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SUPPLEMENTARY MATERIAL: CENTRAL LIMIT THEOREM FOR BIFURCATING MARKOV CHAINS UNDER L^2-ERGODIC CONDITIONS

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7. Supplementary material to Section 3.2 on the critical case

We give a proof to Theorem 3.2. We keep notations from Section 5 on the sub-critical case, and adapt very closely the arguments of this section. We recall that $c_k(\mathfrak{f}) = \sup\{\| f_n \|_{L^k(\mu)},\, n \in \mathbb{N}\}$ for all $k \in \mathbb{N}$. We recall that C denotes any unimportant finite constant which may vary from line to line, which does not depend on n or $\mathfrak{f}$. In this case, the condition (32) is strengthened as follows: for all $\lambda > 0$,

$$p_n < n, \quad \lim_{n\to\infty} p_n/n = 1 \quad \text{and} \quad \lim_{n\to\infty} n - p_n - \lambda \log(n) = +\infty.$$

Lemma 7.1. *Under the assumptions of Theorem 3.2, we have that* $\lim_{n\to\infty} \mathbb{E}[n^{-1} R_0^{k_0}(n)^2] = 0$.

Proof. Mimicking the proof of Lemma 5.2, we get:

$$\lim_{n\to\infty} \mathbb{E}[R_0^{k_0}(n)^2]^{1/2} \leq \lim_{n\to\infty} C c_2(\mathfrak{f})\, \sqrt{n} 2^{-p/2} = 0.$$

This trivially implies the result. □

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Lemma 7.2. *Under the assumptions of Theorem 3.2, we have that* $\lim_{n\to\infty} \mathbb{E}[n^{-1}R_1(n)^2] = 0$.

Proof. Mimicking the proof of Lemma 5.3, we get $\mathbb{E}[R_1(n)^2]^{1/2} \leq Cc_2(\mathfrak{f})\sqrt{n-p}$. As $\lim_{n\to\infty} p/n = 1$, this implies that $\lim_{n\to\infty} \mathbb{E}[n^{-1}R_1(n)^2] = 0$. $\square$

Similarly to Lemma 5.4, we get the following result on $R_2(n)$.

Lemma 7.3. *Under the assumptions of Theorem 3.2, we have that* $\lim_{n\to\infty} \mathbb{E}[n^{-1/2}R_2(n)] = 0$.

We now consider the asymptotics of $V_2(n)$.

Lemma 7.4. *Under the assumptions of Theorem 3.2, we have that* $\lim_{n\to\infty} n^{-1}V_2(n) = \Sigma_2^{\text{crit}}(\mathfrak{f})$ *in probability, where* $\Sigma_2^{\text{crit}}(\mathfrak{f})$, *defined in* (29), *is well defined and finite.*

In the proof, we shall use the analogue of (8) with f replaced by $\hat{f}$ in the left hand-side, whereas $f \in L^4(\mu)$ does imply that $\tilde{f} \in L^4(\mu)$ but does not imply that $\hat{f} \in L^4(\mu)$. Thanks to (8), we get for $f \in L^4(\mu)$ and $g \in L^2(\mu)$, as $\mathcal{R}_j f = \alpha_j^{-1}\mathcal{Q}\mathcal{R}_j f$ and $|\alpha_j| = \alpha$, that:

$$\begin{aligned} \| \mathcal{P}\left(\hat{f} \otimes_{\text{sym}} \mathcal{Q}g\right) \|_{L^2(\mu)} &\leq \| \mathcal{P}\left(\tilde{f} \otimes_{\text{sym}} \mathcal{Q}g\right) \|_{L^2(\mu)} + \alpha^{-1} \sum_{j\in J} \| \mathcal{P}\left(\mathcal{Q}(\mathcal{R}_j f) \otimes_{\text{sym}} \mathcal{Q}g\right) \|_{L^2(\mu)} \\ &\leq C \left(\| f \|_{L^4(\mu)} + \| f \|_{L^2(\mu)} \right) \| g \|_{L^2(\mu)} \\ &\leq C \, \| f \|_{L^4(\mu)} \| g \|_{L^2(\mu)} . \end{aligned} \tag{1}$$

Proof. We keep the decomposition (45) of $V_2(n) = V_5(n) + V_6(n)$ given in the proof of Lemma 5.5. We recall $V_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(H_{6,n})$ with $H_{6,n}$ defined in (46). We set

$$\bar{H}_{6,n} = \sum_{0\leq \ell < k \leq p;\, r\geq 0} \bar{h}_{k,\ell,r}^{(n)} \, \mathbf{1}_{\{r+k<p\}} \quad \text{and} \quad \bar{V}_6(n) = |\mathbb{G}_{n-p}|^{-1} M_{\mathbb{G}_{n-p}}(\bar{H}_{6,n}),$$

where for $0 \leq \ell < k \leq p$ and $0 \leq r < p-k$:

$$\bar{h}_{k,\ell,r}^{(n)} = 2^{r-\ell} \, \alpha^{k-\ell+2r} \, \mathcal{Q}^{p-1-(r+k)}(\mathcal{P}f_{k,\ell,r}) = 2^{-(k+\ell)/2} \, \mathcal{Q}^{p-1-(r+k)}(\mathcal{P}f_{k,\ell,r}),$$

where we used that $2\alpha^2 = 1$. For $f \in L^2(\mu)$, we recall $\hat{f}$ defined in (26). We set:

$$
\begin{aligned}
h_{k,\ell,r}^{(n,1)} &= 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)}(\mathcal{P}(\mathcal{Q}^r(\hat{f}_k) \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}(\hat{f}_\ell))),\\
h_{k,\ell,r}^{(n,2)} &= 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)}(\mathcal{P}(\mathcal{Q}^r(\hat{f}_k) \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}(\sum_{j\in J} \mathcal{R}_j(f_\ell)))),\\
h_{k,\ell,r}^{(n,3)} &= 2^{r-\ell} \mathcal{Q}^{p-1-(r+k)}(\mathcal{P}(\mathcal{Q}^r(\sum_{j\in J} \mathcal{R}_j(f_k)) \otimes_{\text{sym}} \mathcal{Q}^{k-\ell+r}(\hat{f}_\ell))),
\end{aligned}
$$

so that $h_{k,\ell,r}^{(n)} = \bar{h}_{k,\ell,r}^{(n)} + \sum_{i=1}^3 h_{k,\ell,r}^{(n,i)}$. Thanks to (6) for $r \geq 1$ and (1) for $r = 0$, we have using Jensen's inequality, (16) and the fact that the sequence $(\beta_r, r \in \mathbb{N})$ is nonincreasing:

$$
\| h_{k,\ell,r}^{(n,1)} \|_{L^2(\mu)} \leq C 2^{-(k+\ell)/2} \beta_r \| f_\ell \|_{L^2(\mu)} \begin{cases} \| f_k \|_{L^2(\mu)} & \text{for } r \geq 1, \\ \| f_k \|_{L^4(\mu)} & \text{for } r = 0. \end{cases}
$$

Using the same arguments, that $\langle \mu, \mathcal{R}_j(g) \rangle = 0$ for $g \in L^2(\mu)$ (as $\mathcal{R}_j(g)$ is an eigen-vector of $\mathcal{Q}$ associated to α_j) and that $\| \sum_{j\in J} \mathcal{R}_j(f_\ell) \|_{L^2(\mu)} \leq C \| f_\ell \|_{L^2(\mu)}$ (as $\mathcal{R}_j$ are bounded operators on $L^2(\mu)$), we get:

$$
\| h_{k,\ell,r}^{(n,2)} \|_{L^2(\mu)} + \| h_{k,\ell,r}^{(n,3)} \|_{L^2(\mu)} \leq C 2^{-(k+\ell)/2} \beta_r \| f_\ell \|_{L^2(\mu)} \begin{cases} \| f_k \|_{L^2(\mu)} & \text{for } r \geq 1, \\ \| f_k \|_{L^4(\mu)} & \text{for } r = 0. \end{cases}
$$

We deduce that

$$
\sum_{i=1}^3 \| h_{k,\ell,r}^{(n,i)} \|_{L^2(\mu)} \leq C c_2(\mathfrak{f}) c_4(\mathfrak{f}) 2^{-(k+\ell)/2} \beta_r. \tag{2}
$$

Using (36) for the first inequality, Jensen's inequality for the second inequality, the triangular inequality for the third inequality and (2) for the last inequality, we get:

$$\begin{aligned}\mathbb{E}\left[\left(V_6(n)-\bar{V}_6(n)\right)^2\right] &= |\mathbb{G}_{n-p}|^{-2}\mathbb{E}[M_{\mathbb{G}_{n-p}}(H_6(n)-\bar{H}_6(n))^2] \\ &\leq C|\mathbb{G}_{n-p}|^{-1}\sum_{m=0}^{n-p} 2^m \parallel \mathcal{Q}^m(H_6(n)-\bar{H}_6(n)) \parallel_{L^2(\mu)}^2 \\ &\leq C \parallel H_6(n)-\bar{H}_6(n) \parallel_{L^2(\mu)}^2 \\ &\leq C\Big(\sum_{0\leq \ell<k<p}\sum_{r=0}^{p-k-1}\sum_{i=1}^{3} \parallel h_{n,k,\ell,r}^{(n,i)} \parallel_{L^2(\mu)}\Big)^2 \\ &\leq Cc_2(\mathfrak{f})^2 c_4(\mathfrak{f})^2\Big(\sum_{r=0}^{p}\beta_r\Big)^2.\end{aligned}$$

We deduce that

$$\mathbb{E}[(V_6(n)-\bar{V}_6(n))^2] \leq Cc_2(\mathfrak{f})^2 c_4(\mathfrak{f})^2\Big(\sum_{r=0}^{p}\beta_r\Big)^2,$$

and then that

$$\lim_{n\to\infty}\mathbb{E}[n^{-2}(V_6(n)-\bar{V}_6(n))^2]=0. \tag{3}$$

We set $H_6^{[n]}=\sum_{0\leq \ell<k\leq p;\, r\geq 0} h_{k,\ell,r}\,\mathbf{1}_{\{r+k<p\}}$ with for $0\leq \ell<k\leq p$ and $0\leq r<p-k$:

$$h_{k,\ell,r}=2^{-(k+\ell)/2}\langle \mu, \mathcal{P}f_{k,\ell,r}\rangle = \langle \mu, \bar{h}_{k,\ell,r}^{(n)}\rangle.$$

We have that

$$H_6^{[n]}=\sum_{0\leq \ell<k<p}\sum_{r=0}^{p-k-1} h_{k,\ell,r}=\langle \mu, \bar{H}_{6,n}\rangle.$$

We have:

$$\begin{aligned}
\mathbb{E}[(\bar{V}_6(n) - H_6^{[n]})^2] &\leq C|\mathbb{G}_{n-p}|^{-1} \sum_{m=0}^{n-p} 2^m \, \| \mathcal{Q}^m(\bar{H}_{6,n} - H_6^{[n]}) \|_{L^2(\mu)}^2 \\
&\leq C|\mathbb{G}_{n-p}|^{-1} \sum_{m=0}^{n-p} 2^m \left(\sum_{0\leq \ell < k \leq p} \sum_{r=0}^{p-k-1} \alpha^{m+p-r-k} 2^{-(k+\ell)/2} \, \| \mathcal{P} f_{k,\ell,r} \|_{L^2(\mu)} \right)^2 \\
&\leq C(n-p)|\mathbb{G}_{n-p}|^{-1} \left(\sum_{0\leq \ell < k \leq p} \sum_{r=0}^{p-k-1} 2^{-(p+\ell-r)/2} \, \| \mathcal{P}(f_{k,\ell,r}) \|_{L^2(\mu)} \right)^2 \\
&\leq C(n-p)|\mathbb{G}_{n-p}|^{-1} \left(\sum_{0\leq \ell < k < p} 2^{-(\ell+k)/2} \| \sum_{j\in J} \mathcal{R}_j(f_k) \|_{L^2(\mu)} \| \sum_{j\in J} \mathcal{R}_j(f_\ell) \|_{L^2(\mu)} \right)^2 \\
&\leq C(n-p)|\mathbb{G}_{n-p}|^{-1} \, c_2^4(\mathfrak{f}),
\end{aligned}$$

where we used (36) for the first inequality, (15) for the second, $\alpha = 1/\sqrt{2}$ for the third, (6) and the fact that $\mathcal{Q}(\sum_{j\in J} \mathcal{R}_j f) = \sum_{j\in J} \alpha_j \mathcal{R}_j(f)$, with $|\alpha_j| = 1/\sqrt{2}$, for the fourth, $\| \sum_{j\in J} \mathcal{R}_j(f) \|_{L^2(\mu)} \leq \| f \|_{L^2(\mu)}$ for the last. From the latter inequality we conclude that:

$$\lim_{n\to\infty} \mathbb{E}[n^{-2}(\bar{V}_6(n) - H_6^{[n]})^2] = 0. \tag{4}$$

We set for $k, \ell \in \mathbb{N}$: $h^*_{k,\ell} = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P}(f^*_{k,\ell}) \rangle$ and we consider the sums

$$H_0^* = \sum_{0\leq \ell < k} (k+1)|h^*_{k,\ell}| \quad \text{and} \quad H_6^*(\mathfrak{f}) = \sum_{0\leq \ell < k} h^*_{k,\ell} = \Sigma_2^{\text{crit}}(\mathfrak{f}).$$

Using (5), we have:

$$|h^*_{k,\ell}| \leq C 2^{-(k+\ell)/2} \sum_{j\in J} \| \mathcal{R}_j(f_k) \|_{L^2(\mu)} \| \mathcal{R}_j(f_\ell) \|_{L^2(\mu)} \leq C 2^{-(k+\ell)/2} c_2^2(\mathfrak{f}).$$

This implies that $H_0^* \leq C c_2^2(\mathfrak{f})$, $H_6^*(\mathfrak{f}) \leq C c_2^2(\mathfrak{f})$ and then that H_0^* and $H_6^*(\mathfrak{f})$ are well defined. We write:

$$h_{k,\ell,r} = h^*_{k,\ell} + h^\circ_{k,\ell,r}, \quad \text{with} \quad h^\circ_{k,\ell,r} = 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f^\circ_{k,\ell,r} \rangle,$$

where we recall that $f^\circ_{k,\ell,r} = f_{k,\ell,r} - f^*_{k,\ell}$, and

$$H_6^{[n]} = H_6^{[n],*} + H_6^{[n],\circ} \tag{5}$$

with

$$H_6^{[n],*} = \sum_{0\leq \ell<k\leq p} (p-k) h^*_{k,\ell} \quad \text{and} \quad H_6^{[n],\circ} = \sum_{0\leq \ell<k\leq p;\, r\geq 0} h^\circ_{k,\ell,r}\, \mathbf{1}_{\{r+k<p\}}.$$

Recall $\lim_{n\to\infty} p/n = 1$. We have:

$$|n^{-1} H_6^{[n],*} - H_6^*(\mathfrak{f})| \leq |n^{-1}p - 1||H_6^*(\mathfrak{f})| + n^{-1} H_0^* + \sum_{\substack{0\leq \ell<k\\ k>p}} |h^*_{k,\ell}|,$$

so that $\lim_{n\to\infty} |n^{-1} H_6^{[n],*} - H_6^*(\mathfrak{f})| = 0$ and thus:

$$\lim_{n\to\infty} n^{-1} H_6^{[n],*} = H_6^*(\mathfrak{f}). \tag{6}$$

We now prove that $n^{-1} H_6^{[n],\circ}$ converges towards 0. We have:

$$f^\circ_{k,\ell,r} = \sum_{j,j'\in J,\, \theta_j\theta_{j'}\neq 1} (\theta_{j'}\theta_j)^r \theta_{j'}^{k-\ell}\, \mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell. \tag{7}$$

This gives:

$$\begin{aligned} |H_6^{[n],\circ}| &= \Big| \sum_{0\leq \ell<k\leq p,\, r\geq 0} 2^{-(k+\ell)/2} \langle \mu, \mathcal{P} f^\circ_{k,\ell,r}\rangle\, \mathbf{1}_{\{r+k<p\}} \Big| \\ &\leq \sum_{0\leq \ell<k\leq p} 2^{-(k+\ell)/2} \sum_{j,j'\in J,\, \theta_j\theta_{j'}\neq 1} \Big| \langle \mu, \mathcal{P}(\mathcal{R}_j f_k \otimes_{\text{sym}} \mathcal{R}_{j'} f_\ell)\rangle \Big| \Big| \sum_{r=0}^{p-k-1} (\theta_{j'}\theta_j)^r \Big|, \end{aligned} \tag{8}$$

where we used (7) for the inequality. Using (5) in the upper bound (8), we get

$$\Big| \langle \mu, \mathcal{P}(\mathcal{R}_{j'} f_k \otimes_{\text{sym}} \mathcal{R}_j f_\ell)\rangle \Big| \leq 2 \| \mathcal{R}_{j'}(f_k) \|_{L^2(\mu)} \| \mathcal{R}_j(f_\ell) \|_{L^2(\mu)} \leq C \| f_k \|_{L^2(\mu)} \| f_\ell \|_{L^2(\mu)}.$$

This implies that $|H_6^{[n],\circ}| \leq c$, with

$$c = C\, c_2(\mathfrak{f})^2 \sum_{0\leq \ell<k\leq p} 2^{-(k+\ell)/2} \sum_{j,j'\in J,\, \theta_j\theta_{j'}\neq 1} |1-\theta_{j'}\theta_j|^{-1}.$$

Since J is finite, we deduce that c is finite. This gives that $\lim_{n\to\infty} n^{-1} H_6^{[n],\circ} = 0$. Recall that $H_6^{[n]}$ and $H_6^*(\mathfrak{f})$ are complex numbers (*i.e.* constant functions). Use (5) and (6) to get that:

$$\lim_{n\to\infty} n^{-1} H_6^{[n]} = H_6^*(\mathfrak{f}) \tag{9}$$

It follows from (3), (4) and (9) that:

$$\lim_{n\to\infty} \mathbb{E}[(n^{-1}V_6(n) - H_6^*(\mathfrak{f}))^2] = 0. \tag{10}$$

We recall $H_5^{[n]}(\mathfrak{f})$ defined in (53). From (55), we have:

$$\mathbb{E}[n^{-2}V_5(n)^2] \leq 2n^{-2}|\mathbb{G}_{n-p}|^{-2}\,\mathbb{E}\left[M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathfrak{f}))^2\right] + 2n^{-2}H_5^{[n]}(\mathfrak{f})^2.$$

Using (51) with $\alpha = 1/\sqrt{2}$, we get $|H_5^{[n]}(\mathfrak{f})| \leq C\, c_2^2(\mathfrak{f})$ and thus:

$$\lim_{n\to\infty} n^{-2}H_5^{[n]}(\mathfrak{f})^2 = 0.$$

Next, as (56) holds for $\alpha = 1/\sqrt{2}$, we get (57) with the right hand-side replaced by $C\, c_4^4(\mathfrak{f})\,(n-p)2^{-(n-p)}$, and thus:

$$\lim_{n\to\infty} n^{-2}|\mathbb{G}_{n-p}|^{-2}\,\mathbb{E}\left[M_{\mathbb{G}_{n-p}}(A_{5,n}(\mathfrak{f}))^2\right] = 0.$$

It then follows that:

$$\lim_{n\to\infty} \mathbb{E}[n^{-2}V_5(n)^2] = 0.$$

Finally, since $V_2(n) = V_5(n) + V_6(n)$, we get thanks to (7) that in probability $\lim_{n\to\infty} n^{-1}V_2(n) = H_6^*(\mathfrak{f}) = \Sigma_2^{\text{crit}}(\mathfrak{f})$. □

Lemma 7.5. *Under the assumptions of Theorem 3.2, we have that in probability* $\lim_{n\to\infty} V_1(n) = \Sigma_1^{\text{crit}}(\mathfrak{f})$*, where* $\Sigma_1^{\text{crit}}(\mathfrak{f})$*, defined in* (28)*, is well defined and finite.*

Proof. We recall the decomposition (58): $V_1(n) = V_3(n) + V_4(n)$. First, following the proof of (10) in the spirit of the proof of (62), we get:

$$\lim_{n\to\infty} \mathbb{E}[(n^{-1}V_4(n) - H_4^*(\mathfrak{f}))^2] = 0 \quad \text{with} \quad H_4^*(\mathfrak{f}) = \sum_{\ell\geq 0} 2^{-\ell}\langle \mu, \mathcal{P}(\sum_{j\in J} \mathcal{R}_j(f_\ell)\otimes_{\text{sym}}\overline{\mathcal{R}}_j(f_\ell))\rangle = \Sigma_1^{\text{crit}}(\mathfrak{f}).$$

Let us stress that the proof requires to use (4). Since $\sum_{\ell\geq 0} 2^{-\ell}|\langle \mu, \mathcal{P}(\sum_{j\in J} \mathcal{R}_j(f_\ell)\otimes_{\text{sym}} \overline{\mathcal{R}}_j(f_\ell))\rangle| \leq \sum_{\ell\geq 0} 2^{-\ell}c_2^2(\mathfrak{f})$, we deduce that $\Sigma_1^{\text{crit}}(\mathfrak{f})$ is well defined and finite. Next, from (64) we have

$$\mathbb{E}[n^{-2}V_3(n)^2] \leq 2n^{-2}|\mathbb{G}_{n-p}|^{-2}\,\mathbb{E}\left[M_{\mathbb{G}_{n-p}}(A_{3,n}(\mathfrak{f}))^2\right] + 2n^{-2}H_3^{[n]}(\mathfrak{f})^2.$$

It follows from (65) (with an extra term $n-p$ as $2\alpha^2=1$ in the right hand side) and (63) that $\lim_{n\to\infty} \mathbb{E}[n^{-2}V_3(n)^2]=0$. Finally the result of the lemma follows as $V_1=V_3+V_4$. □

We now check the Lindeberg condition using a fourth moment condition. Recall $R_3(n)=\sum_{i\in\mathbb{G}_{n-p_n}} \mathbb{E}\left[\Delta_{n,i}(\mathfrak{f})^4\right]$ defined in (66).

Lemma 7.6. *Under the assumptions of Theorem 3.2, we have that* $\lim_{n\to\infty} n^{-2}R_3(n)=0$.

Proof. Following line by line the proof of Lemma 5.8 with the same notations and taking $\alpha=1/\sqrt{2}$, we get that concerning $|\langle\mu,\psi_{i,p-\ell}\rangle|$ or $\langle\mu,|\psi_{i,p-\ell}|\rangle$, the bounds for $i\in\{1,2,3,4\}$ are the same; the bounds for $i\in\{5,6,7\}$ have an extra $(p-\ell)$ term, the bounds for $i\in\{8,9\}$ have an extra $(p-\ell)^2$ term. This leads to (compare with (73)):

$$R_3(n)\leq C\, n^5\, 2^{-(n-p)}\, c_4^4(\mathfrak{f})$$

which implies that $\lim_{n\to\infty} n^{-2}R_3(n)=0$. □

The proof of Theorem 3.2 then follows the proof of Theorem 3.1.

8. Supplementary material to Section 3.3 on the supercritical case

8.1. Complementary results and proof of Corollary 3.1

Now, we state the main result of this section, whose proof is given in Section 8.3. Recall that $\theta_j=\alpha_j/\alpha$ and $|\theta_j|=1$ and $M_{\infty,j}$ is defined in Lemma 3.1.

Theorem 8.1. *Let X be a BMC with kernel $\mathcal{P}$ and initial distribution ν such that Assumptions 2.2 (ii) and 2.4 are in force with $\alpha\in(1/\sqrt{2},1)$ in (16). We have the following convergence for all sequence $\mathfrak{f}=(f_\ell,\ell\in\mathbb{N})$ uniformly bounded in $L^2(\mu)$ (that is $\sup_{\ell\in\mathbb{N}}\|f_\ell\|_{L^2(\mu)}<+\infty$):*

$$(2\alpha^2)^{-n/2}N_{n,\emptyset}(\mathfrak{f})-\sum_{\ell\in\mathbb{N}}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}M_{\infty,j}(f_\ell)\ \xrightarrow[n\to\infty]{\mathbb{P}}\ 0.$$

Remark 8.1. We stress that if for all $\ell \in \mathbb{N}$, the orthogonal projection of f_ℓ on the eigen-spaces corresponding to the eigenvalues 1 and α_j, $j \in J$, equal 0, then $M_{\infty,j}(f_\ell) = 0$ for all $j \in J$ and in this case, we have

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathfrak{f}) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

As a direct consequence of Theorem 8.1 and Remark 2.5, we deduce the following results. Recall that $\tilde{f} = f - \langle \mu, f \rangle$.

Corollary 8.1. *Under the assumptions of Theorem 8.1, we have for all $f \in L^2(\mu)$:*

$$(2\alpha)^{-n} M_{\mathbb{T}_n}(\tilde{f}) - \sum_{j \in J} \theta_j^n (1 - (2\alpha\theta_j)^{-1})^{-1} M_{\infty,j}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

$$(2\alpha)^{-n} M_{\mathbb{G}_n}(\tilde{f}) - \sum_{j \in J} \theta_j^n M_{\infty,j}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Proof. We first take $\mathfrak{f} = (f, f, \ldots)$ and next $\mathfrak{f} = (f, 0, \ldots)$ in Theorem 8.1, and then use (20). □

We directly deduce the following Corollary.

Corollary 8.2. *Under the hypothesis of Theorem 8.1, if α is the only eigen-value of $\mathcal{Q}$ with modulus equal to α (and thus J is reduced to a singleton), then we have:*

$$(2\alpha^2)^{-n/2} N_{n,\emptyset}(\mathfrak{f}) \xrightarrow[n\to\infty]{\mathbb{P}} \sum_{\ell \in \mathbb{N}} (2\alpha)^{-\ell} M_\infty(f_\ell),$$

where, for $f \in F$, $M_\infty(f) = \lim_{n\to\infty} (2\alpha)^{-n} M_{\mathbb{G}_n}(\mathcal{R}(f))$, and $\mathcal{R}$ is the projection on the eigen-space associated to the eigen-value α.

The Corollary 3.1 is then a direct consequence of Corollary 8.2.

8.2. Proof of Lemma 3.1

Let $f \in L^2(\mu)$ and $j \in J$. Use that $\mathcal{R}_j(L^2(\mu)) \subset \mathbb{C}L^2(\mu)$ to deduce that $\mathbb{E}\left[|M_{n,j}(f)|^2\right]$ is finite. We have for $n \in \mathbb{N}^*$:

$$\begin{aligned}
\mathbb{E}[M_{n,j}(f)|\mathcal{H}_{n-1}] &= (2\alpha_j)^{-n} \sum_{i\in\mathbb{G}_{n-1}} \mathbb{E}[\mathcal{R}_j f(X_{i0}) + \mathcal{R}_j f(X_{i1})|\mathcal{H}_{n-1}] \\
&= (2\alpha_j)^{-n} \sum_{i\in\mathbb{G}_{n-1}} 2\,\mathcal{Q}\mathcal{R}_j f(X_i) \\
&= (2\alpha_j)^{-(n-1)} \sum_{i\in\mathbb{G}_{n-1}} \mathcal{R}_j f(X_i) \\
&= M_{n-1,j}(f),
\end{aligned}$$

where the second equality follows from branching Markov property and the third follows from the fact that $\mathcal{R}_j$ is the projection on the eigen-space associated to the eigen-value α_j of $\mathcal{Q}$. This gives that $M_j(f)$ is a $\mathcal{H}$-martingale. We also have, writing f_j for $\mathcal{R}_j(f)$:

$$\begin{aligned}
\mathbb{E}\left[|M_{n,j}(f)|^2\right] &= (2\alpha)^{-2n}\,\mathbb{E}\left[M_{\mathbb{G}_n}(f_j)M_{\mathbb{G}_n}(\overline{f}_j)\right] \\
&= (2\alpha^2)^{-n}\,\langle \nu, \mathcal{Q}^n(|f_j|^2)\rangle + (2\alpha)^{-2n}\sum_{k=0}^{n-1} 2^{n+k}\langle \nu, \mathcal{Q}^{n-k-1}\mathcal{P}\left(\mathcal{Q}^k f_j \otimes_{\text{sym}} \mathcal{Q}^k\overline{f}_j\right)\rangle \\
&\leq C\,(2\alpha^2)^{-n}\,\langle \mu, \mathcal{Q}^{n-k_0}(|f_j|^2)\rangle + (2\alpha)^{-2n}\sum_{k=0}^{n-1} 2^{n+k}\langle \nu, \mathcal{Q}^{n-k-1}\mathcal{P}\left(|\mathcal{Q}^k f_j|\otimes^2\right)\rangle \\
&\leq C(2\alpha^2)^{-n}\|f_j\|^2_{L^2(\mu)} + C\,(2\alpha^2)^{-n}\sum_{k=0}^{n-k_0} 2^k\,\|\mathcal{Q}^k f_j\|^2_{L^2(\mu)} \qquad (11)
\end{aligned}$$

where we used the definition of $M_{n,j}$ for the first equality, (76) with $m = n$ for the second equality, Assumption 2.2 (ii) for the first term of the first inequality, the fact that $\mathcal{Q}^k f_j \otimes_{\text{sym}} \mathcal{Q}^k\overline{f}_j \leq |\mathcal{Q}^k f_j|\otimes^2$ for the second term of the first inequality and for the last inequality, we followed the lines of the proof of Lemma 5.1. Finally, using that $|\mathcal{Q}^k f_j| = \alpha^k|f_j|$, this implies that $\sup_{n\in\mathbb{N}}\mathbb{E}\left[|M_{n,j}(f)|^2\right] < +\infty$. Thus the martingale $M_j(f)$ converges a.s. and in L^2 towards a limit.

8.3. Proof of Theorem 8.1

Recall the sequence $(\beta_n, n \in \mathbb{N})$ defined in Assumption 2.4 and the σ-field $\mathcal{H}_n = \sigma\{X_u, u \in \mathbb{T}_n\}$. Let $(\hat{p}_n, n \in \mathbb{N})$ be a sequence of integers such that $\hat{p}_n$ is even and (for $n \geq 3$):

$$\frac{5n}{6} < \hat{p}_n < n, \quad \lim_{n\to\infty} (n - \hat{p}_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \alpha^{-(n-\hat{p}_n)} \beta_{\hat{p}_n/2} = 0. \tag{12}$$

Notice such sequences exist. When there is no ambiguity, we shall write $\hat{p}$ for $\hat{p}_n$. Using Remark 5.2, it suffices to do the proof with $N_{n,\emptyset}^{[k_0]}(\mathfrak{f})$ instead of $N_{n,\emptyset}(\mathfrak{f})$. We deduce from (21) that:

$$N_{n,\emptyset}^{[k_0]}(\mathfrak{f}) = R_0^{k_0}(n) + R_4(n) + T_n(\mathfrak{f}), \tag{13}$$

with notations from (34) and (35):

$$R_0^{k_0}(n) = |\mathbb{G}_n|^{-1/2} \sum_{k=k_0}^{n-\hat{p}_n-1} M_{\mathbb{G}_k}(\tilde{f}_{n-k}),$$
$$T_n(\mathfrak{f}) = R_1(n) = \sum_{i\in\mathbb{G}_{n-\hat{p}_n}} \mathbb{E}[N_{n,i}(\mathfrak{f})|\mathcal{H}_{n-\hat{p}_n}],$$
$$R_4(n) = \Delta_n = \sum_{i\in\mathbb{G}_{n-\hat{p}_n}} \left(N_{n,i}(\mathfrak{f}) - \mathbb{E}[N_{n,i}(\mathfrak{f})|\mathcal{H}_{n-\hat{p}_n}]\right).$$

Furthermore, using the branching Markov property, we get for all $i \in \mathbb{G}_{n-\hat{p}_n}$:

$$\mathbb{E}[N_{n,i}(\mathfrak{f})|\mathcal{H}_{n-\hat{p}_n}] = \mathbb{E}[N_{n,i}(\mathfrak{f})|X_i]. \tag{14}$$

We have the following elementary lemma.

Lemma 8.1. *Under the assumptions of Theorem 8.1, we have the following convergence:*

$$\lim_{n\to\infty} (2\alpha^2)^{-n}\, \mathbb{E}\left[R_0^{[k_0]}(n)^2\right] = 0.$$

Proof. We follow the proof of Lemma 5.2. As $2\alpha^2 > 1$ and following the arguments leading to (41) we get that for some constant C which does not depend on n or $\hat{p}$:

$$\mathbb{E}\left[R_0^{k_0}(n)^2\right]^{1/2} \leq C\, 2^{-\hat{p}/2} (2\alpha^2)^{(n-\hat{p})/2}.$$

It follows from the previous inequality that $(2\alpha^2)^{-n}\mathbb{E}\left[R_0(n)^2\right] \leq C(2\alpha)^{-2\hat{p}}$. Then use $2\alpha > 1$ and $\lim_{n\to\infty} \hat{p} = \infty$ to conclude. □

Next, we have the following lemma.

Lemma 8.2. *Under the assumptions of Theorem 8.1, we have the following convergence:*

$$\lim_{n\to\infty} (2\alpha^2)^{-n}\mathbb{E}\left[R_4(n)^2\right] = 0.$$

Proof. First, we have:

$$\begin{aligned}
\mathbb{E}[R_4(n)^2] &= \mathbb{E}\left[\left(\sum_{i\in\mathbb{G}_{n-\hat{p}}} (N_{n,i}(\mathfrak{f}) - \mathbb{E}[N_{n,i}(\mathfrak{f})|X_i])\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i\in\mathbb{G}_{n-\hat{p}}} \mathbb{E}[(N_{n,i}(\mathfrak{f}) - \mathbb{E}[N_{n,i}(\mathfrak{f})|X_i])^2|\mathcal{H}_{n-\hat{p}}]\right] \\
&\leq \mathbb{E}\left[\sum_{i\in\mathbb{G}_{n-\hat{p}}} \mathbb{E}[N_{n,i}(\mathfrak{f})^2|X_i]\right], \qquad (15)
\end{aligned}$$

where we used (14) for the first equality and the branching Markov chain property for the second and the last inequality. Note that for all $i \in \mathbb{G}_{n-\hat{p}}$ we have

$$\mathbb{E}\left[\mathbb{E}[N_{n,i}(\mathfrak{f})^2|X_i]\right] = |\mathbb{G}_n|^{-1}\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{\ell=0}^{\hat{p}} M_{i\mathbb{G}_{\hat{p}-k}}(\tilde{f}_\ell)\right)^2 |X_i\right]\right],$$

where we used the definition of $N_{n,i}(\mathfrak{f})$. Putting the latter equality in (15) and using the first inequality of (36), we get

$$\mathbb{E}[R_4(n)^2] \leq |\mathbb{G}_n|^{-1}\,\mathbb{E}[M_{\mathbb{G}_{n-p}}(h_{\hat{p}})] \leq C\,2^{-\hat{p}}\langle\mu, h_{\hat{p}}\rangle, \quad \text{with} \quad h_{\hat{p}}(x) = \mathbb{E}_x[(\sum_{\ell=0}^{\hat{p}} M_{\mathbb{G}_{\hat{p}-\ell}}(\tilde{f}))^2].$$

Using the second inequality of (36) and (15), we get

$$\langle\mu, h_{\hat{p}}\rangle = \mathbb{E}_\mu[(\sum_{\ell=0}^{\hat{p}} M_{\mathbb{G}_{\hat{p}-\ell}}(\tilde{f}))^2] \leq \left(\sum_{\ell=0}^{p} \mathbb{E}_\mu[(M_{\mathbb{G}_{p-\ell}}(\tilde{f}_\ell))^2]^{1/2}\right)^2 \leq C\,(2\alpha)^{2\hat{p}}.$$

This implies that

$$(2\alpha^2)^{-n}\mathbb{E}\left[R_4(n)^2\right] \leq C\,(2\alpha^2)^{-n}\,(2\alpha^2)^{\hat{p}} \;=\; C\,(2\alpha^2)^{\hat{p}-n}.$$

We then conclude using $2\alpha^2 > 1$ and (12). $\square$

Now, we study the third term of the right hand side of (13). First, note that:

$$\begin{aligned} T_n(\mathfrak{f}) &= \sum_{i\in\mathbb{G}_{n-\hat{p}}} \mathbb{E}[N_{n,i}(\mathfrak{f})|X_i] \\ &= \sum_{i\in\mathbb{G}_{n-\hat{p}}} |\mathbb{G}_n|^{-1/2} \sum_{\ell=0}^{\hat{p}} \mathbb{E}_{X_i}[M_{\mathbb{G}_{\hat{p}-\ell}}(\tilde{f}_\ell)] \\ &= |\mathbb{G}_n|^{-1/2} \sum_{i\in\mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \mathcal{Q}^{\hat{p}-\ell}(\tilde{f}_\ell)(X_i), \end{aligned}$$

where we used (14) for the first equality, the definition (19) of $N_n(\mathfrak{f})$ for the second equality and (74) for the last equality. Next, projecting in the eigen-space associated to the eigenvalue α_j, we get

$$T_n(\mathfrak{f}) = T_n^{(1)}(\mathfrak{f}) + T_n^{(2)}(\mathfrak{f}),$$

where, with $\hat{f} = f - \langle \mu, f\rangle - \sum_{j\in J} \mathcal{R}_j(f)$ defined in (26):

$$\begin{aligned} T_n^{(1)}(\mathfrak{f}) &= |\mathbb{G}_n|^{-1/2} \sum_{i\in\mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \left(\mathcal{Q}^{\hat{p}-\ell}(\hat{f}_\ell)\right)(X_i), \\ T_n^{(2)}(\mathfrak{f}) &= |\mathbb{G}_n|^{-1/2} \sum_{i\in\mathbb{G}_{n-\hat{p}}} \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \alpha^{\hat{p}-\ell} \sum_{j\in J} \theta_j^{\hat{p}-\ell} \mathcal{R}_j(f_\ell)(X_i). \end{aligned}$$

We have the following lemma.

Lemma 8.3. *Under the assumptions of Theorem 8.1, we have the following convergence:*

$$\lim_{n\to\infty} (2\alpha^2)^{-n/2} \mathbb{E}[|T_n^{(1)}(\mathfrak{f})|] = 0.$$

Proof. Recall $\hat{p}$ is even. We set $h_{\hat{p}} = \sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell} \mathcal{Q}^{\hat{p}-\ell}(\hat{f}_\ell)$. We have:

$$\begin{aligned}(2\alpha^2)^{-n/2}\mathbb{E}[|T_n^{(1)}(\mathfrak{f})|] &\leq (2\alpha)^{-n}\mathbb{E}[M_{\mathbb{G}_{n-\hat{p}}}(|h_{\hat{p}}|)] \\ &\leq C\,(2\alpha)^{-n}2^{n-\hat{p}}\langle \mu, |h_{\hat{p}}|\rangle \\ &\leq C\,(2\alpha)^{-n}2^{n-\hat{p}}\|h_{\hat{p}}\|_{L^2(\mu)} \\ &\leq C\,(2\alpha)^{-n}2^{n-\hat{p}}\sum_{\ell=0}^{\hat{p}} 2^{\hat{p}-\ell}\alpha^{\hat{p}-\ell}\beta_{\hat{p}-\ell}\,\|\, f_\ell\,\|_{L^2(\mu)} \\ &= C\sum_{\ell=0}^{\hat{p}} 2^{-\ell}\alpha^{-(n-\hat{p}+\ell)}\beta_{\hat{p}-\ell}\,,\end{aligned}$$

where we used the definition of $T_n^{(1)}(\mathfrak{f})$ for the first inequality, the first equation of (36) for the second, Cauchy-Schwarz inequality for the third and (16) for the last inequality. We have:

$$\sum_{\ell=0}^{\hat{p}/2} 2^{-\ell}\alpha^{-(n-\hat{p}+\ell)}\beta_{\hat{p}-\ell} \leq \alpha^{-(n-\hat{p})}\beta_{\hat{p}/2}\sum_{\ell=0}^{\hat{p}/2}(2\alpha)^{-\ell}.$$

Using the third condition in (12) and that $2\alpha > 1$, we deduce the right hand-side converges to 0 as n goes to infinity. Without loss of generality, we can assume that the sequence $(\beta_n, n \in \mathbb{N}^*)$ is bounded by 1. Since $\alpha > 1/\sqrt{2}$, we also have:

$$\sum_{\ell=\hat{p}/2}^{\hat{p}} 2^{-\ell}\alpha^{-(n-\hat{p}+\ell)}\beta_{\hat{p}-\ell} \leq (1-2\alpha)^{-1}\,2^{-\hat{p}/2}\alpha^{-n+\hat{p}/2} \leq (1-2\alpha)^{-1}\,2^{n/2-3\hat{p}/4}.$$

Using that $n/2 - 3\hat{p}/4 < -n/8$, thanks to the first condition in (12), we deduce the right hand-side converges to 0 as n goes to infinity. Thus, we get that $\lim_{n\to\infty}(2\alpha^2)^{-n/2}\mathbb{E}[|T_n^{(1)}(\mathfrak{f})|] = 0$. □

Now, we deal with the term $T_n^{(2)}(\mathfrak{f})$ in the following result. Recall $M_{\infty,j}$ defined in Lemma 3.1.

Lemma 8.4. *Under the assumptions of Theorem 8.1, we have the following convergence:*

$$(2\alpha^2)^{-n/2}T_n^{(2)}(\mathfrak{f}) - \sum_{\ell\in\mathbb{N}}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}M_{\infty,j}(f_\ell) \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Proof. By definition of $T_n^2(\mathfrak{f})$, we have $T_n^2(\mathfrak{f}) = 2^{-n/2}\sum_{\ell=0}^{\hat{p}}(2\alpha)^{n-\ell}\sum_{j\in J}\theta_j^{n-\ell}M_{n,j}(f_\ell)$ and thus:

$$\begin{aligned}&(2\alpha^2)^{-n/2}T_n^{(2)}(\mathfrak{f}) - \sum_{\ell\in\mathbb{N}}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}M_{\infty,j}(f_\ell)\\ &= \sum_{\ell=0}^{\hat{p}}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}(M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)) - \sum_{\ell=\hat{p}+1}^{\infty}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}M_{\infty,j}(f_\ell).\end{aligned} \tag{16}$$

Using that $|\theta_j| = 1$, we get:

$$\mathbb{E}[|\sum_{\ell=0}^{\hat{p}}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}(M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell))|] \leq \sum_{\ell=0}^{\hat{p}}(2\alpha)^{-\ell}\sum_{j\in J}\mathbb{E}[|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|].$$

Now, using that $(f_\ell, \ell\in\mathbb{N})$ is uniformly bounded in $L^2(\mu)$, a close inspection of the proof of Lemma 3.1, see (11), reveals us that there exists a finite constant C (depending on $\mathfrak{f}$) such that for all $j \in J$, we have:

$$\sup_{\ell\in\mathbb{N}}\sup_{n\in\mathbb{N}}\mathbb{E}[|M_{n,j}(f_\ell)|^2] \leq C.$$

The $L^2(\nu)$ convergence in Lemma 3.1 yields that:

$$\sup_{\ell\in\mathbb{N}}\mathbb{E}[|M_{\infty,j}(f_\ell)|^2] \leq C \quad \text{and} \quad \sup_{\ell\in\mathbb{N}}\sup_{n\in\mathbb{N}}\sum_{j\in J}\mathbb{E}[|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|] < 2|J|\sqrt{C}. \tag{17}$$

Since Lemma 3.1 implies that $\lim_{n\to\infty}\mathbb{E}[|M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell)|] = 0$, we deduce, as $2\alpha > 1$ by the dominated convergence theorem that:

$$\lim_{n\to+\infty}\mathbb{E}[|\sum_{\ell=0}^{\hat{p}}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}(M_{n,j}(f_\ell) - M_{\infty,j}(f_\ell))|] = 0. \tag{18}$$

On the other hand, we have

$$\mathbb{E}[|\sum_{\ell=\hat{p}+1}^{\infty}(2\alpha)^{-\ell}\sum_{j\in J}\theta_j^{n-\ell}M_{\infty,j}(f_\ell)|] \leq \sum_{\ell=\hat{p}+1}^{\infty}(2\alpha)^{-\ell}\sum_{j\in J}\mathbb{E}[|M_{\infty,j}(f_\ell)|] \leq |J|\sqrt{C}\sum_{\ell=\hat{p}+1}^{\infty}(2\alpha)^{-\ell}, \tag{19}$$

where we used $|\theta_j| = 1$ for the first inequality and the Cauchy-Schwarz inequality and (17) for the second inequality. Finally, from (16), (18) and (19) (with $\lim_{n\to\infty}\sum_{\ell=\hat{p}+1}^{\infty}(2\alpha)^{-\ell} = 0$) , we get the result of the lemma. □