Supplement to the Paper:

A Weak Convergence Approach to Inventory Control Using a Long-term Average Criterion

Feller's Branching Diffusion Inventory Model

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September 22, 2018

In the absence of ordering, let the inventory dynamics be described Feller's branching diffusion given by the following stochastic differential equation

$$dX_0(t) = -\tfrac{1}{2}X_0(t)\,dt + \sqrt{X_0(t)}\,dW(t),\ \ X_0(0) = x_0 \in \mathcal{I} := (0,\infty). \tag{1}$$

Suppose without loss of generality that $x_0 = 1$ in this note. The scale and speed densities of (1) are given by $s(x) = e^{x-1}$ and $m(x) = x^{-1}e^{-x+1}$, respectively. Consequently it is straightforward to verify that 0 is an attracting point and ∞ is a non-attracting point. In fact, one can show that 0 is an absorbing point, see, for instance, Theorem 13.1 of Klebaner (2005); and ∞ is a natural point. Moreover, for any $y \in \mathcal{I}$, we have $M[y,\infty) = \int_y^\infty \frac{1}{z}e^{1-z}dz \le \frac{e}{y}\int_y^\infty e^{-z}\,dz < \infty$.This verifies Condition 2.1 of Helmes et al. (2018).

Suppose the holding and ordering costs functions are given by

$$c_0(x) = x^{\gamma_1} + x^{\gamma_2},\ \ x \in \mathcal{I}, \qquad c_1(y,z) := k_1 + \widehat{c}(y,z),\ (y,z) \in \mathcal{R}, \tag{2}$$

in which $\gamma_1 > 0$, $\gamma_2 < 0$, $k_1 > 0$, and $\hat{c} : \mathcal{R} \mapsto \mathbb{R}_+$ is a nonnegative and continuous function. It is immediate to show that for any $y \in \mathcal{I}$, $\int_y^\infty c_0(v)\,dM(v) < \infty$; establishing Condition 2.2 of Helmes et al. (2018). Moreover, Condition 2.3 of Helmes et al. (2018) is trivially satisfied. Therefore, by Theorem 2.1 of Helmes et al. (2018), there exists a pair $(y_0^*, z_0^*) \in \mathcal{R}$ such that $F_0(y_0^*, z_0^*) = F_0^* = \inf\{F_0(y,z) : (y,z) \in \overline{\mathcal{R}}\}$.

Theorem 1. *If $\gamma_2 \le -2$, then there exists an optimal (y_0^*, z_0^*) ordering policy in the class $\mathcal{A}$ for the Branching diffusion model (1) having nonlinear cost structure given by (2).*

Proof. The assertion follows from Theorem 5.1 directly if we can verify Condition 5.1 of Helmes et al. (2018) holds. Using the definitions of ζ, g_0 and G_0 in Helmes et al. (2018), we have

$$G_0(x) = 2\int_1^x \int_u^\infty \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dv \, du, \qquad x \in \mathcal{I}.$$

Let us first study the asymptotic behaviors of $\frac{c_0(x)}{(1+|G_0(x)|)^2}$ and $\frac{(\sigma(x)G_0'(x))^2}{(1+|G_0(x)|)(1+c_0(x))}$ when $x \to \infty$. Since $\lim_{x\to\infty} \int_x^\infty (v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1})e^{-v} \, dv = 0$, we can use L'Hospital's Rule to compute

$$\lim_{x\to\infty} \frac{\int_x^\infty (v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1})e^{-v} \, dv}{x^{\gamma_1-1}e^{-x}} = \lim_{x\to\infty} \frac{-(x^{\gamma_1-1} + x^{\gamma_2-1} - F_0^* x^{-1})e^{-x}}{(\gamma_1-1)x^{\gamma_1-2}e^{-x} - x^{\gamma_1-1}e^{-x}} = 1.$$

Hence there exists some $\delta > 1$ such that

$$\tfrac{1}{2}x^{\gamma_1-1}e^{-x} \le \int_x^\infty (v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1})e^{-v} \, dv \le \tfrac{3}{2}x^{\gamma_1-1}e^{-x}, \text{ for all } x \ge \delta.$$

On the other hand, the integral $2\int_1^\delta \int_u^\infty \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dvdu$ is uniformly bounded by a constant $K = K(\delta, \gamma_1, \gamma_2, F_0^*)$. In the rest of the proof, we shall denote by K a generic positive constant whose exact value may be different from line to line. Thus it follows that for $x \ge \delta$, we have

$$\begin{aligned} |G_0(x)| &\ge 2\int_\delta^x \int_u^\infty \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dv \, du - K \\ &\ge 2\int_\delta^x \tfrac{1}{2}u^{\gamma_1-1}e^{-u}e^u \, du - K = \frac{x^{\gamma_1} - \delta^{\gamma_1}}{\gamma_1} - K. \end{aligned}$$

Likewise, for all $x \ge \delta$, we have

$$G_0'(x) = 2\int_x^\infty \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^x \, dv \le 2 \cdot \tfrac{3}{2}x^{\gamma_1-1}e^{-x}e^x = 3x^{\gamma_1-1}.$$

Hence it follows that

$$\frac{c_0(x)}{(1+|G_0(x)|)^2} \le \frac{x^{\gamma_1} + x^{\gamma_2}}{\left(1 + |\frac{x^{\gamma_1}-\delta^{\gamma_1}}{\gamma_1} - K|\right)^2} \to 0, \qquad \text{as } x \to \infty, \tag{3}$$

$$\frac{(\sigma(x)G_0'(x))^2}{(1+|G_0(x)|)(1+c_0(x))} \le \frac{(x^{\frac{1}{2}}3x^{\gamma_1-1})^2}{\left(1 + |\frac{x^{\gamma_1}-\delta^{\gamma_1}}{\gamma_1} - K|\right)(1 + x^{\gamma_1} + x^{\gamma_2})} \to 0, \qquad \text{as } x \to \infty. \tag{4}$$

Next we consider the asymptotic behaviors of $\frac{c_0(x)}{(1+|G_0(x)|)^2}$ and $\frac{(\sigma(x)G_0'(x))^2}{(1+|G_0(x)|)^3}$ when $x \downarrow 0$. When $0 < x \ll 1$, we can write

$$\begin{aligned} G_0(x) = &- 2\int_x^\kappa \int_u^\kappa \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dv \, du \\ &- 2\int_x^\kappa \int_\kappa^1 \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dv \, du \\ &- 2\int_\kappa^1 \int_u^1 \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dv \, du \\ &+ 2\int_1^x \int_1^\infty \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}e^u \, dv \, du, \end{aligned} \tag{5}$$

where $\kappa = \kappa(F_0^*, \gamma_2) \in (0,1)$ is chosen so that $F_0^* < \frac{1}{2}v^{\gamma_2}$ (and hence $v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1} > \frac{1}{2}v^{\gamma_2-1} > 0$) for all $v \in (0, \kappa]$. It is easy to see that the second, third, and fourth integrals of (5) are uniformly bounded. Thus we have

$$\begin{aligned} |G_0(x)| &\geq 2\int_x^\kappa \int_u^\kappa \tfrac{1}{2}v^{\gamma_2-1}e^{-v}e^u\,dv\,du - K \geq e^{-\kappa}e^x \int_x^\kappa \tfrac{1}{\gamma_2}(\kappa^{\gamma_2} - u^{\gamma_2})\,du - K \\ &= e^{-\kappa}e^x\,\tfrac{1}{\gamma_2}\left(\kappa^{\gamma_2}(\kappa - x) - \frac{\kappa^{\gamma_2+1} - x^{\gamma_2+1}}{\gamma_2+1}\right) - K \geq \frac{e^{-\kappa}x^{\gamma_2+1}}{\gamma_2(\gamma_2+1)} - K. \end{aligned} \tag{6}$$

Then it follows that $\lim_{x\downarrow 0} G_0(x) = -\infty$ and for all $0 < x \ll 1$,

$$\frac{c_0(x)}{(1+|G_0(x)|)^2} \leq \frac{x^{\gamma_1} + x^{\gamma_2}}{\left(1 + |\frac{e^{-\kappa}x^{\gamma_2+1}}{\gamma_2(\gamma_2+1)} - K|\right)^2} \leq \frac{x^{\gamma_1} + x^{\gamma_2}}{Kx^{2\gamma_2+2}} \leq K. \tag{7}$$

Next using the $\kappa \in (0,1)$ chosen before, we write

$$\begin{aligned} \sigma(x)G_0'(x) = 2x^{\frac{1}{2}}e^x &\int_x^\kappa \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}\,dv \\ &+ 2x^{\frac{1}{2}}e^x \int_\kappa^\infty \left(v^{\gamma_1-1} + v^{\gamma_2-1} - F_0^* v^{-1}\right)e^{-v}\,dv. \end{aligned}$$

Observe that the second term above is uniformly bounded for all $x \in (0,1)$. Since $\gamma_2 \leq -2$, $\gamma_1 > 0$ and $F_0^* > 0$, we have

$$|\sigma(x)G_0'(x)| \leq 2e^1 x^{\frac{1}{2}} \int_x^\kappa 2v^{\gamma_2-1}\,dv + K = 4ex^{\frac{1}{2}}\frac{\kappa^{\gamma_2} - x^{\gamma_2}}{\gamma_2} + K \leq \tfrac{4e}{-\gamma_2}x^{\gamma_2+\frac{1}{2}} + K.$$

This, together with (6), implies that for $0 < x \ll 1$,

$$\frac{(\sigma(x)G_0'(x))^2}{(1+|G_0(x)|)^3} \leq \frac{\left(\frac{4e}{-\gamma_2}x^{\gamma_2+\frac{1}{2}} + K\right)^2}{\left(\frac{e^{-\kappa}x^{\gamma_2+1}}{\gamma_2(\gamma_2+1)} - K\right)^3} \leq K\frac{x^{2\gamma_2+1}}{x^{3\gamma_2+3}} = Kx^{-\gamma_2-2} \leq K. \tag{8}$$

Equations (3), (4), (7), and (8) establish Condition 5.1 of Helmes et al. (2018). The proof is therefore complete. □

Remark 2. For $\gamma_2 \in (-2, 0)$, similar computations show that $\lim_{x\to 0} \frac{c_0(x)}{(1+|G_0(x)|)^2} = \infty$. Thus Condition Condition 5.1 (a) of Helmes et al. (2018) fails. Observe that the speed measure is very large in any neighborhood of 0 indicating that the inventory moves slowly while γ_2 determines the penalty rate that the holding cost imposes near 0. It is therefore the delicate interplay between the dynamics of the model and the cost structure which determines whether or not Condition 5.1 holds.

References

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