \end{bmatrix}'$, with $v_{1,t} = 0$ for all $t > T$, where T is the horizon at which the M matrix is being computed (it is a $T \times T$ matrix).

Recall that the 1st column of M lists the impulse response of i_t (at dates $t = 1, ..., T$) to the shock $v_{1,1} = 1$. In general, the jth column of M lists the impulse response of i_t (at dates $t = 1, ..., T$) to the shock $v_{1,j} = 1$; the rows of the matrix are indexed by $i = 1, ..., T$.

The impulse responses and the M matrix are obtained as follows:

$$x_t = \overline{\Omega} x_{t-1} + \hat{\Gamma} \tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{19}$$

where $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_2 (\Psi_{t+1} + \hat{\Gamma} \tilde{e}_{t+1})$ $\forall t \in [1, T]$, with $\Psi_t = 0_{n \times 1}$ for all $t > T$, $\overline{\Omega}$ as in (3), and the M matrix is given by

$$M_{ij} = \frac{\partial x_{1,i}}{\partial v_{1,j}} = i_i|_{v_{1,j}=1} - i_i|_{v_{1,j}=0} \quad \text{for } i, j \in \{1, ..., T\}.$$

Given the parameters $\beta = 0.99$ and $\kappa = \frac{(1-0.85)(1-0.85\beta)}{0.85}(2+\sigma)$, we assign values to ρ_i, σ, θ_π, $\theta_{\Delta y}$, compute the M matrix and then check if it is a P-matrix using a recursive test due to Tsatsomeros and Li (2000).[3] Plots of the P-matrix regions under interest rate smoothing and the baseline value $\sigma = 1$ (now shown in the main paper) are as follows:

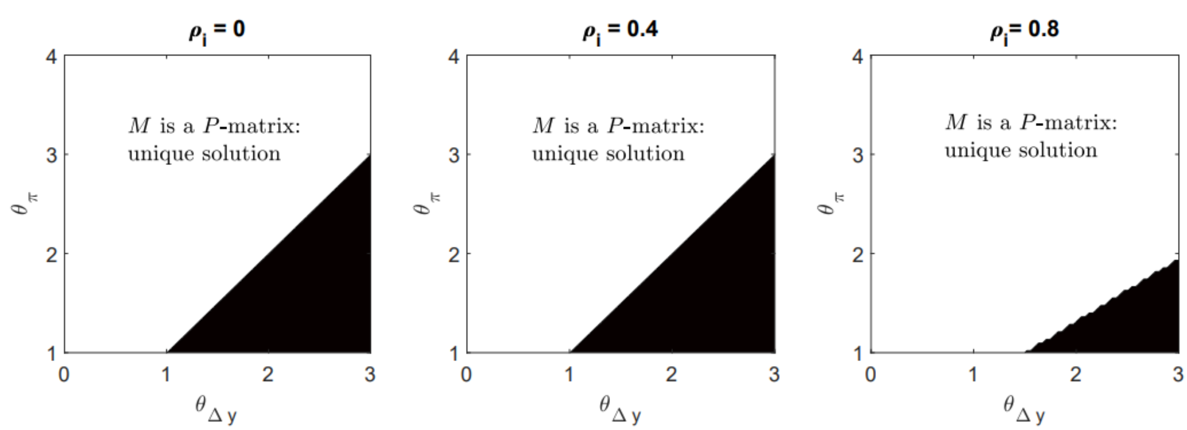

Figure 1: Regions in which M is not a P-matrix (black) for $T = 16$ and various ρ_i.

[3]A MATLAB code is available on the webpage: https://www.math.wsu.edu/faculty/tsat/matlab.html.

3.3 Forward guidance

With forward guidance the shadow interest rate is amended to

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) + e_t^{FG} \tag{20}$$

where $e_t^{FG} < 0$ for all $t \in \mathcal{T}^{FG} \subset \mathbb{N}_+$ and $e_t^{FG} = 0$ otherwise.

Hence, letting $\tilde{i}_t^* := \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1}))$ the interest rate rule (20) is

$$i_t^* = \begin{cases} \tilde{i}_t^* - |e_t^{FG}| & \text{if } t \in \mathcal{T}^{FG} \\ \tilde{i}_t^* & \text{otherwise.} \end{cases} \tag{21}$$

We consider forward guidance horizons of the form $\mathcal{T}^{FG} = \{2, \ldots, t'\}$, where $t' \geq 2$. Hence, forward guidance occurs for consecutive periods $2, .., t'$ and the length of the forward guidance 'horizon' (or spell) is given by $t' - 1$; see also Table 1 in the main text.

Letting $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t \end{bmatrix}'$ as before and $\hat{e}_t = \begin{bmatrix} e_t & e_t^{FG} \end{bmatrix}'$, the reference regime (slack) is

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 \hat{e}_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 \hat{e}_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i, \quad \text{for } i \in \{2,3,4\}, \ \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The M matrix under forward guidance is identical to the model with the baseline interest rate rule, (15). However, since forward guidance is modelled as negative shocks to the shadow interest rate, it affects the initial conditions and we record whether a unique solution or multiple solutions were found in each case. The simulation results in Table 1 of the paper check a large number of initial conditions (800) by varying the news shocks according to

$$e_t^{FG} = -0.01 - \mathcal{U}_t, \quad \text{for } t \in \mathcal{T}^{FG} \text{ and } \mathcal{U}_t = \text{draw from uniform distribution on } (0, 0.01).$$

3.4 Price-level targeting rule

With a price-level targeting interest rate rule, the model is amended to

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{22}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)\left(\theta_p p_t + \theta_{\Delta y}(y_t - y_{t-1})\right) \tag{23}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{24}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{25}$$

$$p_t = p_{t-1} + \pi_t \tag{26}$$

where $\theta_p > 0$ is the reaction coefficient on the (log) price level.

Let $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t & p_t \end{bmatrix}'$ and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & 0 & -(1-\rho_i)\theta_p \\ \sigma^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\kappa & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & 0 & -(1-\rho_i)\theta_p \\ \sigma^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\kappa & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i, \quad \text{for } i \in \{2,3,4\}, \ \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3.4.1 ‘Good’ solutions under price-level targeting

In Figure 2 we plot some unique perfect foresight solutions under price-level targeting and we see the results are robust to modest or strong smoothing of the shadow rate i_t^*.

3.4.2 ‘Bad’ solutions under price-level targeting

We note in the paper that when multiple solutions exist under a price-level targeting rule (23) (i.e. for small enough θ_p), the ‘bad’ solution is sometimes not so bad in terms of stabilization of inflation and the output gap. Below we provide some extra examples of ‘bad solutions’. In Figure 3–5 we plot some ‘good’ and ‘bad’ solutions when the response to the price level θ_p is small enough to give multiple solutions. Robustness is also considered, and we see that the ‘bad’ solution for $\theta_p = 0.2$ has inflation and the output gap highly destabilized.

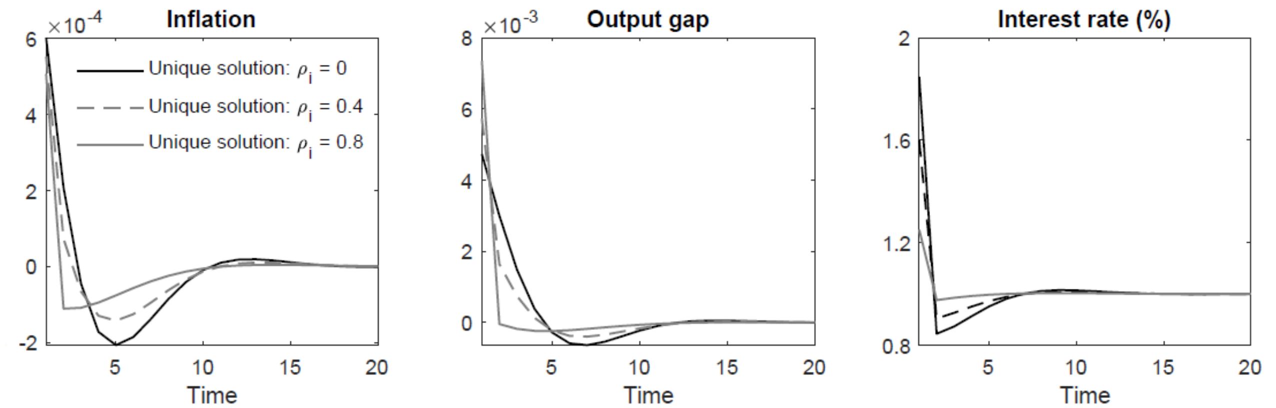

Figure 2: Unique perfect foresight paths under price-level targeting for $\rho_i = 0,\ 0.4,\ 0.8$ when $e_1 = 0.01$, $i_0^* = y_0 = 0$, $\sigma = 1$, $\theta_p = 1.5$ and $\theta_{\Delta y} = 1.6$.

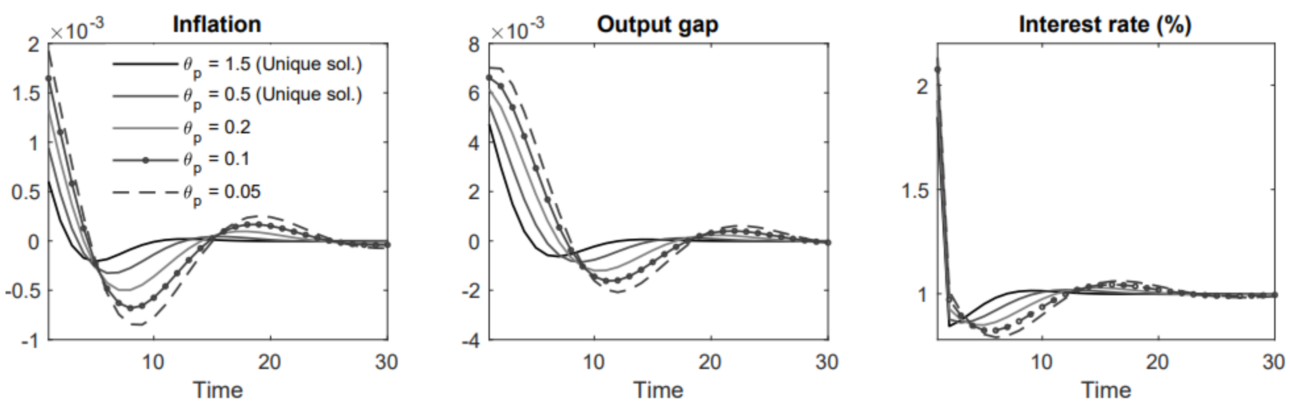

Figure 3: 'Good' solutions under price-level targeting: $\rho_i = 0$, various θ_p, and $\sigma = 1$

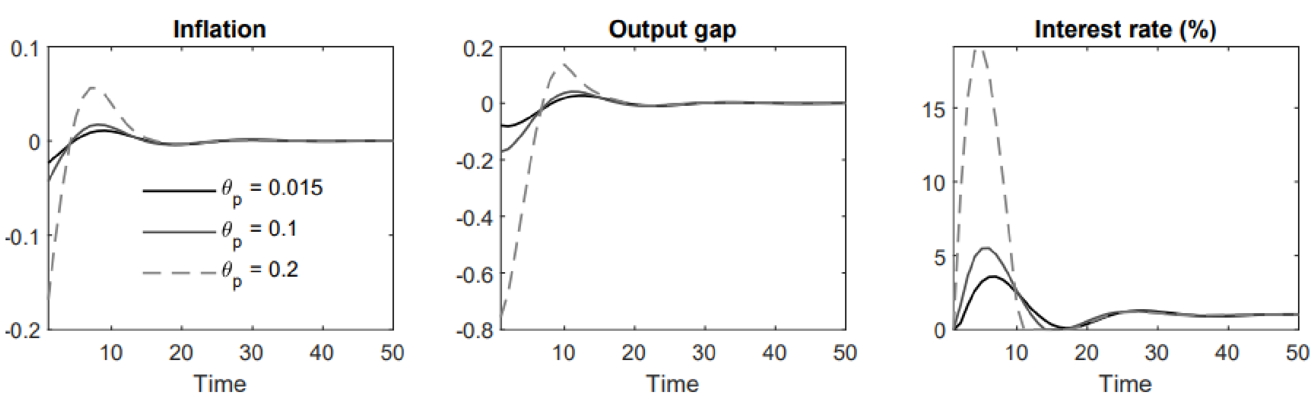

Figure 4: 'Bad' solutions under price-level targeting: $\rho_i = 0$, various θ_p, and $\sigma = 1$

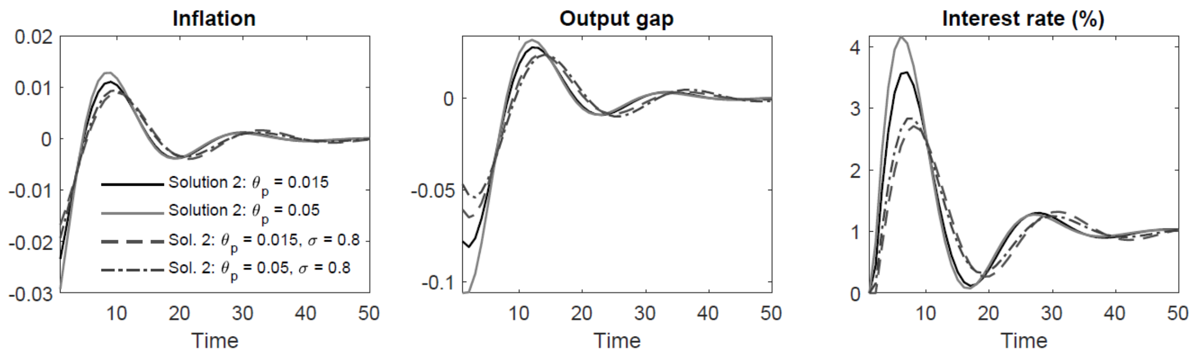

Figure 5: 'Bad' solutions under price-level targeting for various parameter values

4 Foundations of the Algorithm

By assumption, the model returns permanently to the reference regime after some date $T \geq 1$ and escapes the bound (see Assumption 2, main text). The system to be solved is

$$\begin{aligned} B_{1,t}x_t &= B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \quad \forall t \geq 1 \\ B_{i,t} &= \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}}\overline{B}_i + (1 - \mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}})\tilde{B}_i \quad i \in [5] \end{aligned} \tag{27}$$

where $\forall t > T$, $x_t = \overline{\Omega}x_{t-1} + \overline{\Psi}$, $B_{i,t} = \overline{B}_i$, $e_t = 0_{m\times 1}$ and $x^*_{1,t} > \underline{x}_1$, with $x^*_{1,t}$ is defined in (2). Let $\mathbb{1}_t$ denote the guess on the indicator in (27) at date t, and note $\mathbb{1}_t = 1$ $\forall t > T$.

Consider first the periods $1 \leq t \leq T$. Suppose there exist a set of well-defined matrices $\{\Omega_t, \Gamma_t, \Psi_t\}$ such that $x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t$. Shifting this equation forward one period:

$$x_{t+1} = \Omega_{t+1}x_t + \Gamma_{t+1}e_{t+1} + \Psi_{t+1}, \quad 1 \leq t \leq T-1. \tag{28}$$

Substituting (28) into (27) and rearranging gives, for all $t \in \{1, ..., T-1\}$,

$$(B_{1,t} - B_{2,t}\Omega_{t+1})x_t = B_{3,t}x_{t-1} + B_{4,t}e_t + B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}. \tag{29}$$

Provided $\Omega_T, \Gamma_T, \Psi_T$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$, the set $\{\Omega_t, \Gamma_t, \Psi_t\}$ is well-defined for t where these matrices follow the recursive formulas. Therefore, if $\Omega_T, \Gamma_T, \Psi_T$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ $\forall t < T$, $\Omega_t, \Gamma_t, \Psi_t$ are well-defined for $t = 1, ..., T$.

For $t > T$, we have by Assumption 2, $x_t = \overline{\Omega}x_{t-1} + \overline{\Psi}$ where $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ and $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$. Hence, $x_{t+1} = \overline{\Omega}x_t + \overline{\Psi}$, $\forall t \geq T$. Matrices $\Omega_T, \Gamma_T, \Psi_T$ are determined by the first line of (27) and the previous equation at date $t = T$:

$$B_{1,T}x_T = B_{2,T}x_{T+1} + B_{3,T}x_{T-1} + B_{4,T}e_T + B_{5,T}, \qquad x_{T+1} = \overline{\Omega}x_T + \overline{\Psi}$$

or $(B_{1,T} - B_{2,T}\overline{\Omega})x_T = B_{3,T}x_{T-1} + B_{4,T}e_T + B_{2,T}\overline{\Psi} + B_{5,T}$. Provided $\det[B_{1,T} - B_{2,T}\overline{\Omega}] \neq 0$, the matrices $\Omega_T, \Gamma_T, \Psi_T$ are given by the expressions in the Algorithm (see main text).

For the time path $(x_t)_{t=1}^T$ to satisfy the constraint $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ $\forall t \in \{1, ..., T\}$, the guessed structure $(\mathbb{1}_t)_{t=1}^T$ must be verified at all dates. Consider first date $t = 1$. If $\mathbb{1}_t = 1$, then $x_{1,t} = x^*_{1,t}$ and $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}} = 1 = \mathbb{1}_t$ (see (27)) if and only if $x^*_{1,t}|_{\mathbb{1}_t=1} > \underline{x}_1$. On the other hand, if $\mathbb{1}_t = 0$, then $x_{1,t} = \underline{x}_1$ and $\mathbb{1}_{\{x^*_{1,t} > \underline{x}_1\}} = 0 = \mathbb{1}_t$ if and only if $x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1$.

Thus, the guess at $t = 1$ is verified if $\mathbb{1}_t = 1$ and $x^*_{1,t}|_{\mathbb{1}_t=1} > \underline{x}_1$ *or* $\mathbb{1}_t = 0$ and $x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1$. By analogous arguments, the guessed structure for each subsequent t is verified if and only if the above condition holds for this particular t. Hence, the guessed structure $(\mathbb{1}_t)_{t=1}^T$ is verified when the following condition holds for all $t \in \{1, ..., T\}$ and $j = 1, ..., 5$:

$$\begin{cases} B_{j,t} = \overline{B}_j \text{ and } x^*_{1,t} > \underline{x}_1 \text{ for } t \text{ such that } \mathbb{1}_t = 1 \\ B_{j,t} = \tilde{B}_j \text{ and } x^*_{1,t} \leq \underline{x}_1 \text{ for } t \text{ such that } \mathbb{1}_t = 0. \end{cases} \tag{*}$$

Note that a guessed structure $(\mathbb{1}_t)_{t=1}^T$ is rejected if (*) does not hold for some $t \in [T]$.

5 Models with the constraint binding at steady state

In this section we explain how our algorithm applies to models where the constraint binds at the steady state to which any solutions must converge. An example is the borrowing constraint model in Guerrieri and Iacoviello (2015, Online Appendix, Section C.1).

Starting from the general model in (1)–(2), Assumptions 1-2 in the main paper must be adapted for a model that converges to a steady state in which the constraint is binding.

Assumption 1 *We assume* $\det[\tilde{B}_1 - \tilde{B}_2 - \tilde{B}_3] \neq 0$*, such that there exists a unique steady state* $\overline{x} = (\tilde{B}_1 - \tilde{B}_2 - \tilde{B}_3)^{-1}\tilde{B}_5$ *at the alternative regime. This steady state satisfies* $\overline{x}_1 = \underline{x}_1$.

Assumption 2 *For any given initial value, there is a unique stable (terminal) solution at the alternative regime of the form* $x_t = \tilde{\Omega} x_{t-1} + \tilde{\Psi}$*, where* $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}(\tilde{B}_2\tilde{\Psi} + \tilde{B}_5) = (I_n - \tilde{\Omega})\overline{x}$*,* $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_3$ *has eigenvalues in the unit circle, and* $x_t \to \overline{x}$ *as* $t \to \infty$.

Assumption 3 is unchanged relative to the main paper.

We can then restate the Algorithm in the main paper as follows.

1. Pick a $T \geq 1$ and a simulation length $T_s > T$. Guess a sequence $(\mathbb{1}_t)_{t=1}^T$ of 0s and 1s, starting with all 0s (binding in all periods) as an initial guess. Note: $\mathbb{1}_t = 0$ for $t > T$.

2. Find the structural matrices (or 'regimes') implied by the guess:

$$B_{i,t} = \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i, \quad i \in [5]$$

   in periods $t = 1, \ldots, T_s$.

3. Compute $(x_t)_{t=1}^{T_s}$ and the shadow value of the bounded variable $(x_{1,t}^*)_{t=1}^{T_s}$ via

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \le t \le T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases}, \quad x_{1,t}^* = F \begin{bmatrix} x_t' & x_{t+1}' & x_{t-1}' \end{bmatrix}' + Ge_t + H$$

where, for $t = 1, ..., T$ and initial matrices $\Omega_{T+1} = \tilde{\Omega}$, $\Psi_{T+1} = \tilde{\Psi}$, $\Gamma_{T+1} = 0_{n\times m}$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{4,t}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} (B_{2,t}(\Psi_{t+1} + \Gamma_{t+1} e_{t+1}) + B_{5,t}).$$

4. If $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ for $t = 1, \ldots, T$ and $x_{1,t} (= \underline{x}_1) \ge x_{1,t}^*$ $\forall t > T$, accept the guess and store the solution $(x_t)_{t=1}^{T_s}$; else reject. Return to Step 1 and repeat for a new guess.

Note that the only change is to Step 4 and the guesses and the terminal matrices in Step 3, since backward induction now proceeds from the alternative regime not the reference regime.

6 Multiple occasionally-binding constraints

Thus far, we have dealt only with a single occasionally-binding constraint. We now consider *multiple* occasionally-binding constraints, starting with the case of *two* lower bound constraints (such that there are four regimes in total) before turning to an arbitrary number of constraints N, which follows a similar approach to the two-constraint algorithm below.[4]

With two constraints and hence four regimes, there are four different sets of structural matrices which correspond to the members of the set

$$\{\overline{B}_1^s, \overline{B}_2^s, \overline{B}_3^s, \overline{B}_4^s, \overline{B}_5^s : s = 1, \ldots, 4\}$$

and the indicators are: $\mathbb{1}_{\{x_{1,t}^* > \underline{x}_1, x_{2,t}^* > \underline{x}_2\}}$ (both constraints slack, state $s = 1$), $\mathbb{1}_{\{x_{1,t}^* > \underline{x}_1, x_{2,t}^* \le \underline{x}_2\}}$ (constraint 1 slack, state $s = 2$), $\mathbb{1}_{\{x_{1,t}^* \le \underline{x}_1, x_{2,t}^* > \underline{x}_2\}}$ (constraint 2 slack, state $s = 3$) and $\mathbb{1}_{\{x_{1,t}^* \le \underline{x}_1, x_{2,t}^* \le \underline{x}_2\}}$ (both constraints bind, $s = 4$). We use the shorthand $\mathbb{1}_{s,t}$ henceforth.

Indicators $\mathbb{1}_{s,t}$ satisfy $\sum_{s=1}^4 \mathbb{1}_{s,t} = 1$ since regimes are mutually exclusive. Regime $s = 1$ is the terminal structure, and versions of Assumptions 1 and 2 in the main paper must hold.[5]

The system to be solved under perfect foresight is now:

$$\begin{aligned} B_{1,t} x_t &= B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \ge 1 \\ B_{i,t} &= \sum_{s=1}^4 \mathbb{1}_{s,t} \overline{B}_i^s, \quad \forall i \in [5] \end{aligned} \tag{30}$$

[4]For N constraints, there are 2^N different regimes in total and hence 2^N indicator variables.

[5]Hence, the steady state $\overline{x}$ at the reference regime is unique and satisfies $\overline{x}_j > \underline{x}_j$ for variables $j = 1, 2$.

where $\mathbb{1}_{s=1,t} = 1 \ \forall t > T$, $x_0 \in \mathbb{R}^n$ given, and e_t is a vector of known shocks with $e_t = 0_{m\times 1}$ for all $t > T$, and the 'shadow values' of the bounded variables are given by

$$x^*_{1,t} = F_1 \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G_1 e_t + H_1 \tag{31}$$

$$x^*_{2,t} = F_2 \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G_2 e_t + H_2 \tag{32}$$

where $\underline{x}_1, \underline{x}_2, H_1, H_2 \in \mathbb{R}$, F_1, F_2 are $1 \times 3n$ vectors, with vector F_j having entry j equal to zero, and G_1, G_2 are $1 \times m$ vectors.

Variables 1 and 2 could be distinct economic variables or they could be used to impose multiple constraints on the same variable. For example, to put a lower bound $\underline{x}_1$ and an *upper bound* $\overline{\overline{x}}_1$ on variable 1, we set $x_{2,t} = -x_{1,t}$ and $x^*_{2,t} = -x^*_{1,t}$ (so $F_2 = -F_1$, $G_2 = -G_1$, $H_2 = -H_1$).[6] We now show the algorithm for the case of two constraints.

Given two occasionally-binding constraints, our Algorithm must be amended as follows:

1. Pick a $T \geq 1$ and a simulation length $T_{sim} > T$. Guess on the indicators as $(\mathbb{1}^s_t)^T_{t=1}$ for $s = 1, 2, 3$ and $\mathbb{1}^4_t = 1 - \sum^3_{s=1} \mathbb{1}^s_t$ (implied) starting with $\mathbb{1}^1_t = 1$ for all t, $\mathbb{1}^{s\neq 1}_t = 0$ for all t (constraints slack in all periods) as an initial guess. Note: $\mathbb{1}^1_t = 1$ for all $t > T$.

2. Find the structural matrices (or 'regimes') implied by the guess:

$$B_{i,t} = \sum_{s=1}^{4} \mathbb{1}^s_t \overline{B}^s_i, \quad i \in [5]$$

   in periods $t = 1, \ldots, T_{sim}$.

3. Compute $(x_t)^{T_{sim}}_{t=1}$ and shadow values of the bounded variables $(x^*_{1,t})^{T_{sim}}_{t=1}, (x^*_{2,t})^{T_{sim}}_{t=1}$ via

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases}, \quad \begin{aligned} x^*_{1,t} &= F_1 \begin{bmatrix} x'_t & x'_{t+1} & x'_{t-1} \end{bmatrix}' + G_1 e_t + H_1 \\ x^*_{2,t} &= F_2 \begin{bmatrix} x'_t & x'_{t+1} & x'_{t-1} \end{bmatrix}' + G_2 e_t + H_2 \end{aligned}$$

   where, for $t = 1, ..., T$ and initial matrices $\Omega_{T+1} = \overline{\Omega}$, $\Psi_{T+1} = \overline{\Psi}$, $\Gamma_{T+1} = 0_{n\times m}$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{4,t}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} (B_{2,t}(\Psi_{t+1} + \Gamma_{t+1} e_{t+1}) + B_{5,t}).$$

4. If $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$, $x_{2,t} = \max\{\underline{x}_2, x^*_{2,t}\}$ for $t = 1, ..., T$ and $x_{1,t} > \underline{x}_1$, $x_{2,t} > \underline{x}_2$ $\forall t > T$, accept the guess and store the solution $(x_t)^{T_{sim}}_{t=1}$; else reject guess in Step 1. Return to Step 1 and repeat for a new guess.

[6] Note that if $z_t = \min\{\overline{\overline{x}}_1, x^*_{1,t}\}$, then $-z_t = \max\{-\overline{\overline{x}}_1, -x^*_{1,t}\}$, so we set $x^*_{2,t} = -x^*_{1,t}$ and $\underline{x}_2 = -\overline{\overline{x}}_1$.

Case of N constraints

The extension to an arbitrary *finite* number of constraints N is straightforward. In particular, the only non-trivial adjustments are as follows:

- There are $N \leq n$ occasionally-binding constraints and shadow values:[7]

$$x_{i,t} = \max\{\underline{x}_i, x^*_{i,t}\} \ \forall t, \quad x^*_{i,t} = F_i \begin{bmatrix} x'_t & x'_{t+1} & x'_{t-1} \end{bmatrix}' + G_i e_t + H_i, \qquad i = 1, \ldots, N.$$

- The time-varying structural matrices are $B_{i,t} := \sum_{s=1}^{2^N} \mathbb{1}_{s,t} \overline{B}_i^s$, $i \in [5]$, where $\mathbb{1}_{s,t}$ is an indicator variable equal to 1 if regime s occurs at date t and zero otherwise.
- Guess on the indicators $(\mathbb{1}_t^s)_{t=1}^T$ for $s = 1, \ldots, 2^N$ (with $\mathbb{1}_t^1 = 1$, $\mathbb{1}_t^{s \neq 1} = 0$ for $t > T$) is verified if $x_{i,t} = \max\{\underline{x}_i, x^*_{i,t}\}$ for all $i \in [N]$, $t \leq T$ and $x_{i,t} > \underline{x}_i$ for all $t > T$, $i \in [N]$.

Note that we again distinguish between the indicator variable $\mathbb{1}_{s,t}$ for the event of regime s at date t and the guesses on the indicator variables, represented by $\mathbb{1}_t^s$, that will be consistent with the former only when a particular guessed sequence of regimes is verified.

7 Example 1′: Asset pricing model

The model in Guerrieri and Iacoviello (2015, Section 2.4) has the form

$$\begin{aligned} q_t &= \beta(1-\rho)E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\ r_t &= \max\{\underline{r}, \phi q_t\} \\ u_t &= \rho_u u_{t-1} + e_t \end{aligned} \tag{33}$$

where $\beta, \rho \in (0,1)$, $\phi > 0$, $\underline{r} < 0$, $\rho_u \in (0,1)$ and all values of $e_t, e_{t+1}, \ldots$ are known.

Let $x_t = \begin{bmatrix} r_t & q_t & u_t \end{bmatrix}'$, such that the bounded variable is ordered first, and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -\phi & 0 \\ \sigma & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta(1-\rho) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho_u \end{bmatrix}, \ \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5$$

[7] Note that the assumption $N \leq n$ is not restrictive because if, in the original model, one wanted to constrain, say, all n_{orig} variables from below and also constrain n^u of them from above, then one may define new variables $x_{n_{orig}+1}, \ldots, x_{n_{orig}+n^u}$ and let $n = n_{orig} + n^u$ such that x_t has n elements as required.

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ \sigma & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i \text{ for } i \in \{2,3,4\}, \quad \overline{B}_5 = \begin{bmatrix} \underline{r} \\ 0 \\ 0 \end{bmatrix}.$$

Given $x_{1,t} = r_t$, the equation $r_t = \max\{\underline{r}, \phi q_t\}$ can be written in the form $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ by setting $\underline{x}_1 = \underline{r}$ and $x^*_{1,t} = \phi q_t$; note that the latter equation can be written in vector form as in (2) with $F = \begin{bmatrix} 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$.

7.1 Computing the M matrix

To compute the M matrix of impulse responses of the bounded variable, we solve the model in (33) ignoring the bound (i.e. with the max operator removed) and with a 'news shock' $v_{r,t} \in \{0,1\}$ added, such that $r_t = \phi q_t + v_{r,t}$. The resulting model can be written as:

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \tilde{e}_t, \quad \forall t \geq 1 \tag{34}$$

where $\tilde{e}_t := \overline{B}_4 e_t + v_t$ and $v_t := \begin{bmatrix} v_{r,t} & 0 & 0 \end{bmatrix}'$, with $v_{r,t} = 0$ for all $t > T$, where T is the horizon at which the M matrix is being computed.

The 1st column of M lists the impulse response of the policy rate r_t (at dates $t = 1, ..., T$) to the shock $v_{r,1} = 1$. In general, the jth column of M lists the impulse response of r_t (at dates $t = 1, ..., T$) to the shock $v_{r,j} = 1$; the rows of the matrix are indexed by $i = 1, ..., T$.

The impulse responses and the M matrix are obtained as follows:

$$x_t = \overline{\Omega} x_{t-1} + \hat{\Gamma} \tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{35}$$

where $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_2 (\Psi_{t+1} + \hat{\Gamma} \tilde{e}_{t+1})$ $\forall t \in [1, T]$, with $\Psi_t = 0_{n \times 1}$ for all $t > T$, $\overline{\Omega}$ as in (3), and the M matrix is given by

$$M_{ij} = \frac{\partial x_{1,i}}{\partial v_{r,j}} = r_i|_{v_{r,j}=1} - r_i|_{v_{r,j}=0} \quad \text{for } i, j \in \{1, ..., T\}.$$

For the parameters $\beta = 0.99$, $\sigma = 5$, $\phi = 0.2$, $\rho = \rho_u = 0.5$, we found that $M + M'$ is positive definite, which implies that M is a P-matrix (see e.g. Holden, 2023, Appendix: Lemma 1).

7.2 Policy function and perfect foresight paths

To compute the policy function, we set, $x_{t-1} = \begin{bmatrix} 0 & q_{t-1} & u_{t-1} \end{bmatrix}' = 0_{3 \times 1}$, specify values for the news shocks $(e_{t+s})_{s \geq 1}$, and find a perfect foresight solution for linearly-spaced $e_t \in [-0.2, 0.2]$ while holding the other initial conditions fixed. The policy function is computed at 60 different points. Some perfect foresight paths are plotted in Figure 6, where we allow for *non*-zero future shocks and compare to zero future shocks as in Guerrieri and Iacoviello (2015). The policy function plotted in Figure 7 matches the one that is shown in Guerrieri and Iacoviello (2015, Figure 1) for the case of zero future news shocks.

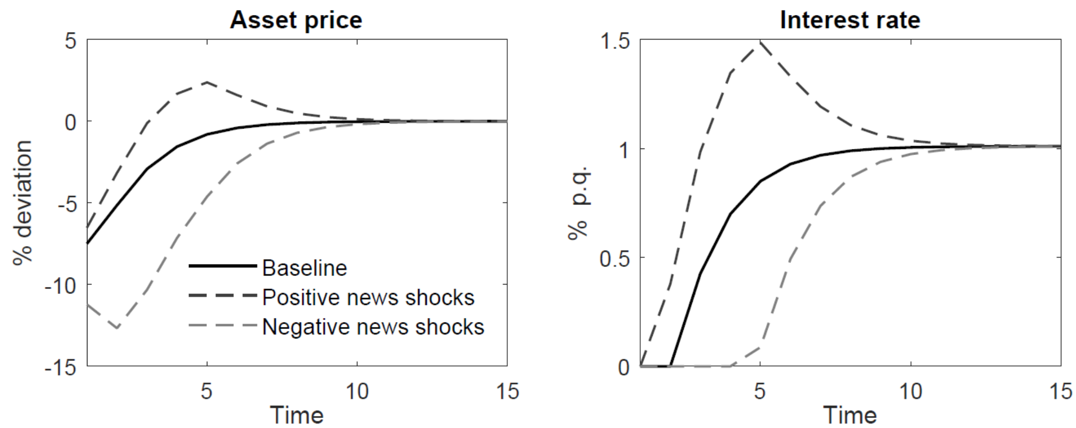

Figure 6: Perfect foresight solutions for different news shocks: $e_1 = -0.1$, $q_0 = u_0 = 0$. In the baseline case, all future (anticipated) shocks are set at 0. In the positive (negative) news case the news shocks are $e_t = 0.02$ ($e_t = -0.02$) for $t = 1, ..., 4$ and zero otherwise.

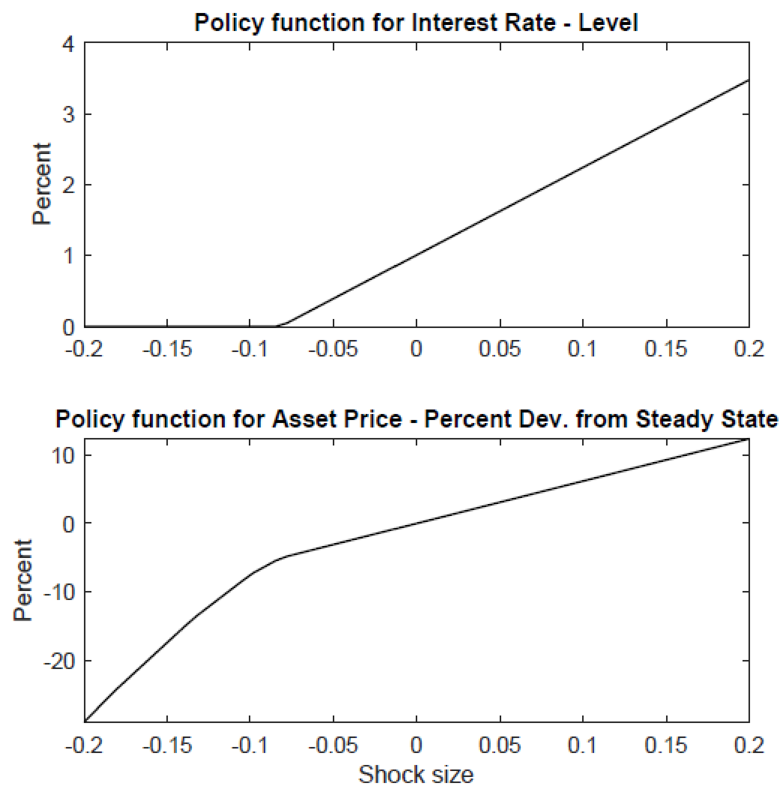

Figure 7: Policy functions for various e_t when $q_{t-1} = u_{t-1} = 0$ and no future news shocks

8 Example 2′: RBC model and investment constraint

We also consider a Real Business Cycle model with a lower bound on investment, as in Guerrieri and Iacoviello (2015, Section 4). This model requires us to log-linearize a non-linear model and to choose an appropriate shadow value $x^*_{1,t}$ in $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$.

A social planner chooses allocations $\{K_t, C_t\}_{t=0}^{\infty}$ to maximize utility $U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}-1}{1-\sigma}\right)$, subject to the following constraints:

$$C_t + I_t = A_t K_{t-1}^{\alpha} \tag{36}$$

$$K_t = (1-\delta)K_{t-1} + I_t \tag{37}$$

$$I_t \geq \phi I_{SS} \tag{38}$$

where $\sigma, I_{SS} > 0$, $\alpha, \phi \in (0,1)$, I_{SS} is the steady-state level of investment, and productivity is $A_t = A_{t-1}^{\rho} exp(\epsilon_t)$, where $\rho \in (0,1)$ and ϵ_t is a shock whose value is known at all dates.

Equations (36)–(38) are, respectively, the resource constraint, the capital accumulation equation, and a constraint that prevents investment from falling below a fraction ϕ of its steady-state value I_{SS}. The necessary conditions for a solution to the planner problem are (36)–(38) plus the consumption Euler equation and the complementary slackness condition:

$$C_t^{-\sigma} - \lambda_t = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta) - (1-\delta)\lambda_{t+1}) \tag{39}$$

$$\lambda_t (I_t - \phi I_{SS}) = 0 \tag{40}$$

where $\lambda_t \geq 0$ is the Lagrange multiplier on the investment constraint.

The investment constraint is slack when $\lambda_t = 0$ and binding when $\lambda_t > 0$. If $\lambda_t > 0$, then $I_t = \phi I_{SS}$ to ensure that the complementary slackness condition (40) holds. If $\lambda_t = 0$, then either $I_t = \phi I_{SS}$ or $I_t > \phi I_{SS}$ (but not $I_t < \phi I_{SS}$, since this would violate condition (38)). The two regimes are as follows. Under the reference regime (slack):

$$I_t = K_t - (1-\delta)K_{t-1}, \quad K_t = A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} - C_t$$

$$C_t^{-\sigma} = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta)), \quad \lambda_t = 0$$

and under the alternative regime (binding):

$$I_t = \phi I_{SS}, \quad K_t = I_t + (1-\delta)K_{t-1}, \quad C_t = A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} - K_t,$$

$$C_t^{-\sigma} - \lambda_t = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta) - (1-\delta)\lambda_{t+1}).$$

To put this non-linear model in the form of (1), we log-linearize the equations under both regimes around the steady state at which the investment constraint is slack.[8] To ease the process, we define the new variables $Y_t := A_t K_{t-1}^{\alpha}$ and $R_t := \alpha A_t K_{t-1}^{\alpha-1} + 1 - \delta$. The two regimes can then be written in terms of deviations from steady state as follows:

$$\hat{i}_t = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1}, \quad \hat{k}_t = (1-\delta)\hat{k}_{t-1} + (Y_{SS}/K_{SS})\hat{y}_t - (C_{SS}/K_{SS})\hat{c}_t$$

[8]The steady state is $I_{SS} = \delta K_{SS}$, $C_{SS} = A_{SS} K_{SS}^{\alpha} - I_{SS}$, $K_{SS} = \left(\frac{\alpha\beta A_{SS}}{1-\beta(1-\delta)}\right)^{1/(1-\alpha)}$ and $A_{SS} = 1$.

$$\hat{c}_t = E_t\hat{c}_{t+1} - (1/\sigma)E_t\hat{r}_{t+1}, \quad \lambda_t = 0, \quad \hat{y}_t = \hat{a}_t + \alpha\hat{k}_{t-1}$$
$$\hat{r}_t = \alpha R_{SS}^{-1}(Y_{SS}/K_{SS})\,\hat{a}_t - \alpha(1-\alpha)R_{SS}^{-1}(Y_{SS}/K_{SS})\,\hat{k}_{t-1}, \quad \hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t$$

under the reference regime, and

$$\hat{i}_t = \phi - 1, \quad \hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{i}_t, \quad C_{SS}\hat{c}_t = Y_{SS}\hat{y}_t + (1-\delta)K_{SS}\hat{k}_{t-1} - K_{SS}\hat{k}_t$$
$$C_{SS}^{\sigma}\lambda_t = -\sigma\hat{c}_t + \sigma E_t\hat{c}_{t+1} - E_t\hat{r}_{t+1} + (1-\delta)(C_{SS}^{\sigma}/R_{SS})E_t\lambda_{t+1}, \quad \hat{y}_t = \hat{a}_t + \alpha\hat{k}_{t-1}$$
$$\hat{r}_t = \alpha R_{SS}^{-1}(Y_{SS}/K_{SS})\,\hat{a}_t - \alpha(1-\alpha)R_{SS}^{-1}(Y_{SS}/K_{SS})\,\hat{k}_{t-1}, \quad \hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t$$

under the alternative regime.

We let $x_t := \begin{bmatrix}\hat{i}_t & \hat{k}_t & \hat{c}_t & \lambda_t & \hat{y}_t & \hat{r}_t & \hat{a}_t\end{bmatrix}'$ and $e_t := \begin{bmatrix}\epsilon_t\end{bmatrix}$, where 'hats' are log deviations from steady-state, i.e. $\hat{z}_t := \ln(Z_t/Z_{SS}) \approx (Z_t - Z_{SS})/Z_{SS}$. Note that $x_{1,t} = \hat{i}_t$. The constraint (38) is $\hat{i}_t \geq \phi - 1$ in deviations and we put this in the form $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ by setting $\underline{x}_1 = \phi - 1$ and $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t$. In the reference regime (slack), $\lambda_t = 0$, so $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1}$ and $\max\{\underline{x}_1, x_{1,t}^*\} = x_{1,t}^*$ if $\delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} (= \hat{i}_t) > \phi - 1$. In the alternative regime (binding), $\lambda_t > 0$, so $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t = (\phi-1) - \lambda_t < \phi - 1$, so $\max\{\underline{x}_1, x_{1,t}^*\} = \underline{x}_1 = \phi - 1$ as required.[9]

The shadow value $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t$ can be written as in (2) by setting $F = \begin{bmatrix}0 & \delta^{-1} & 0 & -1 & 0_{1\times 11} & -(1-\delta)\delta^{-1} & 0_{1\times 5}\end{bmatrix}$ and $G = H = \begin{bmatrix}0\end{bmatrix}$. The matrices $\overline{B}_j, \tilde{B}_j$, $j \in [5]$, under the two regimes have the form:

$$\overline{B}_1 = \begin{bmatrix} 1 & -\delta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{C_{SS}}{K_{SS}} & 0 & -\frac{Y_{SS}}{K_{SS}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\alpha Y_{SS}}{K_{SS}R_{SS}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_3 = \begin{bmatrix} 0 & -(1-\delta)\delta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1-\alpha)Y_{SS}}{K_{SS}R_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[9]Note that if the computed value of λ_t is negative under the alternative regime, then we must reject the guess that the constraint binds in this period. In this case, $x_{1,t}^* > \underline{x}_1$ ($x_{1,t}^*|_{\mathbb{1}_t=0} = \hat{i}_t - \lambda_t = (\phi-1) - \lambda_t > \phi - 1$), so $\max\{\underline{x}_1, x_{1,t}^*\} = x_{1,t}^* > \phi - 1$, and the guess $\mathbb{1}_t = 0$ will be rejected as required.

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{K_{SS}}{C_{SS}} & 1 & 0 & -\frac{Y_{SS}}{C_{SS}} & 0 & 0 \\ 0 & 0 & \sigma & C_{SS}^{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\alpha Y_{SS}}{K_{SS} R_{SS}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & \frac{(1-\delta) C_{SS}^{\sigma}}{R_{SS}} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\delta) K_{SS}}{C_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1-\alpha) Y_{SS}}{K_{SS} R_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \tilde{B}_4 = \overline{B}_4, \quad \tilde{B}_5 = \begin{bmatrix} \phi - 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We set $\beta = 0.96$, $\delta = 0.10$, $\rho = 0.90$, $\phi = 0.975$, $\sigma = 2$, $\alpha = 0.33$, as in Guerrieri and Iacoviello (2015). The policy functions and paths below match Figs. 2-3 in their paper.

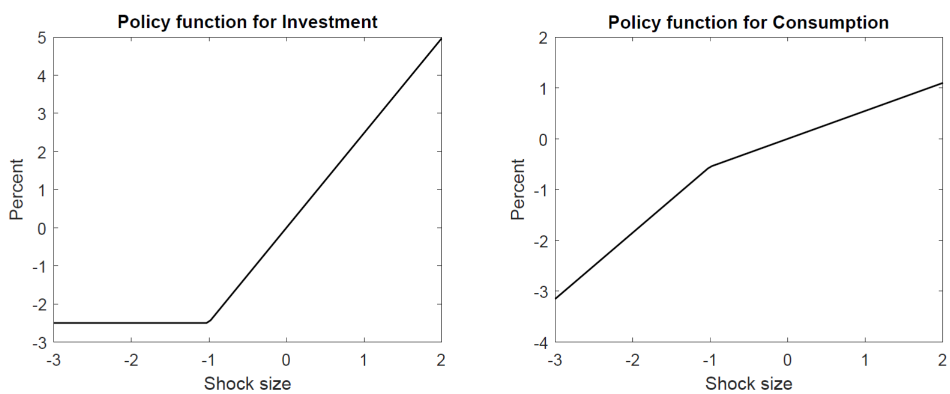

Figure 8: Policy functions for various shocks sizes e_t when $x_{t-1} = \overline{x}$

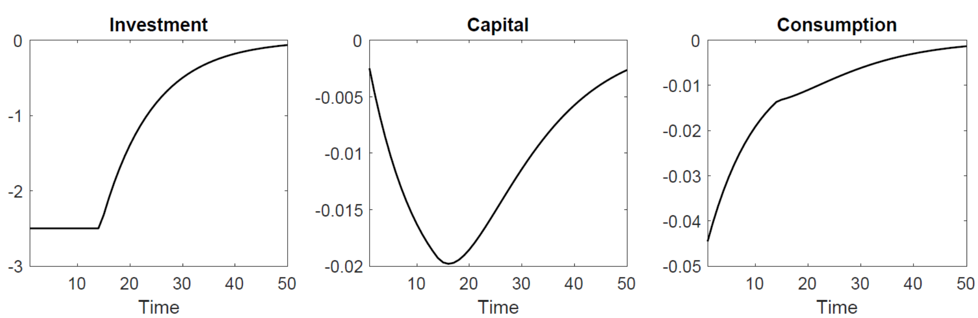

Figure 9: Perfect foresight solution for $e_1 = -0.04$, $x_0 = \overline{x}$

9 Example $3'$: Endogenous business cycles

As a final numerical example, we consider an 'endogenous business cycle' model with boom-bust dynamics of output that can lead to *multiple spells* at the lower bound. To solve for a perfect foresight solution in this case, we ensure the use of guesses that involve 'multiple spells' at the bound in our Algorithm by 'commenting in' the relevant line of code.

We consider a variant of the multiplier-accelerator model of Samuelson (1939) augmented with rational expectations, a countercyclical feedback rule for government expenditure, and an upper bound G_{max} on such expenditure (motivated by austerity policies):[10]

$$C_t = a + b(\beta E_t Y_{t+1} + (1-\beta)Y_{t-1} - \overline{T}) \tag{41}$$

$$I_t = \overline{I} + d(C_t - \beta E_t C_{t+1} - (1-\beta)C_{t-1}) + e_{I,t} \tag{42}$$

$$Y_t = C_t + I_t + G_t, \qquad G_t = \min\{G_{max}, \overline{G} - \theta(Y_{t-1} - \overline{Y})\} \tag{43}$$

where $b \in (0,1)$, $a, d, \theta, \overline{T} > 0$, the intercepts $\overline{I}, \overline{G}$ and $\overline{Y} = \frac{a - b\overline{T} + \overline{I} + \overline{G}}{1-b}$ (all > 0) are steady-state values, and $G_{max} > \overline{G}$. Note that $\overline{T}$ has the interpretation of net taxes.

We set parameters so that steady state output $\overline{Y}$ equals 1 and the model has 'business cycles' (damped oscillations) when started away from steady-state. The intercepts a and $\overline{I}$ are set at 0.025 and 0.20, respectively, and taxes at $\overline{T} = 0.01$. The marginal propensity to consume is set at $b = 0.70$ and the investment function slope at $d = 1.3$ The fraction of rational expectations is set at $\beta = 0.05$, such that 95% of agents have backward-looking expectations based on the past value (one lag). Steady-state government expenditure is set at $\overline{G} = 0.082$, the response coefficient at $\theta = 0.055$, and the upper bound at $G_{max} = 1.035 \times \overline{G}$.

To find a perfect foresight solution, we first write the upper bound constraint in the form $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$, where $x_{1,t} := -G_t$, $\underline{x}_1 = -G_{\max}$, $x^*_{1,t} = \theta(Y_{t-1} - \overline{Y}) - \overline{G}$.[11] We then define the vector of variables as $x_t = [x_{1,t}\ \ C_t\ \ I_t\ \ Y_t]'$ and the vector of shocks as $e_t = [e_{I,t}]$ (see (41)–(43)), such that the matrices in each regime are

$$\overline{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -d & 1 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta b \\ 0 & -\beta d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & \theta \\ 0 & 0 & 0 & (1-\beta)b \\ 0 & -(1-\beta)d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ \overline{B}_5 = \begin{bmatrix} -\overline{G} - \theta\overline{Y} \\ a - b\overline{T} \\ \overline{I} \\ 0 \end{bmatrix}, \ \tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-\beta)b \\ 0 & -(1-\beta)d & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \tilde{B}_5 = \begin{bmatrix} -G_{max} \\ a - b\overline{T} \\ \overline{I} \\ 0 \end{bmatrix}$$

and $\tilde{B}_i = \overline{B}_i$ for $= 1, 2, 4$, and the shadow variable is given by

[10] The original model (which is purely backward-looking) arises as the special case $\beta = a = \overline{I} = \overline{T} = \theta = 0$.

[11] Here, we use the fact that if $G_t = \min\{G_{max}, Z_t\}$, then $-G_t = \max\{-G_{\max}, -Z_t\}$.

$$x^*_{1,t} = F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + Ge_t + H$$

where

$$F = \begin{bmatrix} 0_{1\times 11} & \theta \end{bmatrix}, \quad G = [0], \quad H = \begin{bmatrix} -(\overline{G} + \theta\overline{Y}) \end{bmatrix}.$$

Figure 10 shows the perfect foresight solution for $e_{I,1} = -0.125$ and $e_{I,t} = 0$ for all $t > 1$. Our algorithm finds *one* solution subject to the upper bound (solid black lines) and reports that M is a P-matrix. We also plot for comparison the solution for $G_{\max} \to \infty$ (no upper bound, dashed lines). All variables are given initial values equal to steady-state values.

Investment falls by more than half in period 1 relative to its initial value (top, middle), which pushes output somewhat below its steady-state value (lower panel). Due to forward-looking expectations (perfect foresight), consumption also falls marginally on impact (see top left). Government expenditure is unchanged in period 1, but it rises in period 2 because past output was below the steady-state value, such that the 'fiscal rule' provides stimulus – albeit that this stimulus is truncated because the upper bound on expenditure binds (top right). The underlying dynamics are cyclical; as a result, the upper bound on government expenditure binds a second time (after period 10) when output is again in a trough.

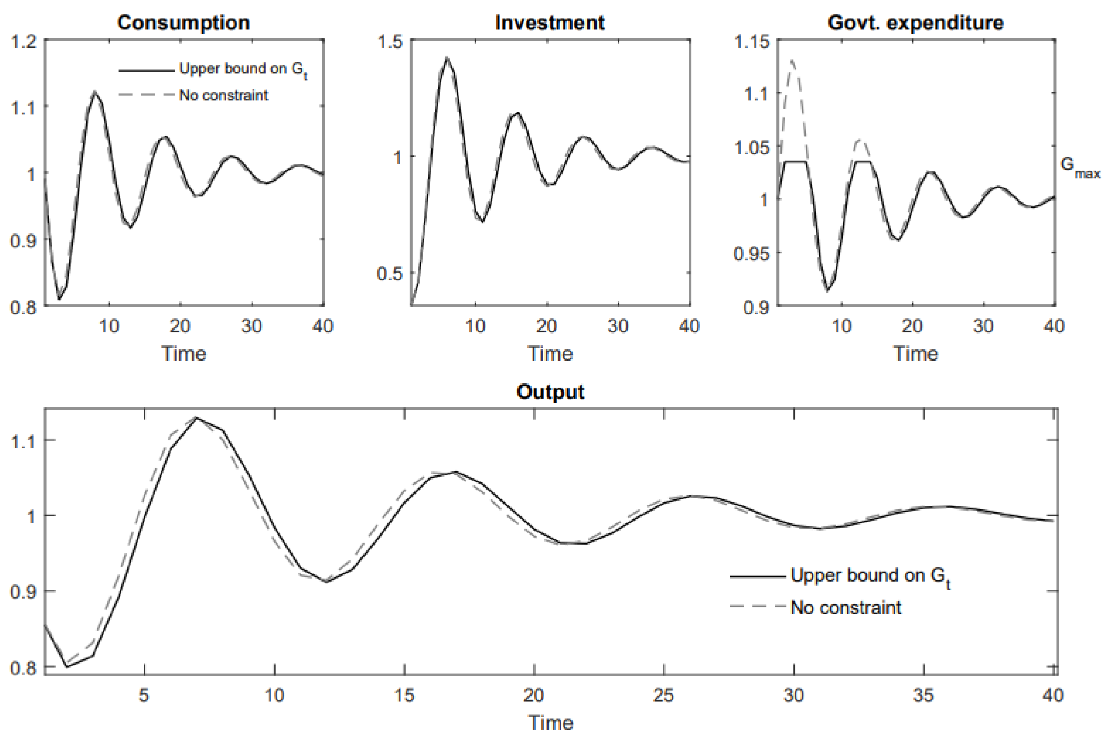

Figure 10: Perfect foresight paths for $G_{\max} = 1.035 \times \overline{G}$ and $G_{max} \to \infty$ when $e_{I,1} = -0.125$, $x_0 = \overline{x}$ (steady state). Variables in the top panel are divided by their steady-state values.

References

Binder, M. and Pesaran, M. H. (1997). Multivariate linear rational expectations models: characterization of the nature of the solutions and their fully recursive computation. *Econometric Theory*, pages 877–888.

Cho, S. and Moreno, A. (2011). The forward method as a solution refinement in rational expectations models. *Journal of Economic Dynamics and Control*, 35(3):257–272.

Guerrieri, L. and Iacoviello, M. (2015). Occbin: A toolkit for solving dynamic models with occasionally binding constraints easily. *Journal of Monetary Economics*, 70:22–38.

Holden, T. D. (2023). Existence and uniqueness of solutions to dynamic models with occasionally binding constraints. *The Review of Economics and Statistics*, 105(6):1481–1499.

Samuelson, P. A. (1939). Interactions between the multiplier analysis and the principle of acceleration. *The Review of Economics and Statistics*, 21(2):75–78.

Sims, C. A. (2002). Solving linear rational expectations models. *Computational Economics*, 20(1-2):1.

Tsatsomeros, M. J. and Li, L. (2000). A recursive test for p-matrices. *BIT Numerical Mathematics*, 40:410–414.