Appendix B: Existence of cooperative equilibrium

Not for Publication

This Appendix is based on the Online Appendix to Camera and Casari (2014). There are four identical players. In each period they are matched in pairs, with uniform probability of selection. In each pair, one player is a seller and the other is a buyer. Seller and buyer are equally likely states for an individual, i.e., the individual is a seller with probability $\alpha = \frac{1}{2}$.

Two outcomes are possible in a match: defection Y, and cooperation Z. In what follows we will say that if the seller chooses Z in a matched pair, then his opponent "consumes" and the seller "produces." For an individual, let $u = 20$ be the stage game payoff from consuming and $-c = d = 2$ the stage game payoff from producing (the seller chooses Z). Set $a = 8$, as the stage game payoff from defection (the seller choose Y).Period payoffs are geometrically discounted at rate $\beta = 0.93$. Payoffs and continuation payoffs in the game are given by expected lifetime utilities.

Equilibrium payoffs

Consider a social norm based on grim trigger. It has a rule for cooperation: a seller must always choose Z. It has also a rule for punishment: If a defection is observed, then Y is selected forever after. Suppose an equilibrium exists based on this social norm. The payoff of the representative player is denoted

$$V = \frac{(1-\alpha)u - \alpha c}{1-\beta}. \tag{1}$$

This is simply the present value of the stream of expected period payoffs, which are time-invariant in equilibrium. To discuss existence of equilibrium we now present individual optimality conditions in and out of equilibrium.

In equilibrium cooperation is a best response for a seller if

$$-c + \beta V \geq a + \beta v_2. \tag{2}$$

The left-hand-side denotes the payoff from cooperating when everyone has always cooperated up to that point. The right-hand-side from defecting when everyone has always cooperated up to that point. The notation v_2 denotes the off-equilibrium continuation payoff in the group where two players have seen a defection and follow the rule of punishment of the social norm (as a seller, choose Y). Since $V > v_2$ for (2) to hold, we rewrite it as

$$\beta \geq \beta_L := \frac{a+c}{V - v_2}.$$

Out of equilibrium payoffs

Consider out of equilibrium actions when everyone follows the social norm. Out of equilibrium we have at least two defectors. Let v_4 denote the continuation payoff for any player in a group with four defectors (everyone defects as a seller). Since both sellers will defect we have

$$v_4 = \frac{a}{1-\beta} \tag{3}$$

and so we call v_4 the **defection** payoff.

Now consider the case where a defection has just taken place for the first time. So there are only two defectors. For concreteness, let player x observe a defection for the first time in period $t-1$. She believes that everyone has played cooperation up to that point. player x may be the one who defected, or her opponent, denoted y. Suppose that everyone will behave according to the social norm from now on. Next period t there will be two defectors (players x and y) and two cooperators (players in the other match who observed nothing).

The continuation payoff for player x at the start of period t is

$$v_2 = \tfrac{1}{3}(a + \beta v_2) + \tfrac{2}{3}[(1-\alpha)(u + \beta \tfrac{v_4+v_3}{2}) + \alpha(a + \beta \tfrac{v_2+v_3}{2})]. \tag{4}$$

To see why note that with probability $\frac{1}{3}$ player x meets again player y (a defector), and with probability $\frac{2}{3}$ player x meets a cooperator.

- If x meets y once again, a is the period payoff, and since no one else observed a defection next period $t+1$ there will still be two defectors. So the discounted continuation payoff is βv_2.
- If x someone other than y, then this player is a cooperator.
  - If x is a seller (with probability $\alpha = \frac{1}{2}$), then x defects and earns a. The defection is seen by her opponent but the continuation payoffs depends also on what happens in the other match. This is because the other pair is also composed of a defector (player y) and a cooperator. If player y is a seller, then he defects (seen by her opponent). Hence, next period we have four defectors (v_4 is the payoff). If, instead, player y is a buyer, then there is no defection in the other match and the following period we have three defectors (v_3 is the payoff). Since y is a seller with probability $\frac{1}{2}$, then a defection occurs the other match with that probability.
  - If x is a buyer (with probability $1-\alpha = \frac{1}{2}$), then he earns u. Again, the continuation payoff depends on events in the other match and, since x does not defect, we cannot have more than three defectors next period. With probability $\frac{1}{2}$ there are three defectors and there are two, otherwise.

Substituting for $\alpha = 1/2$ we rearrange (4) as

$$v_2 = \tfrac{2}{3(2-\beta)}(u + 2a + \beta\tfrac{1}{2}v_4 + \beta v_3). \tag{5}$$

To calculate v_3 consider the case when, at the beginning of some date, player x is one of three defectors (i.e., players who have seen or implemented a defection Y). Suppose that everyone adopts the social norm. The payoff to player x is

$$v_3 = \tfrac{1}{3}\left[\tfrac{1}{2}(u + \beta v_3) + \tfrac{1}{2}(a + \beta v_4)\right] + \tfrac{2}{3}(a + \beta\tfrac{v_4+v_3}{2}), \tag{6}$$

because with probability $\frac{1}{3}$ player x meets a cooperator, and with probability $\frac{2}{3}$ she meets a defector.

- If player x meets a cooperator, then her period payoff depends on whether she is a seller of a buyer. Her continuation payoff depends also on this because only if she produces will the group move to the state with four defectors. Indeed, the other match has two defectors.

- If player x meets a defector. Then she always earns a but the continuation payoff depends on whether the cooperator in the other match is a buyer. If that's the case (with probability 1/2), then the group transitions to a state with four defectors. Otherwise, it will remain in a state with three defectors.

Rearranging (6) we have

$$v_3 = \frac{1}{3(2-\beta)}(u + 5a + 3\beta v_4).$$

Using the above in (4) we have

$$v_2 = \tfrac{2}{3(2-\beta)^2}\{(u + 2a)(2-\beta) + \beta[\tfrac{(2+\beta)a}{2(1-\beta)} + \tfrac{u+5a}{3}]\}. \tag{7}$$

We can now find a condition such that defecting in equilibrium is individually sub-optimal

Lemma 1. *There exists a non-trivial interval* $(\beta_L, 1)$ *such that if* $\beta \in (\beta_L, 1)$*, then (2) holds.*

Proof of Lemma 1. Rewrite (2) as $\frac{a+c}{v_2} \leq \beta(\frac{V}{v_2} - 1)$. As $\beta \to 0$ we have $V \to \frac{u-c}{2}$ and $v_2 \to \frac{u+2a}{3}$. So, clearly, as $\beta \to 0$ then (2) is violated for any $a \geq 0$ and $c < 0$. Notice that $\frac{\partial v_2}{\partial \beta}, \frac{\partial V}{\partial \beta} > 0$. As $\beta \to 1$, we have $v_2 \to \infty$ and $V \to \infty$. It should be clear that as $\beta \to 1$ then $\frac{a+c}{v_2} \to 0$. In addition, the RHS of the inequality converges to a positive quantity since, as $\beta \to 1$, then $\frac{V}{v_2} \to \frac{u-c}{2a} > 1$, given our initial assumption. We conclude that there exists a β_L sufficiently close to one such that (2) holds for all $\beta \in (\beta_L, 1)$, with strict inequality.

Deviating out of equilibrium

Now we find conditions under which it is optimal to follow the rule of punishment after having observed a defection.

Suppose player x observes a deviation for the first time in a match with player y (it does not matter who defects). Consider now the date when player x is a seller, for the first time, after observing the defection in the match with y. This event may happen quite some time after observing the defection (role assignment is probabilistic) so it is possible that everyone else in the group has also observed the defection because y had a chance to defect. It is also possible that y never had a chance to defect, so the group still has two people who observed a defection. This scenario certainly occurs if x is a seller the period after observing the defection.

Consider the following deviation. Player x refuses to choose Y as a seller and, instead, she cooperates. She will follow the social norm for punishment afterward (one-time deviation). The rationale for this is that she can slow down the contagion to full defection, hence enjoy some payoffs u for a little longer.

This deviation is suboptimal if the group has already three defectors since no one will ever cooperate. The best-case scenario is when the group has only two defectors. Hence, consider this case by supposing that player x is a seller the period immediately after observing her first defection,

Choosing to deviate from the social norm out of equilibrium (choosing Y) is a best response if

$$a + \beta(\tfrac{1}{3}v_2 + \tfrac{2}{3}\tfrac{v_3+v_4}{2}) \geq -c + \beta(\tfrac{1}{3}v_2 + \tfrac{2}{3}\tfrac{v_2+v_3}{2}). \tag{8}$$

- Consider the LHS of (8), which is when x follows the social norm, out of equilibrium. Since player x is a seller she will defect, generating a period payoff. The continuation payoff depends on whom she meets. With probability $\frac{1}{3}$ player x meets y, the deviator met earlier. In this case the continuation payoff is v_2 since the other match has two cooperators. If, instead, player x meets a cooperator (probability $\frac{2}{3}$) then the group will have three defectors only if in the other match the defector is not a seller (with probability $\frac{1}{2}$).

- Consider the RHS of (8), which is when x does not defect today (though she should). Instead, she chooses Z today, so her period payoff is $-c$, and will choose Y forever after. Her continuation payoff depends once again on whom she meets. If she meets player y, the other defector, then next period there will be again two defectors (her and player y). This occurs with probability $\frac{1}{3}$. If, instead, player x meets a cooperator, with probability $\frac{2}{3}$ next period the group has 2 or 3 defectors depending on what happens in the other match. With probability $\frac{1}{2}$ a defection occurs in the other match (player y is a seller).

Inequality (8) can be rearranged as

$$a + c \geq \frac{\beta}{3}(v_2 - v_4). \tag{9}$$

Recalling that if it is optimal for player x to defect out of equilibrium after having observed an initial defection (i.e., when there are two defectors, including player x), then it will also be optimal to defect after having observed more than one defection (i.e., when there are more than two defectors, including player x).

Since $v_2 > v_4$ for (9) to hold, we rewrite it as

$$\beta \leq \beta_H := \frac{3(a + c)}{v_2 - v_4}.$$

Inserting $u = 20$, $-c = 2$ and $a = 8$ we numerically find $\beta_L = 0.808$ and $\beta_H = 1.2$. Hence, for the parameterization $u = 20$, $a = 8$ and $-c = 2$ if $\beta \geq 0.808$, then the grim trigger strategy is an equilibrium.