Appendix A Proofs of Theorems

A.1 Theorems and Proofs from Section 3

Lemma 1

Let P be an *NLP*, $\mathcal{I} = \langle T, F \rangle$ an interpretation and $\Omega_P(\mathcal{I}) = \langle T', F' \rangle$ the least 3-valued model of $\frac{P}{\mathcal{I}}$. It holds

(i) $c \in T'$ iff there exists a statement s constructed from P such that $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \subseteq F$.

(ii) $c \in F'$ iff for every statement s constructed from P such that $\texttt{Conc}(s) = c$, we have $\texttt{Vul}(s) \cap T \neq \emptyset$

Proof

- Proving that $c \in T'$ iff there exists a statement s constructed from P such that $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \subseteq F$:

  $\Rightarrow$ Consider $\Psi_{\frac{P}{\mathcal{I}}}^{\uparrow i} = \langle T_i, F_i \rangle$ for each $i \in \mathbb{N}$. It suffices to prove by induction on the value of i that if $c \in T_i$, then there exists a statement s constructed from P such that $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \subseteq F$:

  — *Basis*. For $i = 0$, the result is trivial as $T_0 = \emptyset$.

— *Step.* Assume that for every $c' \in T_n$, there exists a statement s' constructed from P such that $\texttt{Conc}(s') = c'$ and $\texttt{Vul}(s') \subseteq F$. We will prove that if $c \in T_{n+1}$, there exists a statement s constructed from P such that $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \subseteq F$:
If $c \in T_{n+1}$, there exists a rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n (m \geq 0, n \geq 0) \in P$ such that $\{a_1, \ldots, a_m\} \subseteq T_n$ and $\{b_1, \ldots, b_n\} \subseteq F$. It follows via inductive step that for every $j \in \{1, \ldots, m\}$, there exists a statement s_j constructed from P such that $\texttt{Conc}(s_j) = a_j$ and $\texttt{Vul}(s_j) \subseteq F$. But then, we can construct from P a statement s with $\texttt{Conc}(s) = c$ where $\texttt{Vul}(s) = \texttt{Vul}(s_1) \cup \cdots \cup \texttt{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. This implies that $\texttt{Vul}(s) \subseteq F$.

$\Leftarrow$ We will prove by structural induction on the construction of statements that for each statement s constructed from P such that $\texttt{Vul}(s) \subseteq F$, it holds $\texttt{Conc}(s) \in T'$:

— *Basis.* Let s be a statement $c \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$ $(n \geq 0)$ such that $\{b_1, \ldots, b_n\} = \texttt{Vul}(s) \subseteq F$. It follows the fact $c \in \frac{P}{\mathcal{I}}$. Then $c \in T'$.

— *Step.* Assume $s_1, \ldots, s_m$ $(m \geq 1)$ are arbitrary statements constructed from P such that for each $i \in \{1, \ldots, m\}$, if $\texttt{Vul}(s_i) \subseteq F$, then $\texttt{Conc}(s_i) \in T'$. We will prove that if s is a statement $c \leftarrow (s_1), \ldots, (s_m), \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$ $(n \geq 0)$ constructed from P such that $\texttt{Vul}(s) \subseteq F$, then $c \in T'$:
Let s be such a statement. By Definition 8, there exists a rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P$ such that $\texttt{Conc}(s_i) = a_i$ for each $i \in \{1, \ldots, m\}$ and $\texttt{Vul}(s) = \texttt{Vul}(s_1) \cup \cdots \cup \texttt{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. As $\texttt{Vul}(s) \subseteq F$, we obtain $\{b_1, \ldots, b_n\} \subseteq F$ and $\texttt{Vul}(s_i) \subseteq F$ for each $i \in \{1, \ldots, m\}$. By inductive hypothesis, it follows $\{a_1, \ldots, a_m\} \subseteq T'$. Then $c \in T'$.

- Proving that $c \in F'$ iff for every statement s constructed from P such that $\texttt{Conc}(s) = c$, we have $\texttt{Vul}(s) \cap T \neq \emptyset$:

$\Rightarrow$ Firstly, we will prove by structural induction on the construction of statements that for each statement s constructed from P such that $\texttt{Vul}(s) \cap T = \emptyset$, it holds $\texttt{Conc}(s) \notin F'$:

— *Basis.* Let s be a statement $c \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$ $(n \geq 0)$ such that $\{b_1, \ldots, b_n\} \cap T = \emptyset$. It follows the fact $c \in \frac{P}{\mathcal{I}}$ or $c \leftarrow \mathbf{u} \in \frac{P}{\mathcal{I}}$. Then $c \notin F'$.

— *Step.* Assume $s_1, \ldots, s_m$ $(m \geq 1)$ are arbitrary statements constructed from P such that for each $i \in \{1, \ldots, m\}$, if $\texttt{Vul}(s_i) \cap T = \emptyset$, then $\texttt{Conc}(s_i) \notin F'$. We will prove that if s is a statement $c \leftarrow (s_1), \ldots, (s_m), \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$ $(n \geq 0)$ constructed from P such that $\texttt{Vul}(s) \cap T = \emptyset$, then $c \notin F'$:
Let s be such a statement. By Definition 8, there exists a rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P$ such that $\texttt{Conc}(s_i) = a_i$ for each $i \in \{1, \ldots, m\}$ and $\texttt{Vul}(s) = \texttt{Vul}(s_1) \cup \cdots \cup \texttt{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. As $\texttt{Vul}(s) \cap T = \emptyset$, we obtain $\{b_1, \ldots, b_n\} \cap T = \emptyset$ and $\texttt{Vul}(s_i) \cap T = \emptyset$ for each

$i \in \{1, \ldots, m\}$. By inductive hypothesis, it follows $\{a_1, \ldots, a_m\} \cap F' = \emptyset$. Then, $c \notin F'$.

Hence, if $c \in F'$, for every statement s constructed from P such that $\texttt{Conc}(s) = c$, we have $\texttt{Vul}(s) \cap T \neq \emptyset$.

$\Leftarrow$ Assume that for every statement s constructed from P such that $\texttt{Conc}(s) = c$, we have $\texttt{Vul}(s) \cap T \neq \emptyset$. The proof is by contradiction: suppose that $c \notin F'$. Consider $\Psi^{\uparrow i}_{\frac{P}{\mathcal{I}}} = \langle T_i, F_i \rangle$ for each $i \in \mathbb{N}$. It suffices to prove by induction on the value of i that if $c \notin F_i$, then there exists a statement s constructed from P such that $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \cap T = \emptyset$:

— *Basis.* For $i = 0$, the result is trivial as $F_0 = HB_P$.

— *Step.* Assume that for every $c' \notin F_n$, there exists a statement s' constructed from P such that $\texttt{Conc}(s') = c'$ and $\texttt{Vul}(s') \cap T = \emptyset$. We will prove that if $c \notin F_{n+1}$, there exists a statement s constructed from P such that $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \cap T = \emptyset$:
If $c \notin F_{n+1}$, there exists a rule $c \leftarrow a_1, \ldots, a_m, \texttt{not } b_1, \ldots, \texttt{not } b_n (m \geq 0, n \geq 0) \in P$ such that $\{a_1, \ldots, a_m\} \cap F_n = \emptyset$ and $\{b_1, \ldots, b_n\} \cap T = \emptyset$. It follows via inductive step that for every $j \in \{1, \ldots, m\}$, there exists a statement s_j constructed from P such that $\texttt{Conc}(s_j) = a_j$ and $\texttt{Vul}(s_j) \cap T = \emptyset$. But then, we can construct from P a statement s with $\texttt{Conc}(s) = c$ where $\texttt{Vul}(s) = \texttt{Vul}(s_1) \cup \cdots \cup \texttt{Vul}(s_m) \cup \{b_1, \ldots, b_n\}$. This implies that $\texttt{Vul}(s) \cap T = \emptyset$.

□

Theorem 3
Let P be an *NLP* and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated *SETAF*. For any labelling $\mathcal{L}$ of $\mathfrak{A}_P$, it holds $\mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L})) = \mathcal{L}$.

Proof
Let $c \in \mathcal{A}_P$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F \rangle$; there are three possibilities:

- $\mathcal{L}(c) = \texttt{in} \Rightarrow c \in T \Rightarrow \mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))(c) = \texttt{in}$.
- $\mathcal{L}(c) = \texttt{out} \Rightarrow c \in F \Rightarrow \mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))(c) = \texttt{out}$.
- $\mathcal{L}(c) = \texttt{undec} \Rightarrow c \in \overline{T \cup F} \Rightarrow \mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))(c) = \texttt{undec}$.

□

Theorem 4
Let P be an *NLP*, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated *SETAF* and $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of P. It holds that $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M})) = \mathcal{M}$.

Proof
Let $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of P, $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M})) = \langle T', F' \rangle$ and $c \in HB_P$. It suffices to prove the following results:

- $c \in T$ iff $c \in T'$.

- Assume $c \in T$. As $\Omega_P(\mathcal{M}) = \mathcal{M}$, by Lemma 1, there exists a statement s with $\texttt{Conc}(s) = c$ such that $\texttt{Vul}(s) \subseteq F$. In particular, it follows that $c \in \mathcal{A}_P$. This implies $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) = \texttt{in}$ and $c \in T'$.
- Assume $c \in T'$. Then $c \in \mathcal{A}_P$ and $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) = \texttt{in}$. From Definition 11, we obtain $c \in T$.

• $c \in F$ iff $c \in F'$.

- Assume $c \notin F'$. Then $c \in \mathcal{A}_P$ and $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) \neq \texttt{out}$. From Definition 11, we obtain $c \notin F$.
- Assume $c \notin F$. As $\Omega_P(\mathcal{M}) = \mathcal{M}$, by Lemma 1, there exists a statement s with $\texttt{Conc}(s) = c$ such that $\texttt{Vul}(s) \cap T = \emptyset$. In particular, it follows that $c \in \mathcal{A}_P$. This implies $\mathcal{I}2\mathcal{L}_P(\mathcal{M})(c) \neq \texttt{out}$ and $c \notin F'$.

□

Lemma 30
Let P be an *NLP*, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated *SETAF* and $v \in \{\texttt{in}, \texttt{out}, \texttt{undec}\}$. It holds that

- For each $\mathcal{B} \in Att(c)$, $\mathcal{L}(b) = v$ for some $b \in \mathcal{B}$ iff there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) = v$ for every $b \in \mathcal{A}_P \cap V$.
- For each $\mathcal{B} \in Att(c)$, $\mathcal{L}(b) \neq v$ for some $b \in \mathcal{B}$ iff there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) \neq v$ for every $b \in \mathcal{A}_P \cap V$.

Proof
We will prove the result in the first item; the proof of the other result follows a similar path:

$\Rightarrow$ Assume that for each $\mathcal{B} \in Att(c)$, $\mathcal{L}(b) = v$ for some $b \in \mathcal{B}$.
By absurd, suppose that for each $V \in \texttt{Vul}(c)$, it holds that $\mathcal{L}(b) \neq v$ for some $b \in \mathcal{A}_P \cap V$. Then we can construct a set $\mathcal{B}' \subseteq \mathcal{A}_P$ by selecting for each $V \in \texttt{Vul}(c)$, an element $b \in \mathcal{V}$ such that $\mathcal{L}(b) \neq v$. From Definition 9, we know that there exists $\mathcal{B} \subseteq \mathcal{B}'$ such that $(\mathcal{B}, c) \in Att_P$. But then, there exists $\mathcal{B} \in Att(c)$ such that $\mathcal{L}(b) \neq v$ for each $b \in \mathcal{B}$. It is absurd as it contradicts our hypothesis.
$\Leftarrow$ Assume that there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) = v$ for every $b \in \mathcal{A}_P \cap V$.
The result is immediate as according to Definition 9, every set $\mathcal{B}$ of arguments attacking c contains an element $b \in \mathcal{A}_P \cap V$.

□

Theorem 5
Let P be an *NLP* and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated *SETAF*. It holds

- $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P.
- $\mathcal{M}$ is a partial stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of $\mathfrak{A}_P$.

Proof
1. If $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$, then $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P:

Let $\mathcal{M} = \mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F\rangle$. We will show $\mathcal{M}$ is a partial stable model of P, i.e., $\Omega_P(\mathcal{M}) = \langle T', F'\rangle = \langle T, F\rangle$:

- $c \in T$ iff $c \in \mathcal{A}_P$ and $\mathcal{L}(c) = \texttt{in}$ iff for each $\mathcal{B} \in Att(c)$, it holds $\mathcal{L}(b) = \texttt{out}$ for some $b \in \mathcal{B}$ iff (Lemma 30) there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) = \texttt{out}$ for every $b \in \mathcal{A}_P \cap V$ iff there exists a statement s with $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \subseteq F$ iff (Lemma 1) $c \in T'$.
- $c \notin F$ iff $c \in \mathcal{A}_P$ and $\mathcal{L}(c) \neq \texttt{out}$ iff for each $\mathcal{B} \in Att(c)$, it holds $\mathcal{L}(b) \neq \texttt{in}$ for some $b \in \mathcal{B}$ iff (Lemma 30) there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) \neq \texttt{in}$ for every $b \in \mathcal{A}_P \cap V$ iff there exists a statement s with $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \cap T = \emptyset$ iff (Lemma 1) $c \notin F'$.

2. If $\mathcal{M}$ is a partial stable model of P, then $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of $\mathfrak{A}_P$:

   Let $\mathcal{M} = \langle T, F\rangle$ be a partial stable model of P. Then $\Omega_P(\mathcal{M}) = \langle T, F\rangle$. Let c be an argument in $\mathcal{A}_P$. We will prove $\mathcal{L} = \mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of $\mathfrak{A}_P$:

   - $\mathcal{L}(c) = \texttt{in}$ iff $c \in T$ iff (Lemma 1) there exists a statement s with $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \subseteq F$ iff there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) = \texttt{out}$ for every $b \in \mathcal{A}_P \cap V$ iff (Lemma 30) for each $\mathcal{B} \in Att(c)$, it holds $\mathcal{L}(b) = \texttt{out}$ for some $b \in \mathcal{B}$.
   - $\mathcal{L}(c) \neq \texttt{out}$ iff $c \neq F$ iff (Lemma 1) there exists a statement s with $\texttt{Conc}(s) = c$ and $\texttt{Vul}(s) \cap T = \emptyset$ iff there exists $V \in \texttt{Vul}(c)$ such that $\mathcal{L}(b) \neq \texttt{in}$ for every $b \in \mathcal{A}_P \cap V$ iff (Lemma 30) for each $\mathcal{B} \in Att(c)$, it holds $\mathcal{L}(b) \neq \texttt{in}$ for some $b \in \mathcal{B}$.

3. If $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, then $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$:

   It holds that $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of $P \Rightarrow$ according to item 2 above, $\mathcal{I}2\mathcal{L}_P(\mathcal{L}2\mathcal{I}_P(\mathcal{L}))$ is a complete labelling of $\mathfrak{A}_P \Rightarrow$ (via Theorem 3) $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$.
4. If $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of $\mathfrak{A}_P$, then $\mathcal{M}$ is a partial stable model of P:

   It holds that $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a complete labelling of $\mathfrak{A}_P \Rightarrow$ according to item 1 above, $\mathcal{L}2\mathcal{I}_P(\mathcal{I}2\mathcal{L}_P(\mathcal{M}))$ is a partial stable model of $P \Rightarrow$ (via Theorem 4) $\mathcal{M}$ is a partial stable model of P.

□

Lemma 31
Let P be an *NLP*, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated *SETAF*. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be β-complete labellings of $\mathfrak{A}_P$, and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_1) = \langle T_1, F_1\rangle$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2\rangle$. It holds

1. $\texttt{in}(\mathcal{L}_1) \subseteq \texttt{in}(\mathcal{L}_2)$ iff $T_1 \subseteq T_2$;
2. $\texttt{in}(\mathcal{L}_1) = \texttt{in}(\mathcal{L}_2)$ iff $T_1 = T_2$;
3. $\texttt{in}(\mathcal{L}_1) \subset \texttt{in}(\mathcal{L}_2)$ iff $T_1 \subset T_2$.

Proof

1. (⇒): Suppose $\mathtt{in}(\mathcal{L}_1) \subseteq \mathtt{in}(\mathcal{L}_2)$. If $c \in T_1$, by Definition 11, $c \in \mathcal{A}_P$ and $\mathcal{L}_1(A) = \mathtt{in}$. From our initial assumption, it follows $\mathcal{L}_2(c) = \mathtt{in}$. So, by Definition 11, $c \in T_2$.
   (⇐): Suppose $T_1 \subseteq T_2$. If $\mathcal{L}_1(c) = \mathtt{in}$, by Definition 11, $c \in T_1$. From our initial assumption, it follows $c \in T_2$. So, by Definition 11, $\mathcal{L}_2(c) = \mathtt{in}$.
2. It follows directly from point 1.
3. It follows directly from points 1 and 2.

□

Lemma 32
Let P be an *NLP*, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated *SETAF*. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete labellings of $\mathfrak{A}_P$, and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_1) = \langle T_1, F_1 \rangle$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$. It holds

1. $\mathtt{out}(\mathcal{L}_1) \subseteq \mathtt{out}(\mathcal{L}_2)$ iff $F_1 \subseteq F_2$;
2. $\mathtt{out}(\mathcal{L}_1) = \mathtt{out}(\mathcal{L}_2)$ iff $F_1 = F_2$;
3. $\mathtt{out}(\mathcal{L}_1) \subset \mathtt{out}(\mathcal{L}_2)$ iff $F_1 \subset F_2$.

Proof

1. (⇒): Suppose $\mathtt{out}(\mathcal{L}_1) \subseteq \mathtt{out}(\mathcal{L}_2)$. If $c \in F_1$, by Definition 11, there are two possibilities:
   - $c \notin \mathcal{A}_P$. As $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$, we obtain that $c \in F_2$.
   - $c \in \mathcal{A}_P$ and $\mathcal{L}_1(c) = \mathtt{out}$. From our initial assumption, it follows $\mathcal{L}_2(c) = \mathtt{out}$. So, by Definition 11, $c \in F_2$.

   (⇐): Suppose $F_1 \subseteq F_2$. If $\mathcal{L}_1(c) = \mathtt{out}$, by Definition 11, $c \in F_1$. From our initial assumption, it follows $c \in F_2$. So, by Definition 11, $\mathcal{L}_2(c) = \mathtt{out}$.
2. It follows directly from point 1.
3. It follows directly from points 1 and 2.

□

Lemma 33
Let P be an *NLP*, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be its associated *SETAF*. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete labellings of $\mathfrak{A}_P$, and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_1) = \langle T_1, F_1 \rangle$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}_2) = \langle T_2, F_2 \rangle$. It holds

1. $\mathtt{undec}(\mathcal{L}_1) \subseteq \mathtt{undec}(\mathcal{L}_2)$ iff $\overline{T_1 \cup F_1} \subseteq \overline{T_2 \cup F_2}$;
2. $\mathtt{undec}(\mathcal{L}_1) = \mathtt{undec}(\mathcal{L}_2)$ iff $\overline{T_1 \cup F_1} = \overline{T_2 \cup F_2}$;
3. $\mathtt{undec}(\mathcal{L}_1) \subset \mathtt{undec}(\mathcal{L}_2)$ iff $\overline{T_1 \cup F_1} \subset \overline{T_2 \cup F_2}$.

Proof

1. (⇒): Suppose $\mathtt{undec}(\mathcal{L}_1) \subseteq \mathtt{undec}(\mathcal{L}_2)$. If $c \in \overline{T_1 \cup F_1}$, by Definition 11, $c \in \mathcal{A}_P$ and $\mathcal{L}_1(c) = \mathtt{undec}$. From our initial assumption, it follows $\mathcal{L}_2(c) = \mathtt{undec}$. So, by Definition 11, $c \in \overline{T_2 \cup F_2}$.

(⇐): Suppose $\overline{T_1 \cup F_1} \subseteq \overline{T_2 \cup F_2}$. If $\mathcal{L}_1(c) = \texttt{undec}$, by Definition 11, $c \in \overline{T_1 \cup F_1}$. From our initial assumption, it follows $c \in \overline{T_2 \cup F_2}$. So, by Definition 11, $\mathcal{L}_2(c) = \texttt{undec}$.

2. It follows directly from point 1.
3. It follows directly from points 1 and 2.

□

Theorem 6
Let P be an *NLP* and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated *SETAF*. It holds

1. $\mathcal{L}$ is a grounded labelling of $\mathfrak{A}_P$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a well-founded model of P.
2. $\mathcal{L}$ is a preferred labelling of $\mathfrak{A}_P$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a regular model of P.
3. $\mathcal{L}$ is a stable labelling of $\mathfrak{A}_P$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a stable model of P.
4. $\mathcal{L}$ is a semi-stable labelling of $\mathfrak{A}_P$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is an L-stable model of P.

Proof
Let $\mathcal{L}$ be an argument labelling of $\mathfrak{A}_P$ and $\mathcal{L}2\mathcal{I}_P(\mathcal{L}) = \langle T, F \rangle$. The proof is straightforward:

1. $\mathcal{L}$ is a grounded labelling of $\mathfrak{A}_P$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$, and $\texttt{in}(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of $\mathfrak{A}_P$ iff (Theorem 5 and Lemma 31) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $T' \subset T$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a well-founded model of P;
2. $\mathcal{L}$ is a preferred labelling of $\mathfrak{A}_P$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$, and $\texttt{in}(\mathcal{L})$ is maximal (w.r.t. set inclusion) among all complete labellings of $\mathfrak{A}_P$ iff (Theorem 5 and Lemma 31) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $T \subset T'$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a regular model of P;
3. $\mathcal{L}$ is a stable labelling of $\mathfrak{A}_P$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$ such that $\texttt{undec}(\mathcal{L}) = \emptyset$ iff (Theorem 5) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model such that $\overline{T \cup F} = \emptyset$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a stable model of P;
4. $\mathcal{L}$ is a semi-stable labelling of $\mathfrak{A}_P$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}_P$, and $\texttt{undec}(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of $\mathfrak{A}_P$ iff (Theorem 5 and Lemma 33) $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is a partial stable model of P, and there is no partial stable model $\mathcal{M}' = \langle T', F' \rangle$ of P such that $\overline{T' \cap F'} \subset \overline{T \cup F}$ iff $\mathcal{L}2\mathcal{I}_P(\mathcal{L})$ is an L-stable model of P.

□

Corollary 7
Let P be an *NLP* and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ be the associated *SETAF*. It holds

1. $\mathcal{M}$ is a well-founded model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a grounded labelling of $\mathfrak{A}_P$.
2. $\mathcal{M}$ is a regular model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a preferred labelling of $\mathfrak{A}_P$.
3. $\mathcal{M}$ is a stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a stable labelling of $\mathfrak{A}_P$.
4. $\mathcal{M}$ is an L-stable model of P iff $\mathcal{I}2\mathcal{L}_P(\mathcal{M})$ is a semi-stable labelling of $\mathfrak{A}_P$.

Proof
These results come from Theorems 4 and 6. □

A.2 Theorems and Proofs from Section 4

Theorem 8
Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a *SETAF* and $P_{\mathfrak{A}}$ its associated *NLP*.

- For any labelling $\mathcal{L}$ of $\mathfrak{A}$, it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \mathcal{L}$.
- For any interpretation $\mathcal{I}$ of $P_{\mathfrak{A}}$, it holds $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \mathcal{I}$.

Proof
Both results are immediate:

- Proving that for any labelling $\mathcal{L}$ of $\mathfrak{A}$, it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \mathcal{L}$:
  Let $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle T, F\rangle$.
  - $\mathcal{L}(a) = \texttt{in} \Rightarrow a \in T \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))(a) = \texttt{in}$;
  - $\mathcal{L}(a) = \texttt{out} \Rightarrow a \in F \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))(a) = \texttt{out}$;
  - $\mathcal{L}(a) = \texttt{undec} \Rightarrow a \in \overline{T \cup F} \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))(a) = \texttt{undec}$.
- Proving that for any interpretation $\mathcal{I}$ of $P_{\mathfrak{A}}$, it holds $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \mathcal{I}$.
  Let $\mathcal{I} = \langle T, F\rangle$ be an interpretation of $P_{\mathfrak{A}}$, and $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})) = \langle T', F'\rangle$. We will show $T = T'$ and $F = F'$:
  - $a \in T \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})(a) = \texttt{in} \Rightarrow a \in T'$;
  - $a \in F \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})(a) = \texttt{out} \Rightarrow a \in F'$;
  - $a \in \overline{T \cup F} \Rightarrow \mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{I})(a) = \texttt{undec} \Rightarrow a \in \overline{T' \cup F'}$;

□

Theorem 9
Let $\mathfrak{A}$ be a *SETAF* and $P_{\mathfrak{A}}$ be its associated *NLP*. It holds

- $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$.
- $\mathcal{M}$ is a partial stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of $\mathfrak{A}$.

Proof
1. Proving that if $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$, then $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$:
   Let $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle T, F\rangle$ and $\Omega_{P_{\mathfrak{A}}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})) = \langle T', F'\rangle$. It suffices to show $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a fixpoint of $\Omega_{P_{\mathfrak{A}}}$: $T = T'$ and $F = F'$. For any argument $a \in \mathcal{A} = HB_{P_{\mathfrak{A}}}$, there are three possibilities:
   - $a \in T$. Then $\mathcal{L}(a) = \texttt{in}$. From Definition 2, we know that for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{L}(b) = \texttt{out}$ for some $b \in \mathcal{B}$. It follows from Definition 12 that there exists $V \in \mathcal{V}_a$ such that $\mathcal{L}(b) = \texttt{out}$ for every $b \in V$. This means the fact $a \in \dfrac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$, i.e., $a \in T'$.
   - $a \in F$. Then $\mathcal{L}(a) = \texttt{out}$. From Definition 2, we know that there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{L}(b) = \texttt{in}$ for each $b \in \mathcal{B}$. It follows from Definition 12

that for each $V \in \mathcal{V}_a$, there exists $b \in V$ such that $\mathcal{L}(b) = \mathtt{in}$. This means that there exists no rule for a in $\frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$, i.e., $a \in F'$.

- $a \in \overline{T \cup F}$. Then $\mathcal{L}(a) = \mathtt{undec}$. From Definition 2, we know that (i) there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{L}(b) \neq \mathtt{out}$ for each $b \in \mathcal{B}$, and (ii) for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{L}(b) \neq \mathtt{in}$ for some $b \in \mathcal{B}$. It follows from Definition 12 that (i) there does not exist $V \in \mathcal{V}_a$ such that $\mathcal{L}(b) = \mathtt{out}$ for every $b \in V$, and (ii) there exists $V \in \mathcal{V}_a$ such that $\mathcal{L}(b) \neq \mathtt{in}$ for each $b \in V$. This means (i) the fact $a \notin \frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$, and (ii) there exists rule for a in $\frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$. Thus $body(r) = \mathbf{u}$ for any $r \in \frac{P_{\mathfrak{A}}}{\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})}$ such that $head(r) = a$, i.e., $a \in \overline{T' \cup F'}$.

2. Proving that if $\mathcal{M}$ is a partial stable model of $P_{\mathfrak{A}}$, then $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of $\mathfrak{A}$:
   Let $\mathcal{M} = \langle T, F \rangle$ be a partial stable model of $P_{\mathfrak{A}}$. Thus $\mathcal{M}$ is a fixpoint of $\Omega_{P_{\mathfrak{A}}}$, i.e., $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$. We now prove $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of $\mathfrak{A}$. For any $a \in HB_{P_{\mathfrak{A}}} = \mathcal{A}$, there are three possibilities:

   - $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(a) = \mathtt{in}$. Then $a \in T$. As $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$, the fact $a \in \frac{P_{\mathfrak{A}}}{\mathcal{M}}$. This means that there exists a rule $a \leftarrow \mathtt{not}\ b_1, \ldots, \mathtt{not}\ b_n \in P_{\mathfrak{A}}$ $(n \geq 0)$ such that $\{b_1, \ldots, b_n\} \subseteq F$. It follows from Definition 12 that for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) = \mathtt{out}$ for some $b \in \mathcal{B}$;
   - $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(a) = \mathtt{out}$. Then $a \in F$. As $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$, there exists no rule for a in $\frac{P_{\mathfrak{A}}}{\mathcal{M}}$. This means that for every rule $a \leftarrow \mathtt{not}\ b_1, \ldots, \mathtt{not}\ b_n \in P_{\mathfrak{A}}$ $(n \geq 0)$, there exists $b_i \in T$ $(1 \leq i \leq n)$. It follows from Definition 12 that there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) = \mathtt{in}$ for each $b \in \mathcal{B}$;
   - $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(a) = \mathtt{undec}$. Then $a \in \overline{T \cup F}$. As $\Omega_{P_{\mathfrak{A}}}(\mathcal{M}) = \mathcal{M}$, the fact $a \notin \frac{P_{\mathfrak{A}}}{\mathcal{M}}$, but there exists a rule r in $\frac{P_{\mathfrak{A}}}{\mathcal{M}}$ such that $head(r) = a$ and $body(r) = \mathbf{u}$. This means that (i) for each rule $a \leftarrow \mathtt{not}\ b_1, \ldots, \mathtt{not}\ b_n \in P_{\mathfrak{A}}$ $(n \geq 0)$, it holds $\{b_1, \ldots, b_n\} \nsubseteq F$, and (ii) there exists a rule $a \leftarrow \mathtt{not}\ b_1, \ldots, \mathtt{not}\ b_n \in P_{\mathfrak{A}}$ $(n \geq 0)$ such that $\{b_1, \ldots, b_n\} \cap T = \emptyset$. It follows from Definition 12 that (i) there exists $\mathcal{B} \in Att(a)$ such that $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) \neq \mathtt{out}$ for each $b \in \mathcal{B}$, and (ii) for each $\mathcal{B} \in Att(a)$, it holds $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})(b) \neq \mathtt{in}$ for some $b \in \mathcal{B}$.

   Hence, $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of $\mathfrak{A}$.
3. Proving that if $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$, then $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$:
   $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ $\Rightarrow$ according to item 2 above, $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}))$ is a complete labelling of $\mathfrak{A}$ $\Rightarrow$ (Theorem 8) $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$.
4. Proving that if $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a complete labelling of $\mathfrak{A}$, then $\mathcal{M}$ is a partial stable model of $P_{\mathfrak{A}}$:
   $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is complete labelling of $\mathfrak{A}$ $\Rightarrow$ according to item 1 above, $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M}))$ is a partial stable model of $P_{\mathfrak{A}}$ $\Rightarrow$ (Theorem 8) $\mathcal{M}$ is a partial stable model of $P_{\mathfrak{A}}$.

□

Theorem 10
Let $\mathfrak{A}$ be a *SETAF* and $P_{\mathfrak{A}}$ its associated *NLP*. It holds

1. $\mathcal{L}$ is a grounded labelling of $\mathfrak{A}$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a well-founded model of $P_{\mathfrak{A}}$.
2. $\mathcal{L}$ is a preferred labelling of $\mathfrak{A}$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a regular model of $P_{\mathfrak{A}}$.
3. $\mathcal{L}$ is a stable labelling of $\mathfrak{A}$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a stable model of $P_{\mathfrak{A}}$.
4. $\mathcal{L}$ is a semi-stable labelling of $\mathfrak{A}$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is an L-stable model of $P_{\mathfrak{A}}$.

Proof
Let $\mathcal{L}$ be an argument labelling of $\mathfrak{A}$. Recall that $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L}) = \langle \mathtt{in}(\mathcal{L}), \mathtt{out}(\mathcal{L})\rangle$. The proof is straightforward:

1. $\mathcal{L}$ is a grounded labelling of $\mathfrak{A}$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$ and $\mathtt{in}(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of $\mathfrak{A}$ iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ and there is no partial stable model $\mathcal{M}' = \langle T', F'\rangle$ of $P_{\mathfrak{A}}$ such that $T' \subset \mathtt{in}(\mathcal{L})$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a well-founded model of $P_{\mathfrak{A}}$;
2. $\mathcal{L}$ is a preferred labelling of $\mathfrak{A}$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$ and $\mathtt{in}(\mathcal{L})$ is maximal (w.r.t. set inclusion) among all complete labellings of $\mathfrak{A}$ iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ and there is no partial stable model $\mathcal{M}' = \langle T', F'\rangle$ of P such that $\mathtt{in}(\mathcal{L}) \subset T'$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a regular model of $P_{\mathfrak{A}}$;
3. $\mathcal{L}$ is a stable labelling of $\mathfrak{A}$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$ such that $\mathtt{undec}(\mathcal{L}) = \emptyset$ iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ such that $\overline{\mathtt{in}(\mathcal{L}) \cup \mathtt{out}(\mathcal{L})} = \emptyset$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a stable model of $P_{\mathfrak{A}}$;
4. $\mathcal{L}$ is a semi-stable labelling of $\mathfrak{A}$ iff $\mathcal{L}$ is a complete labelling of $\mathfrak{A}$ and $\mathtt{undec}(\mathcal{L})$ is minimal (w.r.t. set inclusion) among all complete labellings of $\mathfrak{A}$ iff (Theorem 9) $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is a partial stable model of $P_{\mathfrak{A}}$ and there is no partial stable model $\mathcal{M}' = \langle T', F'\rangle$ of $P_{\mathfrak{A}}$ such that $\overline{T' \cup F'} \subset \overline{\mathtt{in}(\mathcal{L}) \cup \mathtt{out}(\mathcal{L})}$ iff $\mathcal{L}2\mathcal{I}_{\mathfrak{A}}(\mathcal{L})$ is an L-stable model of $P_{\mathfrak{A}}$.

□

Corollary 11
Let $\mathfrak{A}$ be a *SETAF* and $P_{\mathfrak{A}}$ its associated *NLP*. It holds

1. $\mathcal{M}$ is a well-founded model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a grounded labelling of $\mathfrak{A}$.
2. $\mathcal{M}$ is a regular model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a preferred labelling of $\mathfrak{A}$.
3. $\mathcal{M}$ is a stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a stable labelling of $\mathfrak{A}$.
4. $\mathcal{M}$ is an L-stable model of $P_{\mathfrak{A}}$ iff $\mathcal{I}2\mathcal{L}_{\mathfrak{A}}(\mathcal{M})$ is a semi-stable labelling of $\mathfrak{A}$.

Proof
These results come from Theorems 8 and 10. □

A.3 Theorems and Proofs from Section 5

Proposition 12
Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a *SETAF* and $P_{\mathfrak{A}}$ its associated *NLP*. It holds $P_{\mathfrak{A}}$ is an *RFALP*.

Proof

It follows that

1. Each rule in $P_{\mathfrak{A}}$ has the form $a \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$;
2. for each rule $a \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P_{\mathfrak{A}}$, if $b \in \{b_1, \ldots, b_n\}$, there exists $(\mathcal{B}, a) \in Att$ such that $b \in \mathcal{B}$, i.e,. $b \in \mathcal{A}_P$. Then there exists a rule $r \in P_{\mathfrak{A}}$ such that $b = head(r)$. This suffices to guarantee $HB_{P_{\mathfrak{A}}} = \{head(r) \mid r \in P_{\mathfrak{A}}\}$;
3. A rule $a \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P_{\mathfrak{A}}$ iff there exists a minimal set (w.r.t. set inclusion) $V = \{b_1, \ldots, b_n\}$ such that for each $\mathcal{B} \in Att(a)$, there exists $b \in \mathcal{B} \cap V$. This means there exists no rule $a \leftarrow \texttt{not}\ c_1, \ldots, \texttt{not}\ c_{n'} \in P_{\mathfrak{A}}$ such that $\{c_1, \ldots, c_{n'}\} \subset \{b_1, \ldots, b_n\}$.

Hence, $P_{\mathfrak{A}}$ is an *RFALP*. □

Lemma 34
Let P be an *RFALP*, $Head_P = \{head(r) \mid r \in P\}$ and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ its corresponding *SETAF*. It holds $Head_P = \mathcal{A}_P$.

Proof
The result is straightforward: $c \in Head_P$ iff there exists a rule $c \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P$ $(n \geq 0)$ iff $c \in \mathcal{A}_P$ (Definition 8). □

Theorem 13
Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a *SETAF*, $P_{\mathfrak{A}}$ its associated *NLP* and $\mathfrak{A}_{P_{\mathfrak{A}}}$ the associated *SETAF* of $P_{\mathfrak{A}}$. It holds that $\mathfrak{A} = \mathfrak{A}_{P_{\mathfrak{A}}}$.

Proof
Let $\mathfrak{A} = (\mathcal{A}, Att)$ be a *SETAF* with $\mathcal{A} = \{a_1, \ldots, a_n\}$ and for each $a_i \in \mathcal{A}$, we define $R_i = \{r \in P_{\mathfrak{A}} \mid head(r) = a_i\}$, i.e., $P_{\mathfrak{A}} = R_1 \cup R_2 \cup \cdots \cup R_n$. It follows from Proposition 12 and Lemma 34 that $\mathfrak{A}_{P_{\mathfrak{A}}} = (\mathcal{A}_{P_{\mathfrak{A}}}, Att_{P_{\mathfrak{A}}})$ with $\mathcal{A}_{P_{\mathfrak{A}}} = \{a_1, \ldots, a_n\} = \mathcal{A}$. It remains to prove that $Att = Att_{P_{\mathfrak{A}}}$:

$(\mathcal{B}, a_j) \in Att$ iff $(\mathcal{B}, a_j) \in Att$ and there exists no $\mathcal{B}' \subset \mathcal{B}$ such that $(\mathcal{B}', a_j) \in Att$ iff $\mathcal{B}$ is a minimal set (w. r. t. set inclusion) in which for each rule $r \in R_j$, there exists $b \in \mathcal{B}$ such that $\texttt{not}\ b \in body^-(r)$ iff $\mathcal{B}$ is a minimal set (w. r. t. set inclusion) in which for each $V \in \texttt{Vul}(a_j)$, there exists $b \in \mathcal{B} \cap V$ iff $(\mathcal{B}, a_j) \in Att_{P_{\mathfrak{A}}}$. □

Lemma 35
Let P be an *RFALP*, $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ the corresponding *SETAF* and $c \in \mathcal{A}_P$. If $\{a_1, \ldots, a_n\}$ is a minimal set such that for each $\mathcal{B} \in Att_P(c)$, there exists $a_i \in \mathcal{B}$ $(1 \leq i \leq n)$, then $c \leftarrow \texttt{not}\ a_1, \ldots, \texttt{not}\ a_n \in P$.

Proof
As for each $\mathcal{B} \in Att_P(c)$, there exists $a_i \in \mathcal{B}$ $(1 \leq i \leq n)$, it follows from Definition 9 that there exists $V \in \texttt{Vul}(c)$ such that $V \subseteq \{a_1, \ldots, a_n\}$. Note that for each $\mathcal{B} \in Att_P(c)$, there exists $b \in V \cap \mathcal{B}$. As $\{a_1, \ldots, a_n\}$ is a minimal set with this property, it holds $V = \{a_1, \ldots, a_n\}$. Then (Definition 8) $c \leftarrow \texttt{not}\ a_1, \ldots, \texttt{not}\ a_n \in P$. □

Theorem 14

Let P be an *RFALP*, $\mathfrak{A}_P$ its associated *SETAF* and $P_{\mathfrak{A}_P}$ the associated *NLP* of $\mathfrak{A}_P$. It holds that $P = P_{\mathfrak{A}_P}$.

Proof

Let P be an *RFALP* with $HB_P = \{a_1, \ldots, a_n\}$, and $\mathfrak{A}_P = (\mathcal{A}_P, Att_P)$ the corresponding *SETAF*. For each $a_i \in HB_P$ $(1 \leq i \leq n)$, we define $R_i = \{r \in P_{\mathfrak{A}} \mid head(r) = a_i\}$. It follows that $\mathcal{A}_P = \{a_1, \ldots, a_n\}$. Hence, $HB_{P_{\mathfrak{A}_P}} = \{a_1, \ldots, a_n\}$. We will prove $P = P_{\mathfrak{A}_P}$:

- If $a_i \leftarrow \texttt{not}\ a_{i_1}, \ldots, \texttt{not}\ a_{i_m} \in P$. then $a_i \in \mathcal{A}_P$ and $\{a_{i_1}, \ldots, a_{i_m}\}$ is a minimal set (w.r.t. set inclusion) such that for each $\mathcal{B} \in Att_P(a_i)$, there exists $a_{i_k} \in \mathcal{B}$ $(k \in \{1, \ldots, m\})$. This implies (Definition 12) $a_i \leftarrow \texttt{not}\ a_{i_1}, \ldots, \texttt{not}\ a_{i_m} \in P_{\mathfrak{A}_P}$.
- If $a_i \leftarrow \texttt{not}\ a_{i_1}, \ldots, \texttt{not}\ a_{i_m} \in P_{\mathfrak{A}_P}$, then (Definition 12) $\{a_{i_1}, \ldots, a_{i_m}\}$ is a minimal set (w.r.t. set inclusion) such that for each $\mathcal{B} \in Att_P(a_i)$, there exists $a_{i_k} \in \mathcal{B}$ $(k \in \{1, \ldots, m\})$. Thus (Lemma 35) $a_i \leftarrow \texttt{not}\ a_{i_1}, \ldots, \texttt{not}\ a_{i_m} \in P$.

□

A.4 Theorems and Proofs from Section 6

Theorem 15

The relation $\mapsto_{UTPM}$ is strongly terminating for fair sequences of program transformations, i.e., such fair sequences always lead to irreducible programs.

Proof

Let $P_1 \mapsto_{UTPM} P_2 \mapsto_{UTPM} \cdots \mapsto_{UTPM} P_k \mapsto_{UTPM} \cdots \mapsto_{UTPM} P_{k'} \mapsto_{UTPM} \cdots$ be a fair sequence of $\mapsto_{UTPM}$. This fairness condition implies that for every atom a, there exists a natural number k such that for each *NLP* P_i with $i > k$ in the sequence of $\mapsto_{UTPM}$ above, it holds $a \notin body^+(r)$ for each $r \in P_i$. As each *NLP* is a finite set of rules, from some natural number k' on, $body^+(r) = \emptyset$ for any $r \in P_{k'}$. Then for each $k'' \geq k'$, $\mapsto_U$ and $\mapsto_T$ cannot be applied in $P_{k''}$. It remains the program transformations $\mapsto_P$ and $\mapsto_M$. For each of these $P_{k''}$, there are two possibilities:

- $\mapsto_M$ strictly decreases the number of rules of $P_{k''}$ or
- $\mapsto_P$ strictly decreases the number of negative literals in $body^-(r)$ for some $r \in P_{k''}$.

It follows that the successive application of $\mapsto_M$ or $\mapsto_P$ in these $P_{k''}$s will eventually lead to an irreducible *NLP*. □

Theorem 16

For any *NLP* P, there exists an irreducible *NLP* P^* such that $P \mapsto^*_{UTPM} P^*$.

Proof

A simple method to obtain a fair sequence of program transformations with respect to $\mapsto_{UTPM}$ is to apply $\mapsto_U$ to a rule r only if $\mapsto_T$ is not applicable to r and to ensure that whenever $\mapsto_U$ has been applied to get rid of an occurrence of an atom a, then all such occurrences of a (in other rules of the same program) have also been removed before applying $\mapsto_U$ to another occurrence of an atom $b \neq a$.

As for any *NLP* P, it is always possible to build such a fair sequence of program transformations with respect to $\mapsto_{UTPM}$, we obtain from Theorem 15 that there exists an irreducible *NLP* P^* such that $P \mapsto^*_{UTPM} P^*$. □

Theorem 17
Let P be an *NLP* and P^* be an *NLP* obtained after applying repeatedly the program transformation $\mapsto_{UTPM}$ until no further transformation is possible, i.e., $P \mapsto^*_{UTPM} P^*$ and P^* is irreducible. Then P^* is an *RFALP*.

Proof
To prove it by contradiction, suppose P^* is not an *RFALP*. There are three possibilities:

- A rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P^*$ with $m \geq 1$ and $n \geq 0$. Then
  - The program transformation $\mapsto_U$ (unfolding) can be applied.
  - If $c \in \{a_1, \ldots, a_m\}$, the program transformation $\mapsto_T$ (elimination of tautologies) can be applied.
- A rule $c \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P^*$, but there exists $b \in \{b_1, \ldots, b_n\}$ such that $b \notin \{head(r) \mid r \in P^*\}$. Then the program transformation $\mapsto_P$ (positive reduction) can be applied.
- A rule $c \leftarrow \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P^*$ and there is a rule $c \leftarrow \texttt{not}\ c_1, \ldots, \texttt{not}\ c_p \in P^*$ such that $\{c_1, \ldots, c_p\} \subset \{b_1, \ldots, b_n\}$. Then the program transformation $\mapsto_M$ (elimination of non-minimal rules) can be applied.

It is absurd as in each case, there is still a program transformation to be applied. □

Theorem 18
Let P be an *RFALP*. Then P is irreducible with respect to $\mapsto_{UTPM}$.

Proof
Let P be an *RFALP*. It holds

- The program transformations $\mapsto_U$ and $\mapsto_T$ cannot be applied as they require a rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$ in P with $m \geq 1$.
- The program transformation $\mapsto_P$ cannot be applied as it requires a rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n$ in P such that $b \notin \{head(r) \mid r \in P\}$, but $\{head(r) \mid r \in P\} = HB_P$.
- The program transformation $\mapsto_M$ cannot be applied as it requires two distinct rules r and r' in P such that $head(r) = head(r')$ and $body^-(r') \subset body^-(r)$.

□

Theorem 21
Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_T P_2$. It holds $\mathcal{M}$ is a partial stable model of P_1 iff $\mathcal{M}$ is a partial stable model of P_2.

Proof

Let $P_2 = P_1 - \{r\}$ and $head(r) \in body^+(r)$. We have to show for any interpretation $\mathcal{M} = \langle T, F\rangle$, it holds $\mathcal{M}$ is a partial stable model of P_1 iff $\mathcal{M}$ is a partial stable model of P_2; we distinguish two cases:

- $\{a \mid \texttt{not}\ a \in body^-(r)\} \cap T \neq \emptyset$: Then $\frac{P_1}{\mathcal{M}} = \frac{P_2}{\mathcal{M}}$. This trivially implies that $\mathcal{M}$ is a partial stable model of P_1 iff it is a partial stable model of P_2.
- $\{a \mid \texttt{not}\ a \in body^-(r)\} \cap T = \emptyset$: Then it is clear $\frac{P_1}{\mathcal{M}} \mapsto_T \frac{P_2}{\mathcal{M}}$. As both $\frac{P_1}{\mathcal{M}}$ and $\frac{P_2}{\mathcal{M}}$ are positive programs, according to Lemma 19, it holds $\mathcal{M}$ is the least model of $\frac{P_1}{\mathcal{M}}$ iff $\mathcal{M}$ is the least model of $\frac{P_2}{\mathcal{M}}$. Hence, $\mathcal{M}$ is a partial stable model of P_1 iff it is a partial stable model of P_2.

□

Theorem 22

Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_P P_2$. It holds $\mathcal{M}$ is a partial stable model of P_1 iff $\mathcal{M}$ is a partial stable model of P_2.

Proof

Let

$$P_2 = P_1 - \{c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n\} \\ \cup \{c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n\}$$

such that r is the rule $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P_1$ and $b \notin \{head(r') \mid r' \in P_1\}$. We have to show that for any interpretation $\mathcal{M} = \langle T, F\rangle$, it holds $\mathcal{M}$ is a partial stable model of P_1 iff $\mathcal{M}$ is a partial stable model of P_2; we distinguish two cases:

- $(\{a \mid \texttt{not}\ a \in body^-(r)\} - \{b\}) \cap T \neq \emptyset$ or $b \in F$: Then $\frac{P_1}{\mathcal{M}} = \frac{P_2}{\mathcal{M}}$. This trivially implies that $\mathcal{M}$ is a partial stable model of P_1 iff it is a partial stable model of P_2.
- $(\{a \mid \texttt{not}\ a \in body^-(r)\} - \{b\}) \cap T = \emptyset$ and $b \notin F$. Let $\langle T_1, F_1\rangle$ and $\langle T_2, F_2\rangle$ be respectively the least models of $\frac{P_1}{\mathcal{M}}$ and $\frac{P_2}{\mathcal{M}}$. As $b \notin \{head(r') \mid r' \in P_1\}$, it is clear that $b \in F_1$ and $b \in F_2$. Given that $b \notin F$, we obtain $\mathcal{M} = \langle T, F\rangle$ is different from both $\langle T_1, F_1\rangle$ and $\langle T_2, F_2\rangle$. Hence, $\mathcal{M}$ is neither a partial stable model of P_1 nor of P_2. This implies that $\mathcal{M}$ is a partial stable model of P_1 iff it is a partial stable model of P_2.

□

Theorem 23

Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_M P_2$. It holds $\mathcal{M}$ is a partial stable model of P_1 iff $\mathcal{M}$ is a partial stable model of P_2.

Proof

Suppose that there are two distinct rules r and r' in P_1 such that $head(r) = head(r')$, $body^+(r') \subseteq body^+(r)$, $body^-(r') \subseteq body^-(r)$ and $P_2 = P_1 - \{r\}$. We have to show that for any interpretation $\mathcal{M} = \langle T, F\rangle$, it holds that $\mathcal{M}$ is a partial stable model of P_1 iff $\mathcal{M}$ is a partial stable model of P_2; we distinguish two cases:

- $\{a \mid \texttt{not}\ a \in body^{-}(r)\} \cap T \neq \emptyset$ or $(\{a \mid \texttt{not}\ a \in body^{-}(r)\} \cap T = \emptyset$ and $body^{+}(r) = body^{+}(r'))$: Then $\frac{P_1}{\mathcal{M}} = \frac{P_2}{\mathcal{M}}$. This trivially implies that $\mathcal{M}$ is a partial stable model of P_1 iff it is a partial stable model of P_2.
- $\{a \mid \texttt{not}\ a \in body^{-}(r)\} \cap T = \emptyset$ and $body^{+}(r') \subset body^{+}(r)$: Then it is clear that $\frac{P_1}{\mathcal{M}} \mapsto_M \frac{P_2}{\mathcal{M}}$. As both $\frac{P_1}{\mathcal{M}}$ and $\frac{P_2}{\mathcal{M}}$ are positive programs, according to Lemma 19, it holds that $\mathcal{M}$ is the least model of $\frac{P_1}{\mathcal{M}}$ iff $\mathcal{M}$ is least model $\frac{P_2}{\mathcal{M}}$. Hence, $\mathcal{M}$ is a partial stable model of P_1 iff it is a partial stable model of P_2.

□

Theorem 24
Let P be an *NLP* and P^* be an irreducible *NLP* such that $P \mapsto^*_{UTPM} P^*$. It holds $\mathcal{M}$ is a partial stable model of P iff $\mathcal{M}$ is a partial stable model of P^*.

Proof
If $P \mapsto^*_{UTPM} P^*$, then there exists a finite sequence of program transformations $P = P_1 \mapsto_{UTPM} \cdots \mapsto_{UTPM} P_n = P^*$. According to Theorems 20, 21, 22 and 23, $\mathcal{M}$ is a partial stable model of P_i iff $\mathcal{M}$ is a partial stable model of P_{i+1} with $1 \leq i < n$. Thus by transitivity, $\mathcal{M}$ is a partial stable model of P iff $\mathcal{M}$ is a partial stable model of P^*. □

Corollary 25
Let P be an *NLP* and P^* be an irreducible *NLP* such that $P \mapsto^*_{UTPM} P^*$. It holds $\mathcal{M}$ is a well-founded, regular, stable, L-stable model of P iff $\mathcal{M}$ is respectively a well-founded, regular, stable, L-stable model of P^*.

Proof
As P and P^* share the same set of partial stable models (Theorem 24), the result is straightforward. □

Corollary 26
For any *NLP* P, there exists an *RFALP* P^* such that $\mathcal{M}$ is a partial stable, well-founded, regular, stable, L-stable model of P iff $\mathcal{M}$ is respectively a partial stable, well-founded, regular, stable, L-stable model of P^*.

Proof
From Theorem 16, we know that for any *NLP* P, there exists an irreducible *NLP* P^* such that $P \mapsto^*_{UTPM} P^*$. From Theorem 17, we obtain P^* is an *RFALP*. Besides, from Theorem 24 and Corollary 25, we infer $\mathcal{M}$ is a partial stable, well-founded, regular, stable, L-stable model of P iff $\mathcal{M}$ is respectively a partial stable, well-founded, regular, stable, L-stable model of P^*. □

Theorem 27
*NLP*s and *RFALP*s have the same expressiveness for partial stable, well-founded, regular, stable, and L-stable semantics.

Proof
We have

- For any *NLP* P, there exists an *RFALP* P^* such that $\mathcal{M}$ is a partial stable, well-founded, regular, stable, L-stable model of P iff $\mathcal{M}$ is respectively a partial stable, well-founded, regular, stable, L-stable model of P^* (Corollary 26).
- Obviously, any *RFALP* is an *NLP*.

Hence, *NLP*s and *RFALP*s have the same expressiveness for partial stable, well-founded, regular, stable and L-stable semantics. □

Lemma 36
Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_U P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof
Let P_1 and P_2 be *NLP*s such that

$$\begin{aligned} P_2 = & P_1 - \{c \leftarrow a, a_1, \ldots, a_m, \texttt{not } b_1, \ldots, \texttt{not } b_n\} \\ & \cup \{c \leftarrow a'_1, \ldots, a'_p, a_1, \ldots, a_m, \texttt{not } b'_1, \ldots, \texttt{not } b'_q, \texttt{not } b_1, \ldots, \texttt{not } b_n \mid \\ & \quad a \leftarrow a'_1, \ldots, a'_p, \texttt{not } b'_1, \ldots, \texttt{not } b'_q \in P_1\}, \end{aligned}$$

$\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$ and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

- For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that $\texttt{Conc}(s) = \texttt{Conc}(s')$, and $\texttt{Vul}(s) = \texttt{Vul}(s')$.
- For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that $\texttt{Conc}(s') = \texttt{Conc}(s)$, and $\texttt{Vul}(s') = \texttt{Vul}(s)$.

Hence, $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$, and $Att_{P_1} = Att_{P_2}$. □

Lemma 37
Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_T P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof
Let $P_2 = P_1 - \{r\}$, where there exists a rule $r \in P_1$ such that $head(r) \in body^+(r)$. In addition, let $\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$ and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

- For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that $\texttt{Conc}(s) = \texttt{Conc}(s')$, and for each $V \in \texttt{Vul}(s)$, there exists $V' \in \texttt{Vul}(s) \cap \texttt{Vul}(s')$ such that $V' \subseteq V$.
- For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that $\texttt{Conc}(s') = \texttt{Conc}(s)$, and $\texttt{Vul}(s') = \texttt{Vul}(s)$.

Hence, $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$, and for each $c \in \mathcal{A}_{P_1}$, V is a minimal set (w.r.t. set inclusion) in $\texttt{Vul}_{P_1}(c)$ iff V is a minimal set (w.r.t. set inclusion) in $\texttt{Vul}_{P_2}(c)$; it holds that $Att_{P_1} = Att_{P_2}$. □

Lemma 38
Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_P P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof

Let $c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n \in P_1$ be a rule such that $b \notin \{head(r) \mid r \in P_1\}$,

$$P_2 = (P_1 - \{c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n\}) \\ \cup \{c \leftarrow a_1, \ldots, a_m, \texttt{not}\ b_1, \ldots, \texttt{not}\ b_n\},$$

$\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$ and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

- For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that $\texttt{Conc}(s) = \texttt{Conc}(s')$, and $\texttt{Vul}(s) = \{V \mid \exists V' \in \texttt{Vul}(s')\ \textit{such that}\ V = V'\ \textit{or}\ V = V' \cup \{b\}\}$.
- For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that $\texttt{Conc}(s') = \texttt{Conc}(s)$, and $\texttt{Vul}(s') = \{V' \mid \exists V \in \texttt{Vul}(s)\ \textit{such that}\ V' = V\ \textit{or}\ V' = V - \{b\}\}$.

Hence, $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$, and as $b \notin \mathcal{A}_{P_1} \cup \mathcal{A}_{P_2}$, it holds that $Att_{P_1} = Att_{P_2}$. □

Lemma 39
Let P_1 and P_2 be *NLP*s such that $P_1 \mapsto_M P_2$. It holds that $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$.

Proof
Let $P_2 = P_1 - \{r\}$, where there are two distinct rules r and r' in P_1 such that $head(r) = head(r')$, $body^+(r') \subseteq body^+(r)$, $body^-(r') \subseteq body^-(r)$. In addition, let $\mathfrak{A}_{P_1} = (\mathcal{A}_{P_1}, Att_{P_1})$ and $\mathfrak{A}_{P_2} = (\mathcal{A}_{P_2}, Att_{P_2})$. Note that

- For each statement $s \in \mathfrak{S}_{P_1}$, there exists $s' \in \mathfrak{S}_{P_2}$ such that $\texttt{Conc}(s) = \texttt{Conc}(s')$, and for each $V \in \texttt{Vul}(s)$, there exists $V' \in \texttt{Vul}(s) \cap \texttt{Vul}(s')$ such that $V' \subseteq V$.
- For each statement $s' \in \mathfrak{S}_{P_2}$, there exists $s \in \mathfrak{S}_{P_1}$ such that $\texttt{Conc}(s') = \texttt{Conc}(s)$, and $\texttt{Vul}(s') = \texttt{Vul}(s)$.

Hence, $\mathcal{A}_{P_1} = \mathcal{A}_{P_2}$, and for each $c \in \mathcal{A}_{P_1}$, V is a minimal set (w.r.t. set inclusion) in $\texttt{Vul}_{P_1}(c)$ iff V is a minimal set (w.r.t. set inclusion) in $\texttt{Vul}_{P_2}(c)$; it holds that $Att_{P_1} = Att_{P_2}$. □

Theorem 28
For any *NLP*s P_1 and P_2, if $P_1 \mapsto_{UTPM} P_2$, then $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_2}$

Proof
It follows straightforwardly from Lemmas 36, 37, 38 and 39. □

Theorem 29
The relation $\mapsto_{UTPM}$ is confluent, i.e., for any *NLP*s P, P' and P'', if $P \mapsto^*_{UTPM} P'$ and $P \mapsto^*_{UTPM} P''$ and both P' and P'' are irreducible, then $P' = P''$.

Proof
From Theorem 28, we know that $\mathfrak{A}_P = \mathfrak{A}_{P'} = \mathfrak{A}_{P''}$. Thus $P_{\mathfrak{A}_{P'}} = P_{\mathfrak{A}_{P''}}$. As P' and P'' are *RFALP*s (Theorem 17), it holds (Theorem 14) that $P' = P_{\mathfrak{A}_{P'}} = P_{\mathfrak{A}_{P''}} = P''$. □