Voting Equilibria Under Proportional Representation

Online Appendix

Seok-ju Cho

Proof of Lemma 1

Let $v_t = i$. Take any $j \in L$ and consider profile (j, v_{-t}). Since t is neither majority-pivotal nor median pivotal, $k(j, v_{-t}) = k(v)$, and, thus, $p_h(j, v_{-t}) = p_h(v)$ for every $h \in L$. Moreover, $k(v)$ is a majority party in (j, v_{-t}) if and only if it is so in v.

Suppose v_t is not strategically sincere for t in v. There exists $j \in L \setminus \{i\}$ such that $u(p_j(v); t) > u(p_i(v); t)$. Take any $\epsilon \in (0, 1)$. From the discussion in the previous paragraph, we conclude that

$$U(j, v_{-t}; t|\epsilon) - U(v_t, v_{-t}; t|\epsilon) \geq \frac{\epsilon}{n}[u(p_j(v); t) - u(p_i(v); t)] > 0,$$

which implies that v_t is not a robust best response to v_{-t}.

Suppose v_t is strategically sincere for t in v. Then, for every $j \in L$, $u(p_i(v); t) \geq u(p_j(v); t)$. Thus,

$$U(v_t, v_{-t}; t|\epsilon) - U(j, v_{-t}; t|\epsilon) \geq \frac{\epsilon}{n}[u(p_i(v); t) - u(p_j(v); t)] \geq 0$$

for every $j \in L$ and every $\epsilon \in [0, 1)$. Hence, v_t is a robust best response to v_{-t}. ■

Proof of Proposition 1

1. Suppose $|X_m| > M$. Define the voting profile v by the following:

$$v_t = \begin{cases} m & \text{if } t \in X_m, \\ \min\left(\arg\max \left\{ u(\theta_i; t) \,\middle|\, i \in \arg\max\{u(s_j^m; t) | j \in L\} \right\} \right) & \text{otherwise.} \end{cases} \tag{9}$$

Since $v_t = m$ for every $t \in X_m$, $b_m(v) > M$. Thus, $k(v) = m$, and no voter is majority- or median-pivotal. By construction, v is strategically sincere. Lemma 1 then implies that $v \in V(T, \theta, q)$.

2. Let $v \in V(T, \theta, q)$ and $b_{k(v)}(v) > M$. Since $b_{k(v)}(v) > M$, no voter is majority- or median-pivotal. By Lemma 1, v is strategically sincere. Suppose $k(v) < m$. Then $t_M > \overline{y}_k$ which implies $T_{k(v)}(v) \subseteq \{t_1, \ldots, t_{M-1}\}$, contradicting $b_{k(v)}(v) > M$. Suppose $k(v) > m$. Then $t_{M-1} < \underline{y}_{k(v)}$, which implies $T_{k(v)}(v) \subseteq \{t_M, \ldots, t_\ell\}$, contradicting $b_{k(v)}(v) > M$. Thus, $k(v) = m$. Since v is strategically sincere, $T_m(v) \subseteq X_m$. Hence, $|X_m| > M$. ■

Proof of Proposition 2

Suppose $|X_m| < M - 1$ and $\{t_{M-1}, t_M, t_{M+1}\} \subseteq X_m$. Let v be as defined in (9). Note that, by definition of X_m, $m \in \arg\max\{u(s_i^m; t) | i \in L\}$ if and only if $t \in X_m$. Thus, $T_m(v) = X_m$, implying $T_m(v) < M - 1$. Also, for every t with $v_t \neq m$, either $t < \underline{y}_m$ or $t > \overline{y}_m$. If $t < \underline{y}_m$, then $t < \theta_m < s_i^m$ for every $i > m$. Thus, there is no $i \in L$ such that $i > m$ and $i \in \arg\max\{u(s_j^m; t) | j \in L\}$. Therefore, $v_t < m$. Similarly, if $t > \overline{y}_m$, then $v_t > m$. Then since $\underline{y}_m \leq t_{M-1} < t_{M+1} \leq \overline{y}_m$, $\sum_{i=1}^{m-1} b_i(v) < M - 1$ and $\sum_{i=m+1}^{\ell} b_i(v) < M - 1$. This implies that $k(v) = m$ and no voter is majority- or median-pivotal. Then, by construction, v is strategically sincere, and, by Lemma 1, $v \in V(T, \theta, q)$. ■

Proof of Proposition 3

1. Assume $|X_m| = M$. Define the voting profile v as in (9). By construction, $T_m(v) = X_m$ and $T^*(v) = T$. We just have to show that $v \in V(T, \theta, q)$. For any $t \in T \setminus X_m$, t is neither majority-pivotal, nor median-pivotal. Thus, by Lemma 1, v_t is a robust best response. Take any $t \in X_m$ and take any $\epsilon \in [0, 1]$. Since m is the majority party in v,

$$U(v; t|\epsilon) = (1 - \epsilon)u(\theta_m; t) + \frac{\epsilon}{n}\left[Mu(\theta_m; t) + \sum_{i \in L \setminus \{m\}} b_i(v)u(s_i^m; t)\right]. \tag{10}$$

Consider voter t's deviation by voting for some $j \neq m$ and let $v' = (j, v_{-t})$. Since $t_{M-1} \in X_m$, $\sum_{i=1}^{m-1} b_i(v) < M - 1$. Since $t_{M+1} \in X_m$, $\sum_{i=m+1}^{\ell} b_i(v) < M - 1$. Thus, $k(v') = m$, implying $p_i(v') = p_i(v) = s_i^m$ for every $i \in L$. Since m is not a majority party in v',

$$U(v'; t|\epsilon) = \frac{1}{n}\left[(M - 1)u(\theta_m; t) + \sum_{i \in L \setminus \{m\}} b_i(v)u(s_i^m; t) + u(s_j^m; t)\right]. \tag{11}$$

Subtracting (11) from (10), we obtain

$$\begin{aligned} U(v; t|\epsilon) - U(v'; t|\epsilon) &= \frac{1 - \epsilon}{n}\left[(M - 1)u(\theta_m; t) - \sum_{i \in L \setminus \{m\}} b_i(v)u(s_i^m; t)\right] \\ &\quad + \frac{1}{n}[u(\theta_m; t) - u(s_j^m; t)]. \end{aligned}$$

Since $t \in X_m$, $u(\theta_m; t) \geq u(s_i^m; t)$ for every $i \in L$. Also, $\sum_{i \in L \setminus \{m\}} b_i(v) = M - 1$. Hence, $U(v; t|\epsilon) \geq U(v'; t|\epsilon)$. Therefore, v is a robust equilibrium.

2. Assume $|X_m| = M - 1$. If $m = 1$, then $t_1 \in X_m$, implying $t_{M+1} \notin X_m$, a contradiction. If $m = \ell$, then $t_n \in X_m$, implying $t_{M-1} \notin X_m$, a contradiction. Thus, $2 \leq m \leq \ell - 1$. Let

$r_m = \frac{n-1}{2n}$. For each $i \in L \setminus \{m\}$, let

$$r_i = \frac{1}{n}\left| \left\{ t \in T \setminus X_m \;\middle|\; i = \min\left(\arg\max \left\{ u(\theta_i; t) \;\middle|\; i \in \arg\max\{u(s_j^m; t) | j \in L\} \right\} \right) \right\} \right|.$$

Note that $\sum_{i \in L} r_i = 1$. Either $\sum_{i \in L} r_i s_i^m \geq \theta_m$ or $\sum_{i \in L} r_i s_i^m < \theta_m$. If the former is true, then let $t^* = \max\{t \in T | t < \underline{y}_m\}$. If the latter is true, then let $t^* = \min\{t \in T | t > \overline{y}_m\}$. Note that $t^* \notin X_m$. Define the voting profile v by the following:

$$v_t = \begin{cases} m & \text{if } t \in X_m \cup \{t^*\}, \\ \min\left(\arg\max \left\{ u(\theta_i; t) \;\middle|\; i \in \arg\max\{u(s_j^m; t) | j \in L\} \right\} \right) & \text{otherwise.} \end{cases} \tag{12}$$

Note that $T_m(v) = X_m \cup \{t^*\}$ and so $|T_m(v)| = M$. By construction, for every $t \in T \setminus \{t^*\}$, v_t is strategically sincere in v. For any $t \in T \setminus T_m(v)$, t is neither majority-pivotal, nor median-pivotal. So, v_t is a robust best response by Lemma 1. For every $t \in X_m$, the argument in the proof of the first statement of Proposition 3 holds true. Lastly, consider voter t^*'s deviation by voting for some $j \neq m$, and let $v' = (j, v_{-t^*})$. Since $k(v') = m$, $p_i(v') = p_i(v) = s_i^m$ for every $i \in L$. Let

$$i^* = \min\left(\arg\max \left\{ u(\theta_i; t^*) \;\middle|\; i \in \arg\max\{u(s_i^m; t^*) | i \in L\} \right\} \right).$$

Note that if $j = i^*$, then $T_i(v') = nr_i$ for every $i \in L$; and that if $j \neq i^*$, then $T_i(v') = nr_i$ for every $i \in L \setminus \{j, i^*\}$, $T_j(v') = nr_j + 1$, and $T_{i^*}(v) = nr_{i^*} - 1$. Take any $\epsilon \in [0, 1)$. Then,

$$\sum_{i \in L} r_i u(s_i^m; t^*) - U(v'; t^* | \epsilon) = \frac{1}{n}[u(s_{i^*}^m; t^*) - u(s_j^m; t^*)] \geq 0 \tag{13}$$

since $u(s_{i^*}^m; t^*) = \max\{u(s_i^m; t^*) | i \in L\}$. Note that, by construction, either $t^* < \theta_m \leq \sum_{i \in L} r_i s_i^m$ or $\sum_{i \in L} r_i s_i^m < \theta_m < t^*$. Then, since f is strictly concave, $u(\theta_m; t^*) > \sum_{i \in L} r_i u(s_i^m; t^*)$.

Then, from (13), we conclude that $u(\theta_m; t^*) > U(v'; t^*|\epsilon)$. Then, for sufficiently small ϵ,

$$U(v;t^*|\epsilon)-U(v';t^*|\epsilon) = (1-\epsilon)[u(\theta_m;t^*)-U(v';t^*|\epsilon)]+\epsilon\left[\sum_{i\in L}\frac{b_i(v)}{n}u(s_i^m;t^*) - U(v';t^*|\epsilon)\right] > 0.$$

Thus, v is a robust equilibrium of $G(T, \theta, q, 0)$. ■

Proof of Proposition 4

Assume $\{t_{M-1}, t_M, t_{M+1}\} \not\subseteq X_m$. For each $t \in T$, let

$$\alpha(t) = \min\left(\arg\max\left\{u(\theta_i;t)\middle| i \in \arg\max\left\{u(s_j^{m+1};t)\middle| j \in \arg\max\{u(s_h^m;t)|h \in L\}\right\}\right\}\right),$$

and define voting profile $\hat{v}$ by the following.

$$\hat{v}_t = \begin{cases} m & \text{if } t \in [\underline{y}_m, t_M], \\ \alpha(t) & \text{otherwise;} \end{cases} \tag{14}$$

Recall that $\theta_1 < t_M < \theta_\ell$ and $\theta_m \leq t_M$. This implies that $m \leq \ell - 1$. For every $t > \max\{t_M, \overline{y}_m\}$, $\hat{v}_t = \alpha(t) \geq m + 1$. If $m = 1$, then $\underline{y}_m = t_1$, so $\{t_1, \ldots, t_M\} \subseteq T_m(\hat{v})$. Otherwise, for every $t < \underline{y}_m$, $\hat{v}_t = \alpha(t) \leq m - 1$. Thus, when $m = 1$, party m is the majority party, and when $m > 1$, party m is the median party. Then, for every $i \in L$, $p_i(\hat{v}) = s_i^m$. For every $\epsilon \in [0, 1]$ and every $t \in T$,

$$U(\hat{v};t|\epsilon) = (1-\epsilon)u(\theta_1;t) + \frac{\epsilon}{n}\sum_{i\in L} b_i(\hat{v})u(s_i^m;t) \tag{15}$$

if $m = 1$; and

$$U(\hat{v};t|\epsilon) = \frac{1}{n}\sum_{i\in L} b_i(\hat{v})u(s_i^m;t) \tag{16}$$

if $m > 1$.

The proof will include a series of lemmas. The first lemma shows that, for voters who vote for the median party or any party to the right of the median in profile $\hat{v}$, a deviation by voting for any party to the left of the median is not profitable.

Lemma 3 *Assume $m > 1$. For every $t \geq \underline{y}_m$ and every $j \leq m-1$, there exists $\bar{\epsilon} > 0$ such that $U(\hat{v}; t|\epsilon) \geq U(j, \hat{v}_{-t}; t|\epsilon)$ for every $\epsilon \in [0, \bar{\epsilon}]$.*

Proof: Take any $t \geq \underline{y}_m$ and let $h = \hat{v}_t$. Note that $h \geq m$. Take any $j \leq m-1$. First, suppose $t_{M-1} \geq \underline{y}_m$. Then $\sum_{i=1}^{m-1} b_i(\hat{v}) < M-1$, implying $k(j, \hat{v}_{-t}) = m$. Then, for every $\epsilon \in [0, 1]$,

$$U(\hat{v}; t|\epsilon) - U(j, \hat{v}_{-t}; t|\epsilon) = \frac{1}{n}[u(s_h^m; t) - u(s_j^m; t)]. \tag{17}$$

If $t \geq t_{M+1}$, then $h = \alpha(t) \in \arg\max\{u(s_i^m; t)|i \in L\}$. So, $u(s_h^m; t) \geq u(s_j^m; t)$, implying (17) is nonnegative. If $t \in [\underline{y}_m, t_M]$, then $h = m$. Since $t \geq \underline{y}_m = \frac{s_{m-1}^m + s_m^m}{2}$, $u(s_m^m; t) \geq u(s_j^m; t)$. Thus, (17) is nonnegative.

Now suppose $t_{M-1} < \underline{y}_m$. Then, $\sum_{i=1}^{m-1} b_i(\hat{v}) = M-1$. A1 implies that, for each $i = 2, \ldots, \ell$, $(\frac{\theta_{i-1}+\theta_i}{2}, \theta_i) \cap T \neq \emptyset$. Since $t_M \geq \theta_m$, it must be that $t_{M-1} > \frac{\theta_{m-1}+\theta_m}{2}$. Since $\underline{y}_m > t_{M-1}$, $\underline{y}_m > \frac{\theta_{m-1}+\theta_m}{2}$, which implies $\underline{x}_m(q) > \theta_{m-1}$. Then, it must be that $q > \theta_{m-1}$. Since $t_{M-1} \in (\frac{\theta_{m-1}+\theta_m}{2}, \underline{y}_m)$, $\alpha(t_{M-1}) = m-1$, so $b_{m-1}(\hat{v}) > 0$. This implies that $k(j, \hat{v}_{-t}) = m-1$, and, so, $p_i(j, \hat{v}_{-t}) = s_i^{m-1}$ for every $i \in L$. We consider two cases separately: $m > 2$ and $m = 2$.

First, suppose $m > 2$. Since $t_1 < \theta_1$, $\alpha(t_1) = 1$. So, $b_1(\hat{v}) > 0$, implying $b_{m-1}(j, \hat{v}_{-t}) < M$. Then, for every $\epsilon \in [0, 1]$,

$$U(j, \hat{v}_{-t}; t|\epsilon) = \frac{1}{n}\left(\sum_{i \in L} b_i(\hat{v})u(s_i^{m-1}; t) + [u(s_j^{m-1}; t) - u(s_h^{m-1}; t)]\right). \tag{18}$$

Suppose $q > \theta_m$, i.e., $\underline{x}_m(q) = 2\theta_m - q$. Then, $A(j, \hat{v}_{-t}) = [2\theta_{m-1} - q, q]$. Since $\underline{x}_m(q) \in (\theta_{m-1}, \theta_m)$, for every $i \geq m$, $s_i^m = s_i^{m-1}$. For every $i \leq m-1$, $s_i^m = 2\theta_m - q > s_i^{m-1}$. Since $t \geq t_M > 2\theta_m - q$, $u(2\theta_m - q; t) > u(s_i^{m-1}; t)$ for every $i \leq m-1$. Also, since $t > \theta_m$,

$s_j^{m-1} < 2\theta_m - q$, and $s_h^{m-1} \in [\theta_m, q]$, it must be the case that $u(s_h^{m-1};t) > u(s_j^{m-1};t)$. Then,

$$\begin{aligned} U(\hat{v};t|\epsilon) - U(j,\hat{v}_{-t};t|\epsilon) &= \frac{1}{n}\left(\sum_{i=1}^{m-1} b_i(\hat{v})[u(2\theta_m - q;t) - u(s_i^{m-1};t)] + [u(s_h^{m-1};t) - u(s_j^{m-1};t)]\right) \\ &> 0 \end{aligned}$$

for every $\epsilon \in [0,1]$. Suppose $q \in (\theta_{m-1}, \theta_m)$. Then, for every $i \geq m$, $s_i^{m-1} = q$, and, for every $i \leq m-1$, $s_i^m = q$. Thus, for every $\epsilon \in [0,1]$,

$$\begin{aligned} U(\hat{v};t|\epsilon) - U(j,\hat{v}_{-t};t|\epsilon) &= \frac{1}{n}\left(\sum_{i=1}^{m-1} b_i(\hat{v})[u(q;t) - u(s_i^{m-1};t)]\right. \\ &\left. + \sum_{i=m}^{\ell} b_i(\hat{v})[u(s_i^m;t) - u(q;t)] + [u(q;t) - u(s_j^{m-1};t)]\right) \end{aligned} \tag{19}$$

Since $s_i^{m-1} < q < t$, $u(q;t) > u(s_i^{m-1};t)$ for every $i \leq m-1$ (including j). Since $t > \theta_m$ and $s_i^m \in [\theta_m, 2\theta_m - q]$, $u(s_i^m;t) > u(q;t)$ for every $i \geq m$. Thus, (19) is positive.

Now suppose $m = 2$. Then, $b_1(\hat{v}) = M - 1$ and $j = 1$. Thus, party 1 is the majority party in $(j, \hat{v}_{-t})$. Let $C(t) = \frac{1}{n}\sum_{i \in L} b_i(\hat{v}) u(s_i^m;t) - u(\theta_1;t)$ and let

$$G(t) = \frac{1}{n}\left(\sum_{i \in L} b_i(\hat{v}) u(s_i^m;t) - \sum_{i \in L} b_i(j,\hat{v}_{-t}) u(s_i^{m-1};t)\right).$$

Then, for every $\epsilon \in [0,1]$,

$$U(\hat{v};t|\epsilon) - U(j,\hat{v}_{-t};t|\epsilon) = (1-\epsilon)C(t) + \epsilon G(t). \tag{20}$$

Note that $t > \theta_m$, $s_i^m \in [\underline{x}_m(q), \overline{x}_m(q)]$, and $\theta_1 < \underline{x}_m(q)$. This implies that $u(s_i^m;t) > u(\theta_1;t)$ for every $i \in L$. Hence, $C(t) > 0$. If $G(t) \geq 0$, then (20) is positive for every $\epsilon \in [0,1]$. If $G(t) < 0$, then let $\bar{\epsilon} = \frac{C(t)}{C(t) - G(t)}$. Then, for every $\epsilon \in [0, \bar{\epsilon}]$, $U(\hat{v};t|\epsilon) \geq U(j,\hat{v}_{-t};t|\epsilon)$, which completes the proof of the lemma. ∎

We are ready to prove the first statement of the proposition. Suppose $t_{M+1} \in X_m$. Since $\theta_m \le t_M < t_{M+1} \le \overline{y}_m$, $t_M \in X_m$. Then, it must be the case that $t_{M-1} < \underline{y}_m$, implying $m > 1$. Note that $[\underline{y}_m, t_M] \cap T = \{t_M\}$. Since $t_M \in X_m$, $\hat{v}_{t_M} = m \in \arg\max\{u(s_i^m; t_M) | i \in L\}$. Also, by construction, $\hat{v}_t = \alpha(t) \in \arg\max\{u(s_i^m; t) | i \in L\}$ for every $t \in T \setminus \{t_M\}$. Thus, $\hat{v}$ is strategically sincere, i.e., $T^*(\hat{v}) = T$.

I now will show that $\hat{v} \in V(T, \theta, q)$. Since $t_{M+1} \in X_m$, $\hat{v}_{t_{M+1}} = \alpha(t_{M+1}) = m$. This implies that $\sum_{i=m+1}^{\ell} b_i(\hat{v}) < M - 1$. Then, for every $t \le t_{M-1}$, voter t is neither majority-pivotal nor median-pivotal. Thus, by Lemma 1, $\hat{v}_t$ is a robust best response to $\hat{v}_{-t}$ for every $t \le t_{M-1}$. Take any $t \ge t_M$, and consider voter t's deviation by voting for any $j \ne \hat{v}_t$. If $j \ge t_M$, the deviation would not change the median party, i.e., $k(j, \hat{v}_{-t}) = m$. Then, for every $i \in L$, $p_i(j, \hat{v}_{-t}) = s_i^m = p_i(\hat{v})$. Since $\hat{v}_t \in \arg\max\{u(s_i^m; t) | i \in L\}$, $U(\hat{v}; t|\epsilon) \ge U(j, \hat{v}_{-t}; t|\epsilon)$ for every $\epsilon \in [0, 1]$. Suppose $j \le m - 1$. By Lemma 3, there exists $\bar{\epsilon} > 0$ such that $U(\hat{v}; t|\epsilon) \ge U(j, \hat{v}_{-t}; t|\epsilon)$ for every $\epsilon \in [0, \bar{\epsilon}]$. Thus, $\hat{v} \in V(T, \theta, q)$, which completes the proof of the first statement in Proposition 4.

To prove the second statement, assume $t_{M+1} \notin X_m$. Let

$$\beta(t) = \min\left(\arg\max\left\{u(\theta_i; t) \middle| i \in \arg\max\left\{u(s_j^m; t) \middle| j \in \arg\max\{u(s_h^{m+1}; t) | h \in L\}\right\}\right\}\right).$$

Define voting profile $\tilde{v}$ by

$$\tilde{v}_t = \begin{cases} m+1 & \text{if } t \in [t_M, \overline{y}_{m+1}], \\ \beta(t) & \text{otherwise.} \end{cases} \tag{21}$$

I will prove that either $\hat{v}$ or $\tilde{v}$ is a robust voting equilibrium of $G(T, \theta, q, 0)$. Define voting profiles $\hat{v}'$ and $\tilde{v}'$ by the following.

$$\hat{v}'_t = \begin{cases} m+1 & \text{if } t = t_M, \\ \hat{v}_t & \text{otherwise;} \end{cases} \tag{22}$$

and

$$\tilde{v}'_t = \begin{cases} m & \text{if } t = t_M, \\ \tilde{v}_t & \text{otherwise.} \end{cases} \tag{23}$$

That is, $\hat{v}'$ is the voting profile in which the median voter unilaterally deviates from $\hat{v}$ by voting for $m+1$, and $\tilde{v}'$ is the voting profile in which the median voter unilaterally deviates from $\tilde{v}$ by voting for m. For each $t \in T$ and each $\epsilon \in [0,1]$, let $\hat{\Delta}(t|\epsilon) = U(\hat{v}; t|\epsilon) - U(\hat{v}'; t|\epsilon)$ and $\tilde{\Delta}(t|\epsilon) = U(\tilde{v}; t|\epsilon) - U(\tilde{v}'; t|\epsilon)$.

Note that, for every $t \leq \underline{y}_m$, $\alpha(t) \leq m$. Also, since $t_{M+1} \geq \overline{y}_m$, for every $t \geq t_{M+1}$, $\alpha(t) \geq m+1$. Thus,

$$\sum_{i=1}^{m} b_i(\hat{v}) = M \quad \text{and} \quad \sum_{i=1}^{m} b_i(\hat{v}') = M - 1. \tag{24}$$

Since $t_{M-1} < \frac{\theta_m + \theta_{m+1}}{2} \leq \frac{s_m^{m+1} + s_{m+1}^{m+1}}{2}$, $\beta(t) \leq m$ for every $t \leq t_{M-1}$. Clearly, for every $t \geq \overline{y}_{m+1}$, $\beta(t) \geq m+1$. Hence,

$$\sum_{i=1}^{m} b_i(\tilde{v}) = M - 1 \quad \text{and} \quad \sum_{i=1}^{m} b_i(\tilde{v}') = M. \tag{25}$$

An implication of (24) and (25) is that $k(\hat{v}) = k(\tilde{v}') = m$ and $k(\hat{v}') = k(\tilde{v}) = m+1$. Thus, for every $i \in L$, $p_i(\hat{v}) = p_i(\tilde{v}') = s_i^m$ and $p_i(\hat{v}') = p_i(\tilde{v}) = s_i^{m+1}$. I now present a series of lemmas.

Lemma 4 *For each given $\epsilon \in [0,1]$, $\hat{\Delta}(t|\epsilon)$ is decreasing in t and $\tilde{\Delta}(t|\epsilon)$ is increasing in t.*

Proof: For each $t \in T$, let

$$\hat{D}(t) = \frac{1}{n} \left[\sum_{i \in L} b_i(\hat{v}) u(s_i^m; t) - \sum_{i \in L} b_i(\hat{v}') u(s_i^{m+1}; t) \right] \tag{26}$$

and

$$\tilde{D}(t) = \frac{1}{n}\left[\sum_{i\in L} b_i(\tilde{v})u(s_i^{m+1};t) - \sum_{i\in L} b_i(\tilde{v}')u(s_i^m;t)\right] \tag{27}$$

I first claim $\hat{D}$ is decreasing and $\tilde{D}$ is increasing in t. Since $\hat{v}$ and $\hat{v}'$ differ only in that $\hat{v}_{t_M} = m$ and $\hat{v}'_{t_M} = m+1$, we write

$$\begin{aligned} \hat{D}(t) &= \frac{1}{n}\left(\sum_{i\in L} b_i(\hat{v})[u(s_i^m;t) - u(s_i^{m+1};t)] + [u(s_m^{m+1};t) - u(s_{m+1}^{m+1};t)]\right) \\ &= \frac{1}{n}\left(\sum_{i\in L} b_i(\hat{v})[f(|s_i^m - t|) - f(|s_i^{m+1} - t|)] + [f(|s_m^{m+1} - t|) - f(|s_{m+1}^{m+1} - t|)]\right) \end{aligned} \tag{28}$$

Note that, for each $i \in L$, $s_i^m \leq s_i^{m+1}$ and $s_m^{m+1} \leq s_{m+1}^{m+1}$. Then, since f is decreasing and concave, for each $i \in L$, $f(|s_i^m - t|) - f(|s_i^{m+1} - t|)$ is decreasing in t and $f(|s_m^{m+1} - t|) - f(|s_{m+1}^{m+1} - t|)$ is decreasing in t. Hence $\hat{D}$ is decreasing in t. A symmetric argument proves that $\tilde{D}$ is increasing in t.

Let $\epsilon \in [0,1]$. First, suppose that $b_m(\hat{v}) < M$ and $b_{m+1}(\hat{v}') < M$. Then, $\hat{\Delta}(t|\epsilon) = \hat{D}(t)$, implying $\hat{\Delta}(t|\epsilon)$ is decreasing in t. Second, suppose $b_m(\hat{v}) = M$. Since $t_1 < \theta_1$, $\hat{v}_{t_1} = 1$. This, together with (24), implies that $m = 1$. Since $t_n \geq \theta_\ell$, $\hat{v}_{t_n} = \ell$, implying $b_{m+1}(\hat{v}') < M$. Then,

$$\hat{\Delta}(t|\epsilon) = (1-\epsilon)\left[u(\theta_1;t) - \frac{1}{n}\sum_{i\in L} b_i(\hat{v}')u(s_i^{m+1};t)\right] + \epsilon\hat{D}(t).$$

But since $\theta_1 \leq s_i^{m+1}$ for every $i \in L$, the expression in the square bracket is decreasing in t. Thus, $\hat{\Delta}(t|\epsilon)$ is decreasing in t. Lastly, suppose $b_{m+1}(\hat{v}') = M$. Again since $\hat{v}_{t_n} = \hat{v}'_{t_n} = \ell$, it must be the case that $m+1 = \ell$. Then since $\hat{v}_{t_1} = 1$, $b_m(\hat{v}) < M - 1$. Then,

$$\hat{\Delta}(t|\epsilon) = (1-\epsilon)\left[\frac{1}{n}\sum_{i\in L} b_i(\hat{v})u(s_i^m;t) - u(\theta_\ell;t)\right] + \epsilon\hat{D}(t).$$

But since $s_i^m \leq \theta_\ell$ for every $i \in L$, the expression in the square bracket is decreasing in t. Thus, $\hat{\Delta}(t|\epsilon)$ is decreasing in t. A symmetric argument proves $\tilde{\Delta}(t|\epsilon)$ is increasing in t. ∎

Lemma 5 *The following is true.*

1. *If* $\hat{\Delta}(t_M|0) > 0$*, then* $\hat{v}$ *is a robust equilibrium of* $G(T, \theta, q, 0)$.

2. *If* $\tilde{\Delta}(t_M|0) > 0$*, then* $\tilde{v}$ *is a robust equilibrium of* $G(T, \theta, q, 0)$.

3. *If* $\hat{\Delta}(t_M|0) = \tilde{\Delta}(t_M|0) = 0$*, then either* $\hat{v}$ *or* $\tilde{v}$ *is a robust equilibrium of* $G(T, \theta, q, 0)$.

Proof: 1. Suppose $\hat{\Delta}(t_M) > 0$. Take any $t \in T$, and let $h = \hat{v}_t$. Assume $t \geq t_{M+1}$ and notice that $\hat{v}_t \geq m+1$. Consider voter t's deviation by voting for j. Suppose $j \geq m$. Then the deviation does not change the majority or the median status of party m. Since $h \in \arg\max\{u(s_i^m; t)|i \in L\}$, $U(\hat{v}; t|\epsilon) \geq U(j, \hat{v}_{-t}; t|\epsilon)$ for every $\epsilon \in [0,1]$. Suppose $j \leq m-1$. Then, by Lemma 3, $U(\hat{v}; t|\epsilon) \geq U(j, \hat{v}_{-t}; t|\epsilon)$ for sufficiently small ϵ. Thus, $\hat{v}_t$ is a robust best response to $\hat{v}_{-t}$.

Assume $t \leq t_M$. Again, consider voter t's deviation from $\hat{v}$ by voting for any $j \neq h$, i.e., we consider the profile $(j, \hat{v}_{-t})$. If $j \leq m$, then the deviation would not change the identity of the median or majority party. So, $k(j, \hat{v}_{-t}) = k(\hat{v}) = m$, and, for every $i \in L$, $p_i(j, \hat{v}_{-t}) = p_i(\hat{v}) = s_i^m$ And since, by construction, $h \in \arg\max\{u(s_i^m; t)|i \leq m\}$, $U(j, \hat{v}_{-t}; t|\epsilon) \leq U(\hat{v}; t|\epsilon)$ for every $\epsilon \in [0,1]$.

Now suppose $j \geq m+1$. Then $k(j, \hat{v}_{-t}) = m+1$ and $p_i(j, \hat{v}_{-t}) = s_i^{m+1}$ for every $i \in L$. Note that the only possible difference between $\hat{v}'$ and $(j, \hat{v}_{-t})$ is that, in $(j, \hat{v}_{-t})$, one vote for h in $\hat{v}$ is transferred to j, and, in $\hat{v}'$, one vote for m is transferred to $m+1$. I claim $U(\hat{v}'; t|\epsilon) \geq U(j, \hat{v}_{-t}; t|\epsilon)$ for every $\epsilon \in [0,1]$. To see this, first, suppose $m < \ell - 1$. Then, there is no majority party in $\hat{v}'$ or $(j, \hat{v}_{-t})$. So, for every $\epsilon \in [0,1]$,

$$U(\hat{v}'; t|\epsilon) - U(j, \hat{v}_{-t}; t|\epsilon) = \frac{1}{n}\bigg([u(s_{m+1}^{m+1}; t) - u(s_j^{m+1}; t)] + [u(s_h^{m+1}; t) - u(s_m^{m+1}; t)]\bigg).$$

Since $t \leq t_M$ and $m+1 \leq j$, we have $t < \theta_{m+1} = s_{m+1}^{m+1} \leq s_j^{m+1}$, implying $u(s_{m+1}^{m+1}; t) \geq u(s_j^{m+1}; t)$. If $h = m$, then clearly $u(s_h^{m+1}; t) = u(s_m^{m+1}; t)$. Suppose $h < m$. Then, since

$h = \alpha(t)$, $t \le \frac{s_h^m+\theta_m}{2}$. But since $s_h^m \le s_h^{m+1} \le s_m^{m+1}$ and $\theta_m \le s_m^{m+1}$, $\frac{s_h^m+\theta_m}{2} \le \frac{s_h^{m+1}+s_m^{m+1}}{2}$, implying $u(s_h^{m+1};t) \ge u(s_m^{m+1};t)$. Therefore, $U(\hat{v}';t|\epsilon) \ge U(j,\hat{v}_{-t};t|\epsilon)$. Second, suppose $m = \ell - 1$. Then, $m+1$ is the majority party in $\hat{v}'$ and $(j,\hat{v}_{-t})$, and $j = m+1 = \ell$. Then,

$$U(\hat{v}';t|\epsilon) - U(j,\hat{v}_{-t};t|\epsilon) = \frac{\epsilon}{n}[u(s_h^{m+1};t) - u(s_m^{m+1};t)] \ge 0.$$

Hence, the claim is true. This implies that, if $U(\hat{v};t|\epsilon) \ge U(\hat{v}';t|\epsilon)$ for sufficiently small ϵ, then $\hat{v}_t$ is a robust best response to $\hat{v}_{-t}$. Thus, it suffices to show that $\hat{\Delta}(t|\epsilon) \ge 0$ for sufficiently small ϵ. But since $t \le t_M$ and $\hat{\Delta}(t|\epsilon)$ is decreasing in t by Lemma 4, it suffices to show $\hat{\Delta}(t_M|\epsilon) \ge 0$ for sufficiently small ϵ. Suppose $1 < m < \ell - 1$. Then, for every $\epsilon \in [0,1]$, $\hat{\Delta}(t_M|\epsilon) = \hat{\Delta}(t_M|0) > 0$. Suppose $m = 1$ or $m = \ell - 1$. Then,

$$\hat{\Delta}(t_M|\epsilon) = (1-\epsilon)\hat{\Delta}(t_M|0) + \epsilon\hat{D}(t_M). \tag{29}$$

If $\hat{D}(t_M)$ is nonnegative, then $\hat{\Delta}(t_M|\epsilon) \ge 0$ for every $\epsilon \in [0,1]$. If $\hat{D}(t_M) < 0$, then $\hat{\Delta}(t_M|\epsilon) \ge 0$ for every $\epsilon \in [0, \frac{\hat{\Delta}(t_M|0)}{\hat{\Delta}(t_M|0) - \hat{D}(t_M)}]$. Therefore, $\hat{v} \in V(T,\theta,q)$.

2. A symmetric argument proves the second statement.

3. Suppose $\hat{\Delta}(t_M|0) = \tilde{\Delta}(t_M|0) = 0$. Again, note that, if $\hat{\Delta}(t_M|\epsilon) \ge 0$ for sufficiently small ϵ, then $\hat{v}$ is a robust equilibrium, and that, if $\tilde{\Delta}(t_M|\epsilon) \ge 0$ for sufficiently small ϵ, then $\tilde{v}$ is a robust equilibrium. If $1 < m < \ell - 1$, then $\hat{\Delta}(t_M|\epsilon) = \hat{\Delta}(t_M|0) = 0$ for every $\epsilon \in [0,1]$. Thus, $\hat{v}$ is a robust equilibrium. Suppose $m = 1$. Since $\hat{\Delta}(t_M|0) = 0$, we obtain from (29) that $\hat{\Delta}(t_M|\epsilon) = \epsilon\hat{D}(t_M)$. Similarly, because $\tilde{\Delta}(t_M|0) = 0$, $\tilde{\Delta}(t_M|\epsilon) = \epsilon\tilde{D}(t_M)$. So, it suffices to prove that either $\hat{D}(t_M) \ge 0$ or $\tilde{D}(t_M) \ge 0$.

Note that

$$\hat{\Delta}(t_M|0) = u(\theta_1;t_M) - \frac{1}{n}\sum_{i \in L} b_i(\hat{v}')u(s_i^2;t_M) \tag{30}$$

and

$$\tilde{\Delta}(t_M|0) = \frac{1}{n}\sum_{i\in L} b_i(\tilde{v})u(s_i^2;t_M) - u(\theta_1;t_M). \tag{31}$$

Since $\hat{\Delta}(t_M|0) = \tilde{\Delta}(t_M|0) = 0$, we have

$$\frac{1}{n}\sum_{i\in L} b_i(\hat{v}')u(s_i^2;t_M) = \frac{1}{n}\sum_{i\in L} b_i(\tilde{v})u(s_i^2;t_M) = u(\theta_1;t_M).$$

Then, from (26) and (27), we obtain that

$$\hat{D}(t_M) = \frac{1}{n}\sum_{i\in L} b_i(\hat{v})u(s_i^1;t_M) - u(\theta_1;t_M) \tag{32}$$

and

$$\tilde{D}(t_M) = u(\theta_1;t_M) - \frac{1}{n}\sum_{i\in L} b_i(\tilde{v}')u(s_i^1;t_M). \tag{33}$$

Since $t_M \leq \frac{\theta_1+\theta_2}{2}$, $u(\theta_1;t_M) \geq u(s_2^2;t_M)$ and, for every $i \geq 3$, $u(\theta_1;t_M) > u(s_i^2;t_M)$. Then since $\hat{\Delta}(t_M|0) = 0$, (30) implies $u(\theta_1;t_M) < u(s_1^2;t_M)$. Then, it must be that $s_1^2 = \underline{x}_2(q) \in (\theta_1,\theta_2)$. Suppose $\underline{x}_2(q) = q$. Then, for every $i \geq 2$, $s_i^1 = q$. And since $u(q;t_M) > u(\theta_1;t_M)$, we conclude $\hat{D}(t_M) > 0$ from (32). This implies $\hat{\Delta}(t_M|\epsilon) \geq 0$ for every $\epsilon \in [0,1]$. Hence, $\hat{v} \in V(T,\theta,q)$. Now suppose $\underline{x}_2(q) = 2\theta_2 - q$. This implies $q > \theta_2$. Comparing the definitions of $\hat{v}$ and $\tilde{v}'$, we first conclude that $T_1(\hat{v}) = T_1(\tilde{v}') = \{t_1,\ldots,t_M\}$. Note that, since $q > \theta_2$, $\overline{x}_1(q) = \overline{x}_2(q) = q$, which implied that, for every $i \geq 2$, $s_i^1 = s_i^2$. Consider a voter $t \in [t_{M+1}, \overline{y}_2]$. By the definition of $\tilde{v}'$, $\tilde{v}'_t = 2$. Since $t \geq t_{M+1} > \overline{y}_1 = \frac{\theta_1+\theta_2}{2}$, $u(s_1^1;t) = u(\theta_1;t) < u(\theta_2;t) = u(s_2^1;t)$. Since $t \leq \overline{y}_2$, $u(s_2^1;t) = u(s_2^2;t) \geq u(s_3^2;t) = u(s_3^1;t)$. Thus, $\hat{v}_t = \alpha(t) = 2$ as well. Since $s_i^1 = s_i^2$ for every $i \geq 3$, $\alpha(t) = \beta(t)$ for every $t > \overline{y}_2$. Thus, we conclude that $\hat{v}_t = \tilde{v}'_t$ for every $t \in T$, which implies $b_i(\hat{v}) = b_i(\tilde{v}')$ for every $i \in L$. Then, from (32) and (33), we conclude that either $\hat{D}(t_M) \geq 0$ or $\tilde{D}(t_M) \geq 0$. Thus, either $\hat{v}$ or $\tilde{v}$ is a robust equilibrium. A symmetric argument proves the statement for the case that $m = \ell - 1$. ∎

Lemma 6 $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) \geq 0$.

Proof: We consider three mutually exclusive and jointly exhaustive cases.

CASE 1: Assume $m = 1$.

Note that party 1 is the majority party in $\hat{v}$ and $\tilde{v}'$, and party 2 is the median party in $\hat{v}'$ and $\tilde{v}$. Then, by definition,

$$\hat{\Delta}(t_M|0) = u(\theta_1; t_M) - \frac{1}{n}\sum_{i\in L} b_i(\hat{v}')u(s_i^2; t_M) \tag{34}$$

and

$$\tilde{\Delta}(t_M|0) = \frac{1}{n}\sum_{i\in L} b_i(\tilde{v})u(s_i^2; t_M) - u(\theta_1; t_M). \tag{35}$$

By adding (34) and (35), we write

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{1}{n}\sum_{i\in L}[b_i(\tilde{v}) - b_i(\hat{v}')]u(s_i^2; t_M). \tag{36}$$

From (21), we write

$$\tilde{v}_t = \begin{cases} 2 & \text{if } t \in [t_M, \overline{y}_2], \\ \beta(t) & \text{otherwise.} \end{cases} \tag{37}$$

Also, from (14) and (22), we write

$$\hat{v}'_t = \begin{cases} 1 & \text{if } t \leq t_{M-1} \\ 2 & \text{if } t = t_M \\ \alpha(t) & \text{otherwise.} \end{cases} \tag{38}$$

Since $t_{M-1} < \frac{\theta_1+\theta_2}{2}$ and $\theta_1 \leq s_1^2 < s_2^2 = \theta_2$, for every $t \leq t_{M-1}$, $\arg\max\{u(s_i^2;t)|i \in L\} = \{1\}$. Thus, for every $t \leq t_{M-1}$, $\tilde{v}_t = \beta(t) = 1$. For any $t > \overline{y}_2$, clearly $\beta(t) \neq 1$. Thus, $T_1(\tilde{v}) = \{t_1, \ldots, t_{M-1}\}$. Also, since $t_{M+1} > \overline{y}_1$, for any $t \geq t_{M+1}$, $\alpha(t) \neq 1$. So,

$T_1(\hat{v}') = \{t_1, \ldots, t_{M-1}\}$. Therefore, $b_1(\tilde{v}) = b_1(\hat{v}')$. Then, (36) is reduced to

$$\Delta(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{1}{n}\sum_{i=2}^{\ell}[b_i(\tilde{v}) - b_i(\hat{v}')]u(s_i^2; t_M). \tag{39}$$

First, suppose $q < \theta_1$. Let $L^- = \{i \in L|\theta_i < 2\theta_1 - q\}$. Suppose $L^- = L$. Then, $s_i^1 = s_i^2 = \theta_i$ for every $i \in L$. Then, $\alpha(t) = 2$ if and only if $t \in (\frac{\theta_1+\theta_2}{2}, \frac{\theta_2+\theta_3}{2}] = (\overline{y}_1, \overline{y}_2]$. Since $t_{M+1} > \overline{y}_1$, $T_2(\tilde{v}) = T_2(\hat{v}')$. Also, since $s_i^1 = s_i^2 = \theta_i$ for every $i \in L$, $\alpha(t) = \beta(t)$ for every $t \in L$. Hence, $b_i(\tilde{v}) = b_i(\hat{v}')$ for every $i = 2, \ldots, \ell$. Therefore, $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$. Suppose $L^- \neq L$. Let $\underline{j} = \max L^-$.

Suppose $\underline{j} \geq 2$. For every $t \leq \frac{\theta_{\underline{j}}+2\theta_1-q}{2}$,

$$\alpha(t) = \beta(t) = \min\left(\arg\max\{u(\theta_i; t)|i = 1, \ldots, \underline{j}\}\right).$$

For every $t > \frac{\theta_{\underline{j}}+s^2_{\underline{j}+1}}{2}$,

$$\alpha(t) = \beta(t) = \min\left(\arg\max\left\{u(\theta_i; t)|i \in \arg\max\{u(s_j^2; t)|j = \underline{j}+1, \ldots, \ell\}\right\}\right).$$

Let $\tilde{T} = \{t \in T|\frac{\theta_{\underline{j}}+2\theta_1-q}{2} < t \leq \frac{\theta_{\underline{j}}+s^2_{\underline{j}+1}}{2}\}$. For every $t \in \tilde{T}$, $\hat{v}'_t = \underline{j}+1$ and $\tilde{v}_t = \underline{j}$. Hence,

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{|\tilde{T}|}{n}[u(\theta_{\underline{j}}; t_M) - u(s^2_{\underline{j}+1}; t_M)] \geq 0$$

because $t_M < \theta_{\underline{j}} < s^2_{\underline{j}+1}$.

Suppose $\underline{j} = 1$. Then, for every $t > t_M$,

$$\alpha(t) = \beta(t) = \min\left(\arg\max\left\{u(\theta_i; t)|i \in \arg\max\{u(s_j^2; t)|j = 2, \ldots, \ell\}\right\}\right). \tag{40}$$

This implies that $T_2(\hat{v}') = \{t \in T|t \in [t_M, \overline{y}_2]\}$, and so $b_2(\tilde{v}) = b_2(\hat{v}')$. Also, for every

$i = 3, \ldots, \ell$, $b_i(\hat{v}') = b_i(\tilde{v})$. Thus, $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$.

Suppose $\theta_1 < q < \theta_2$. Then, for every $i = 2, \ldots, \ell$, $s_i^1 = q$, which implies that, for every $t > t_M$, (40) is true. Then, $b_2(\tilde{v}) = b_2(\hat{v}) + 1$, and, for every $i = 3, \ldots, \ell$, $b_i(\hat{v}) = b_i(\tilde{v})$. Thus, $\Delta(t_M|0) + \tilde{\Delta}(t_M|0) = 0$.

Lastly, suppose $q > \theta_2$. Then, for every $i = 2, \ldots, \ell$, $s_i^1 = s_i^2$. Then, again, for every $t > t_M$, (40) is true, implying $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$.

CASE 2: Assume $m = \ell - 1$. A symmetric argument can prove the statement for this case.

CASE 3: Assume $1 < m < \ell - 1$.

Party m is the median party in $\hat{v}$ and $\tilde{v}'$, and party $m+1$ is the median party in $\hat{v}'$ and $\tilde{v}$. Then,

$$\hat{\Delta}(t_M|0) = \frac{1}{n}\left(\sum_{i \in L} b_i(\hat{v})[u(s_i^m; t_M) - u(s_i^{m+1}; t_M)] + u(s_m^{m+1}; t_M) - u(s_{m+1}^{m+1}; t_M) \right) \quad (41)$$

and

$$\tilde{\Delta}(t_M|0) = \frac{1}{n}\left(\sum_{i \in L} b_i(\tilde{v})[u(s_i^{m+1}; t_M) - u(s_i^m; t_M)] + u(s_{m+1}^m; t_M) - u(s_m^m; t_M) \right). \quad (42)$$

By adding (41) and (42), we obtain

$$\begin{aligned} \hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) \;&=\; \frac{1}{n}\Bigg(\sum_{i \in L}[b_i(\hat{v}) - b_i(\tilde{v})][u(s_i^m; t_M) - u(s_i^{m+1}; t_M)] \qquad (43) \\ &\quad + u(s_m^{m+1}; t_M) - u(s_{m+1}^{m+1}; t_M) + u(s_{m+1}^m; t_M) - u(s_m^m; t_M) \Bigg). \end{aligned}$$

First, assume $q < \theta_m$. Let $L^- = \{i \in L | \theta_i < 2\theta_m - q\}$. Note that, for every $i \in L^-$, $s_i^m = s_i^{m+1}$. In particular, $m \in L^-$. Also, if $i \notin L^-$, then $s_i^m = 2\theta_m - q$. Then, (43) is reduced to

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{1}{n}\Bigg(\sum_{i \in L \setminus L^-} [b_i(\hat{v}) - b_i(\tilde{v})][u(2\theta_m - q; t_M) - u(s_i^{m+1}; t_M)] \\ + u(s_{m+1}^m; t_M) - u(\theta_{m+1}; t_M). \Bigg) \quad (44)$$

Suppose $L^- = L$. Then, $s_{m+1}^m = \theta_{m+1}$, so $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$. Suppose $L^- \neq L$. Let $\underline{j} = \max L^-$. Suppose $\underline{j} \geq m+1$. If $t \leq \frac{\theta_{\underline{j}} + 2\theta_m - q}{2}$, then $\hat{v}_t \in L^-$ and $\tilde{v}_t \in L^-$. If $t > \frac{\theta_{\underline{j}} + s_{\underline{j}+1}^{m+1}}{2}$, then

$$\alpha(t) = \beta(t) = \min\left(\arg\max \left\{ u(\theta_i; t) | i \in \arg\max\{u(s_j^{m+1}; t) | j = \underline{j}+1, \ldots, \ell\} \right\} \right),$$

so $\hat{v}_t = \tilde{v}_t$. Let $\tilde{T} = \{t \in T | \frac{\theta_{\underline{j}} + 2\theta_m - q}{2} < t \leq \frac{\theta_{\underline{j}} + s_{\underline{j}+1}^{m+1}}{2}\}$. For every $t \in \tilde{T}$, $\hat{v}_t = \underline{j}+1$ and $\tilde{v}_t = \underline{j}$. This implies that for every $i > \underline{j}+1$, $b_i(\hat{v}) = b_i(\tilde{v})$, and $b_{\underline{j}+1}(\hat{v}) - b_{\underline{j}+1}(\tilde{v}) = |\tilde{T}|$. Then, from (44), we have

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{|\tilde{T}|}{n}\left[u(2\theta_m - q; t_M) - u(s_{\underline{j}+1}^{m+1}; t_M) \right] \geq 0,$$

because $t_M < 2\theta_m - q < s_{\underline{j}+1}^{m+1}$. Now suppose $\underline{j} = m$. For every $t \leq \frac{\theta_{m+1} + s_{m+2}^{m+1}}{2}$, $\alpha(t) \leq m+1$ and $\beta(t) \leq m+1$. For every $t > \frac{\theta_{m+1} + s_{m+2}^{m+1}}{2}$,

$$\alpha(t) = \beta(t) = \min\left(\arg\max \left\{ u(\theta_i; t) | i \in \arg\max\{u(s_j^{m+1}; t) | j = m+2, \ldots, \ell\} \right\} \right),$$

so $\hat{v}_t = \tilde{v}_t$. This implies that for every $t > m+1$, $T_i(\hat{v}) = T_i(\tilde{v})$, so $b_i(\hat{v}) = b_i(\tilde{v})$. Also, from the strategies, $T_{m+1}(\hat{v}) = \{t \in T | \alpha(t) = m+1\} = \{t \in T | t_{M+1} \leq t \leq \overline{y}_{m+1}\}$, and $T_{m+1}(\tilde{v}) = \{t \in T | t_M \leq t \leq \overline{y}_{m+1}\}$, implying $b_{m+1}(\hat{v}) - b_{m+1}(\tilde{v}) = -1$. Then, from (44), we conclude that $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$.

Second, assume $\theta_m < q < \theta_{m+1}$. Then, for every $i = 1, \ldots, m$, $s_i^{m+1} = q$, and, for every

$i = m+1, \ldots, \ell$, $s_i^m = q$. Then from (43) we have

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{1}{n}\Bigg(\sum_{i=1}^{m} [b_i(\hat{v}) - b_i(\tilde{v})][u_i(s_i^m; t_M) - u_i(q; t_M)] + \sum_{i=m+1}^{\ell} [b_i(\hat{v}) - b_i(\tilde{v})][u_i(q; t_M) - u_i(s_i^{m+1}; t_M)] + 2u(q; t_M) - u(\theta_m; t_M) - u(\theta_{m+1}; t_M)\Bigg). \tag{45}$$

For every $t < \underline{y}_m$,

$$\alpha(t) = \beta(t) = \min\left(\arg\max \left\{ u(\theta_i; t) | i \in \arg\max\{u(s_j^m; t) | j = 1, \ldots, m-1\} \right\} \right).$$

For every $t > \overline{y}_{m+1}$,

$$\alpha(t) = \beta(t) = \min\left(\arg\max \left\{ u(\theta_i; t) | i \in \arg\max\{u(s_j^{m+1}; t) | j = m+2, \ldots, \ell\} \right\} \right).$$

Also, for every $t \in [\underline{y}_m, \overline{y}_{m+1}]$, $\{\hat{v}_t, \tilde{v}_t\} = \{m, m+1\}$. Therefore, for every $i \in L \setminus \{m, m+1\}$, $T_i(\hat{v}) = T_i(\tilde{v})$, implying $b_i(\hat{v}) = b_i(\tilde{v})$. Note that $t_{M+1} > \overline{y}_m$. So, $T_m(\hat{v}) = \{t \in T | t \in [\underline{y}_m, t_M]\}$ and $T_m(\tilde{v}) = \{t \in T | t \in [\underline{y}_m, t_{M-1}]\}$. This implies $b_m(\hat{v}) - b_m(\tilde{v}) = 1$. Also, $T_{m+1}(\hat{v}) = \{t \in T | t \in [t_{m+1}, \overline{y}_{m+1}]\}$ and $T_{m+1}(\tilde{v}) = \{t \in T | t \in [t_M, \overline{y}_{m+1}]\}$, implying $b_{m+1}(\hat{v}) = b_{m+1}(\tilde{v}) = -1$. Then, from (45), we conclude that $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$.

Lastly, assume $q > \theta_{m+1}$. Let $L^+ = \{i \in L | \theta_i > 2\theta_{m+1} - q\}$. Then, for every $i \in L^+$, $s_i^m = s_i^{m+1}$, and in particular $m+1 \in L^+$. For every $i \notin L^+$, $s_i^{m+1} = 2\theta_{m+1} - q$. Then, we have

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = \frac{1}{n}\Bigg(\sum_{i \in L \setminus L^+} [b_i(\hat{v}) - b_i(\tilde{v})][u(s_i^m; t_M) - u(2\theta_{m+1} - q; t_M)] + u(s_m^{m+1}; t_M) - u(\theta_m; t_M)\Bigg). \tag{46}$$

First, if $L^+ = L$, then clearly $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$. Suppose $L^+ \neq L$. Let $\underline{j} = \min L^+$. Suppose $\underline{j} \leq m$. Then, if $t \leq \frac{\theta_{\underline{j}} + s^m_{\underline{j}-1}}{2}$, then $\alpha(t) = \beta(t) \leq \underline{j}-1$. Let $\tilde{T} = \{t \in T | \frac{\theta_{\underline{j}} + s^m_{\underline{j}-1}}{2} < t \leq \frac{\theta_{\underline{j}} + 2\theta_{m+1} - q}{2}\}$. If $t \in \tilde{T}$, then $\hat{v}_t = \alpha(t) = \underline{j}$ and $\tilde{v}_t = \beta(t) = \underline{j} - 1$. Then, for every $i < \underline{j} - 1$, $b_i(\hat{v}) - b_i(\tilde{v}) = 0$, and $b_{\underline{j}-1}(\hat{v}) - b_{\underline{j}-1}(\tilde{v}) = -|\tilde{T}|$. Then, from (46), we have

$$\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = -\frac{|\tilde{T}|}{n}\left[u(s^m_{\underline{j}-1}; t_M) - u(2\theta_{m+1} - q; t_M)\right] \geq 0$$

because $s^m_{\underline{j}-1} > 2\theta_{m+1} - q > \theta_m > t_M$. Suppose $\underline{j} = m + 1$. If $t < \underline{y}_m$, then $\alpha(t) = \beta(t)$. Thus, for every $i < m$, $b_i(\hat{v}) - b_i(\tilde{v}) = 0$. From the strategies, $T_m(\hat{v}) = \{t \in T | t \in [\underline{y}_m, t_M]\}$ and $T_m(\tilde{v}) = \{t \in T | t \in [\underline{y}_m, t_{M+1}]\}$. So, $b_m(\hat{v}) - b_m(\tilde{v}) = 1$. Then, clearly from (46) $\hat{\Delta}(t_M|0) + \tilde{\Delta}(t_M|0) = 0$. ■

Lemma 6 implies that either $\hat{\Delta}(t_M|0) > 0$, or $\tilde{\Delta}(t_M|0) > 0$, or $\hat{\Delta}(t_M|0) = \tilde{\Delta}(t_M|0) = 0$. Then, by Lemma 5, either $\hat{v}$ or $\tilde{v}$ is a robust equilibrium of $G(T, \theta, q, 0)$.

Suppose $\hat{v} \in V(T, \theta, q)$. By construction, for every $t \notin [\underline{y}_m, t_M]$, $\hat{v}_t = \alpha(t)$, so $\hat{v}_t$ is strategically sincere in $\hat{v}$. For any $t \in [\underline{y}_m, t_M]$, $\hat{v}_t = m$, and $m \notin \arg\max\{u(s^m_i; t) | i \in L\}$ if and only if $t > \overline{y}_m$. Therefore, $T \setminus T^*(\hat{v}) = \{t \in T | \overline{y}_m < t \leq t_M\}$. Suppose $\tilde{v}$ is a robust equilibrium. By construction, for every $t \notin [t_M, \overline{y}_{m+1}]$, $\tilde{v}_t = \beta(t)$, so $\tilde{v}_t$ is strategically sincere in $\tilde{v}$. For any $t \in [t_M, \overline{y}_{m+1}]$, $\tilde{v}_t = m + 1$, and $m + 1 \notin \arg\max\{u(s^{m+1}_i; t) | i \in L\}$ if and only if $t < \underline{y}_{m+1}$. Therefore, $T \setminus T^*(\tilde{v}) = \{t \in T | t_M \leq t < \underline{y}_{m+1}\}$. ■

Proof of Proposition 5

Assume that v and v' are strategically sincere robust equilibria of $G(T, \theta, q)$. By Proposition 6, $k(v) = k(v') = m$. Then, since v and v' are strategically sincere and A2 holds, $b_m(v) = b_m(v') = |X_m|$. If $|X_m| \geq M$, then $\lambda^v = \lambda^{v'}$ as both of them are the degenerate lottery on

θ_m. Suppose $|X_m| < M$. Then, for each $x \in \mathbb{R}$,

$$\lambda^v(x) = \frac{1}{n} \sum_{\{i \in L | s_i^m = x\}} b_i(v) \text{ and } \lambda^{v'}(x) = \frac{1}{n} \sum_{\{i \in L | s_i^m = x\}} b_i(v').$$

Since A2 holds, and v and v' are strategically sincere, for every x with $\{i \in L | s_i^m = x\} \neq \emptyset$,

$$\frac{1}{n} \sum_{\{i \in L | s_i^m = x\}} b_i(v) = \frac{1}{n} \sum_{\{i \in L | s_i^m = x\}} b_i(v') = \left| \left\{ t \in T \,\middle|\, u(x;t) = \max\{u(s_j^m;t) | j \in L\} \right\} \right|,$$

which completes the proof. ■

Proof of Lemma 2

Let $v \in V(T, \theta, q)$ and let $t \in T \backslash T^*(v)$. Let $k = k(v)$ and $i = v_t$. Then, $i \notin \arg\max\{u(s_h^k;t) | h \in L\}$. Suppose $i \neq k$. Since L is finite, $\arg\max\{u(s_h^k;t) | h \in L\} \neq \emptyset$. Let $j \in \arg\max\{u(s_h^k;t) | h \in L\}$ and let $v' = (j, v_{-t})$. First, suppose $k(v') = k$. Then, $p_h(v') = s_h^k$ for every $h \in L$. If k is not the majority party in both v and v', then, for every $\epsilon \in [0, 1]$

$$U(v;t|\epsilon) - U(v';t|\epsilon) = \frac{1}{n}[u(s_i^k;t) - u(s_j^k;t)] < 0,$$

contradicting that v is a robust equilibrium. If k is the majority party in both v and v', then, for every $\epsilon \in (0, 1]$,

$$U(v;t|\epsilon) - U(v';t|\epsilon) = \frac{\epsilon}{n}[u(s_i^k;t) - u(s_j^k;t)] < 0,$$

a contradiction. If k is not the majority party in v, but it is in v', then it must be the case that $b_k(v) = M - 1$ and $j = k$. Then, for every $\epsilon \in [0, 1]$,

$$U(v;t|\epsilon) - U(v';t|\epsilon) = (1-\epsilon) \left[\sum_{h \in L} \frac{b_h(v)}{n} u(s_h^k;t) - u(s_j^k;t) \right] + \frac{\epsilon}{n}[u(s_i^k;t) - u(s_j^k;t)] < 0,$$

a contradiction.

Secondly, suppose $k(v') \neq k$. Suppose $i < k$. Then, it must be the case that $\sum_{h=1}^{k} b_h(v) = M$ and $j > k$. Since $j > k$, $t \geq \overline{y}_k$. Then, since $s_i^k < s_k^k = \theta_k < t$, we have $u(s_i^k; t) < u(s_k^k; t)$. I also claim that $b_k(v) < M-1$. To see this, suppose $b_k(v) = M-1$. Note that $b_i(v)+b_k(v) = M$ and $t \neq t_1$. This implies $v_{t_1} \geq k > 1$. But since $t_1 < \theta_1$, $u(s_1^k; t_1) > u(s_{v_{t_1}}; t_1)$. Also, party k would remain as the median party even after voter t_1's deviation by voting for party 1. Then, for every $\epsilon \in [0, 1)$, $U(v; t_1|\epsilon) < U(1, v_{-t_1}; t_1|\epsilon)$, a contradiction that implies that the claim is true. Then, for every $\epsilon \in [0, 1)$,

$$U(v; t|\epsilon) - U(k, v_{-t}; t|\epsilon) = \frac{1}{n}[u(s_i^k; t) - u(s_k^k; t)] < 0,$$

a contradiction. A symmetric argument will lead to a contradiction when $i > k$. ■

Proof of Proposition 6

Let v be a strategically sincere robust equilibrium. Let $k = k(v)$. Since v is strategically sincere, for every $t < \underline{y}_k$, $v_t < k$; and for every $t > \overline{y}_k$, $v_t > k$. Then, for k to be decisive, it must be that $t_M \in X_k$. Since $t_M \in [\theta_m, \frac{\theta_m+\theta_{m+1}}{2}]$, either $k = m$ or $k = m+1$. Suppose $k = m+1$. Then, $t_M = \frac{\theta_m+\theta_{m+1}}{2}$ and $\underline{x}_{m+1}(q) = \theta_m$. Since $s_m^{m+1} = \theta_m$ and $s_{m+1}^{m+1} = \theta_{m+1}$, we have $\max\{u(s_i^{m+1}; t_M)|i \in L\} = u(s_m^{m+1}; t_M) = u(s_{m+1}^{m+1}; t)$, contradicting A2. Thus, $k(v) = m$. ■

Proof of Proposition 7

Let $v \in V(T, \theta, q)$. Suppose v is strategically sincere and satisfies C1. Suppose v_t is strategic. Let $j = v_t$ and $k = k(v)$. Suppose $t \in (\underline{y}_k, \overline{y}_k)$, then $j = k$ since v is strategically sincere. By definition, $\frac{\theta_{k-1}+\theta_k}{2} \leq \underline{y}_k$ and $\overline{y}_k \leq \frac{\theta_k+\theta_{k+1}}{2}$. Then, $\arg\max\{u(\theta_h; t)|h \in L\} = \{k\}$, contradicting that v_t is strategic. Thus, either $t \leq \underline{y}_k$ or $t \geq \overline{y}_k$.

Suppose $t \leq \underline{y}_k$. Since v is strategically sincere, $j \leq k-1$. I claim that $p_j(v) = \underline{x}_k(q)$. Suppose not. Then $p_j(v) = \theta_j > \underline{x}_k(q)$. Suppose $t \geq \theta_j$. Since $j < k$, $p_{j+1}(v) = \theta_{j+1}$. Since v is strategically sincere, $t \in [\theta_j, \frac{\theta_j+\theta_{j+1}}{2}]$, implying v_t is sincere, a contradiction. suppose $t < \theta_j$. If $j = 1$, clearly $\arg\max\{u(\theta_h; t)|h \in L\} = \{1\}$. So, v_t is sincere, a contradiction. So, $j > 1$. Since v is strategically sincere, $t \in [\frac{p_{j-1}(v)+\theta_j}{2}, \theta_j)$. But $p_{j-1}(v) = \max\{\theta_{j-1}, \underline{x}_k(q)\} \geq \theta_{j-1}$, which implies $t \in [\frac{\theta_{j-1}+\theta_j}{2}, \theta_j)$. Thus, v_t is sincere, a contradiction.

Thus, the claim is true, $p_j(v) = \underline{x}_k(q)$, which implies $\theta_j \leq \underline{x}_k(q)$. I now claim that $\theta_{j+1} > \underline{x}_k(q)$. Suppose not. Then, $p_j(v) = p_{j+1}(v) = \underline{x}_k(q)$. By C1, $t \leq \frac{\theta_j+\theta_{j+1}}{2}$. If $j = 1$, then v_t is sincere, a contradiction. If $j \geq 2$, then $p_{j-1}(v) = \underline{x}_k(q)$. Then, C1 implies that $t \geq \frac{\theta_{j-1}+\theta_j}{2}$. Thus, v_t is sincere, a contradiction. Hence, the claim is true.

Since v is strategically sincere, $t \leq \frac{\underline{x}_k(q)+\theta_{j+1}}{2}$. If $t \leq \frac{\theta_j+\theta_{j+1}}{2}$, then v_t is sincere. Thus, $\frac{\theta_j+\theta_{j+1}}{2} < t \leq \frac{\underline{x}_k(q)+\theta_{j+1}}{2} < \theta_{j+1}$. Then, $\arg\max\{u(\theta_h; t)|h \in L\} = \{j+1\}$. Thus, $i(t) = j+1$, and we have $\theta_j < t < \theta_{i(t)} \leq \theta_k$.

I now prove that $k \geq m$. Suppose $k \leq m-1$. Since v is strategically sincere, $\bigcup_{h=1}^{k} T_h(v) \subseteq [t_1, \overline{y}_k]$. But since $\theta_m \leq t_M$ and $k \leq m-1$, $t_M > \overline{y}_k$. Then $\sum_{h=1}^{k} b_h(v) < M$, contradicting $k = k(v)$. Thus, $k \geq m$. Therefore, $\theta_j < t < \theta_{i(t)} \leq \theta_m$. A symmetric argument will prove that when $t \geq \overline{y}_k$, then $\theta_m \leq \theta_i(t) < t < \theta_j$. ■