Taking Sides in Wars of Attrition: Online Appendix

Online Appendix

Proof of Lemma 1: An atom of types of one faction or the other may drop out at $t = 0$. Thereafter play moves along $\theta(\rho_0)$ with w_2 and $w_1 = \widetilde{w}_1(w_2) \equiv \overline{w}_1(w_2/\overline{w}_2)^{1/\rho_0}$ drop out at the same time. Conditional on at least one type being strategic, faction 1's probability of winning at $(\widetilde{w}_1(z), z)$ is

$$\begin{aligned}
\Pi_1(\widetilde{w}_1(z), z) &= \frac{1}{R(\widetilde{w}_1(z), z)} \int_z^{\overline{w}_2} \int_{\widetilde{w}_1(z)}^{\infty} e^{\underline{w}_1 + \underline{w}_2 - w_1 - w_2} dw_1 dw_2 \\
&= \frac{1}{R(\widetilde{w}_1(z), z)} \int_z^{\overline{w}_2} e^{\underline{w}_1 + \underline{w}_2 - \widetilde{w}_1(w_2) - w_2} dw_2 \qquad \text{(A5)} \\
\frac{d\Pi_1}{dz} &= \frac{e^{-z-\widetilde{w}_1(z)}}{R(\widetilde{w}_1(z), z)} \left[\left(1 + \frac{\overline{w}_1}{\rho_0 \overline{w}_2} \left(\frac{z}{\overline{w}_2} \right)^{1/\rho_0 - 1} \right) \Pi_1(\widetilde{w}_1(z), z) - 1 \right].
\end{aligned}$$

To show that the factor in brackets is negative, define the line

$$y_1(w_2) \equiv \widetilde{w}_1(z) + \frac{\overline{w}_1}{\rho_0 \overline{w}_2} \left(\frac{z}{\overline{w}_2} \right)^{1/\rho_0 - 1} (w_2 - z).$$

Note that $y_1(z) = \widetilde{w}_1(z)$ and the slope of y_1 is the slope of $\widetilde{w}_1(w_2)$ at $w_2 = z$. The concavity of $\widetilde{w}_1$ then implies

$$\begin{aligned}
\Pi_1(\widetilde{w}_1(z), z) &< \frac{1}{R(\widetilde{w}_1(z), z)} \int_z^{\overline{w}_2} e^{\underline{w}_1 + \underline{w}_2 - y_1(w_2) - w_2} dw_2 \\
&< -\frac{1}{R(\widetilde{w}_1(z), z)} \left[1 + \frac{\overline{w}_1}{\rho_0 \overline{w}_2} \left(\frac{z}{\overline{w}_2} \right)^{1/\rho_0 - 1} \right]^{-1} e^{\underline{w}_1 + \underline{w}_2 - y_1(w_2) - w_2} \Big|_z^{\overline{w}_2} \\
&< \left[1 + \frac{\overline{w}_1}{\rho_0 \overline{w}_2} \left(\frac{z}{\overline{w}_2} \right)^{1/\rho_0 - 1} \right]^{-1} \left[\frac{e^{-z-\widetilde{w}_1(z)} - e^{-y_1(\overline{w}_2) - \overline{w}_2}}{e^{-z-\widetilde{w}_1(z)} - e^{-\overline{w}_1 - \overline{w}_2}} \right].
\end{aligned}$$

The concavity of $\widetilde{w}_1$ also implies $y_1(\overline{w}_2) < \overline{w}_1$. Hence the second factor is less than one and

$$\Pi_1(\widetilde{w}_1(z), z) < \left[1 + \frac{\overline{w}_1}{\rho_0 \overline{w}_2} \left(\frac{z}{\overline{w}_2} \right)^{1/\rho_0 - 1} \right]^{-1}.$$

Using this bound in the expression for $d\Pi_1/dz$ ensures $d\Pi_1/dz < 0$.

Applying L'Hopital's to Eq A5 gives

$$\begin{aligned}\lim_{z\to\overline{w}_2} \Pi_1(\widetilde{w}_1(z), z) &= \lim_{z\to\overline{w}_2} -e^{-\widetilde{w}_1(z)-z}\left[-\left(1+\frac{\overline{w}_1}{\rho_0\overline{w}_2}\left(\frac{z}{\overline{w}_2}\right)^{1/\rho_0-1}\right)e^{-\widetilde{w}_1(z)-z}\right]^{-1} \\ &= \left[1+\frac{\overline{w}_1}{\rho_0\overline{w}_2}\right]^{-1}\end{aligned}$$

which is greater than 1/2 when $\rho_0 > \overline{w}_1/\overline{w}_2$. $\square$

Proof of Lemma 2: The effects of changes in the cost ratio and total cost on Π_j follow from the discussion preceding the statement of the lemma. To see how these changes affect the expected duration, observe that the probability that the fighting ends at or before t is $F(t,\rho,k) \equiv [1-e^{\underline{w}_1+\underline{w}_2-\tau_1(t)-\tau_2(t)}]/R(\underline{w}_1,\underline{w}_2)$. Let $D(\underline{w}_1,\underline{w}_2,\rho,k)$ be the expected duration given that at least one faction is strategic. A change in ρ affects the size of the atom that drops out at $t=0$ and this affects the expected duration. To abstract away from these effects, take $\underline{w}_1=\underline{w}_2=0$. Then the size of the atom dropping out at $t=0$ is the same for all ρ (as no atom of either faction drops out at $t=0$.) We show that $\partial D(0,0,\rho,k)/\partial\rho > 0$.

Let $F^R(t,\rho,k) \equiv F(t,\rho,k)/R(0,0)$ be the distribution of stop times given that at least one faction is strategic and $\underline{w}_1=\underline{w}_2=0$. Then the expected duration of the conflict is $\int_0^\infty t dF^R(t,\rho,k)$. It suffices to show that $\partial F^R/\partial\rho < 0$ before the highest-payoff types quit at $t < \sigma_j(\overline{w}_j) = \overline{w}_1\overline{w}_2/k$. If so, then $F^R(t,\rho,k) > F^R(t,\rho',k)$ when $\rho' > \rho$ before the highest-payoff strategic types quit and $F^R(t,\rho,k) = F^R(t,\rho',k) = 1$ at later times. This implies $F^R(t,\rho',k)$ first-order stochastically dominates $F^R(t,\rho,k)$ and establishes the claim.

Differentiation gives

$$\operatorname{sgn}\left\{\frac{\partial F^R}{\partial\rho}\right\} = \operatorname{sgn}\left\{\overline{w}_1\left(\frac{kt}{\overline{w}_1\overline{w}_2}\right)^{1/(1+\rho)} - \overline{w}_2\left(\frac{kt}{\overline{w}_1\overline{w}_2}\right)^{\rho/(1+\rho)}\right\}.$$

The expression on the right is negative when

$$\left(\frac{\overline{w}_1\overline{w}_2}{kt}\right)^{\frac{1-\rho}{1+\rho}} > \frac{\overline{w}_1}{\overline{w}_2}.$$

Using $1 > \rho > \overline{w}_1/\overline{w}_2$ from Assumption 1 and $t < \overline{w}_1\overline{w}_2/k$ yields

$$\left(\frac{\overline{w}_1\overline{w}_2}{kt}\right)^{\frac{1-\rho}{1+\rho}} > 1 > \rho > \frac{\overline{w}_1}{\overline{w}_2}. \ \square$$

Formal statement of comparative statics:

PROPOSITION 2: *Let $\overline{\alpha}_1$ and $\underline{\alpha}_1$ respectively denote the equilibrium probabilities that M supports 1 in the equilibrium in which M is most likely to support 1 and least likely. Then both $\overline{\alpha}_1$ and $\underline{\alpha}_1$ are weakly increasing in v_1 and f_1 and weakly deceasing in v_2 and f_2. The higher k_j, the shorter the conflict if M supports j. This in turn makes M more likely to support j when fighting is costly ($f_j < 0$) and the less likely M is to support j when it is profitable ($f_j > 0$). The effects of an increase in γ on $\overline{\alpha}_1$ and $\underline{\alpha}_1$ are ambiguous and depend on the sign of $\Pi_1(z_1, z_2, \rho_1) - \Pi_2(z_1, z_2, \rho_2)$.*[1]

The expression for $\Delta_{02}(z_1, z_2)$:

$$\begin{aligned}
\Delta_{02}(z_1, z_2) &\equiv S_0(z_1, z_2) - S_2(z_1, z_2) \\
&= (v_1 + \gamma)\Pi_1(z_1, z_2, \rho_0) + (v_2 - \gamma)\Pi_2(z_1, z_2, \rho_0) + f_1 D(z_1, z_2, \rho_0, k_0) \\
&\quad -[v_1\Pi_1(z_1, z_2, \rho_2) + v_2\Pi_2(z_1, z_2, \rho_2) - f_2 D(z_1, z_2, \rho_2, k_2)] \\
&= (v_1 - v_2)[\Pi_1(z_1, z_2, \rho_0) - \Pi_1(z_1, z_2, \rho_2)] + \gamma[\Pi_1(z_1, z_2, \rho_0) - \Pi_2(z_1, z_2, \rho_2)] \\
&\quad + f_1 D(z_1, z_2, \rho_0, k_0) - f_2 D(z_1, z_2, \rho_2, k_2).
\end{aligned}$$

The hazard-rate example: The equilibrium of the counter-example game has a very simple structure. Since $\rho_0 = \rho_1 = \rho_2$, the curves $\theta(\rho_0)$, $\theta(\rho_1)$, and $\theta(\rho_2)$ are identical. As a result, the lowest-payoff types still active at T must lie on $\theta(\rho_1)$ and satisfy $z_2 = \overline{w}_2(z_1/\overline{w}_1)^{\rho_1}$ regardless of what M does at T. There is also no interval of pure fighting ($\lambda = 0$).[2] The first phase must therefore last until T which implies $T = z_1 z_2/k_0$. The previous two equations can be solved for z_1 and z_2 and used to obtain an expression for $H_{2|2}(t)/H_{2|0}(t)$:

[1] The comparative static relations are strict when the equilibrium probabilites are interior, e.g., $\overline{\alpha}_1$ is strictly increasing in v_1 when $\overline{\alpha}_1 \in (0, 1)$.

[2] Since $\theta(\rho_0)$, $\theta(\rho_1)$, and $\theta(\rho_2)$ coincide, no atoms of either type drop out regardless of what M does. This in turn implies $\lambda = 0$ via Eq A4 and $z_2 = \overline{w}_2(z_1/\overline{w}_1)^{\rho_1}$.

$$\begin{aligned}
\frac{H_{2|2}(t)}{H_{2|0}(t)} &= \frac{\tau'_{1|2}(\widehat{t})e^{\tau'_{2|2}(\widehat{t})}}{\tau'_{1|0}(\widehat{t})e^{\tau'_{2|0}(\widehat{t})}} \\
&= \frac{k_2}{1+\rho_2}\left[\frac{k_2\widehat{t}}{\overline{w}_1\overline{w}_2}+\left(\frac{z_1}{\overline{w}_1}\right)^{1+\rho_2}\right]^{\frac{-\rho_2}{1+\rho_2}} \times e^{\overline{w}_2\left[\frac{k_2\widehat{t}}{\overline{w}_1\overline{w}_2}+\left(\frac{z_1}{\overline{w}_1}\right)^{1+\rho_2}\right]^{\frac{\rho_2}{1+\rho_2}}} \\
&\quad \times \left(\frac{1+\rho_0}{k_0}\right)\left[\frac{k_0\widehat{t}}{\overline{w}_1\overline{w}_2}+\left(\frac{z_1}{\overline{w}_1}\right)^{1+\rho_0}\right]^{\frac{\rho_0}{1+\rho_0}} \times e^{-\overline{w}_2\left[\frac{k_0\widehat{t}}{\overline{w}_1\overline{w}_2}+\left(\frac{z_1}{\overline{w}_1}\right)^{1+\rho_0}\right]^{\frac{\rho_0}{1+\rho_0}}}
\end{aligned}$$

To see that the $H_{2|2}(t)/H_{2|0}(t) > 1$, observe first that $H_{2|2}(t)/H_{2|0}(t) = 1$ if $k_2 = k_0$. It follows that $H_{2|2}(t)/H_{2|0}(t) > 1$ if $H_{2|2}(t)/H_{2|0}(t)$ is increasing in k_2. To see that it is, differentiate $H_{2|2}(t)/H_{2|0}(t)$ with respect to k_2. Pulling out common factors shows that the sign of $\partial(H_{2|2}(t)/H_{2|0}(t))/\partial k_2$ is the same as the sign of Λ where

$$\Lambda = 1 - \left(\frac{\rho_2}{1+\rho_2}\right)\left(1+\frac{\overline{w}_1\overline{w}_2}{k_2\widehat{t}}\left(\frac{z_1}{\overline{w}_1}\right)^{1+\rho_2}\right)^{-1}\left(1-\overline{w}_2\left[\frac{k_2\widehat{t}}{\overline{w}_1\overline{w}_2}+\left(\frac{z_1}{\overline{w}_1}\right)^{1+\rho_2}\right]\right).$$

To see that this is positive, observe that if the third factor in parentheses is negative, Λ is clearly positive. If the third factor is positive, it is clearly less than one. The other two factors in parentheses are also less than one. Hence the product is less than one and $\Lambda > 0$.

Derivation of Equilibrium with two-decision times: To describe the equilibria when there are two decision times, let (r_1, r_2) be the lowest-payoff types still active at T' and take $(z_{1|j}, z_{2|j})$ to be the lowest-payoff types still active at T'' given M supported $j \in \{1,2\}$ at T'. The probabilities that M supports 1 at T' and T'' are α'_1 and $\alpha''_{1|j}$ respectively. In equilibrium, $w_j < r_j$ fight a war of attrition with ρ_0 and k_0. Type w_j stops at $\sigma_j(w_j, \underline{w}_1, \underline{w}_2, \rho_0, k_0, r_1, r_2)$ (see Eq A3 in the article). No types drop out and the fighting continues from $\sigma_j(r_j, \underline{w}_1, \underline{w}_2, \rho_0, k_0, r_1, r_2)$ to T' at which time M first takes sides.

Suppose M joins 1 at T'. We can think of the continuation game following M's decision at T' as a game in which M has one exogenous decision time at $T''-T'$, the lowest-payoff types at the start of the game are (r_1, r_2) rather than $(\underline{w}_1, \underline{w}_2)$, and the cost ratio and total cost prior to M's decision are ρ_1 and k_1 rather than ρ_0 and k_0. Equilibrium play in this continuation game follows from Proposition 1. Types $w_j \in (r_j, z_{j|1}]$ fight a war of attrition in which w_j stops at time $T' + \sigma_j(w_j, r_1, r_2, \rho_1, k_1, z_{1|1}, z_{2|1})$. The fighting then continues to T'' when M again decides what to do. The pattern of play is similar if M supports 2 at T'. Characterizing the equilibrium amounts to finding (r_1, r_2), $(z_{1|1}, z_{2|1})$, $(z_{1|2}, z_{2|2})$, α'_1, $\alpha''_{1|1}$, and $\alpha''_{1|2}$.

To sketch the derivation, note that regardless of what M did at T', Lemma 4 implies that a necessary condition for $(z_{1|j}, z_{2|j})$ to be consistent with equilibrium play is that it is in F_{12}, i.e., $(z_{1|j}, z_{2|j})$ must be weakly between $\theta(\rho_1)$ and $\theta(\rho_2)$ and one of the following three conditions must hold: (i) $\Delta_{12}(z_{1|j}, z_{2|j}) \geq 0$ when $(z_{1|j}, z_{2|j})$ is on $\theta(\rho_1)$, (ii) $\Delta_{12}(z_{1|j}, z_{2|j}) \leq 0$ when $(z_{1|j}, z_{2|j})$ is on $\theta(\rho_2)$, and (iii) $\Delta_{12}(z_{1|j}, z_{2|j}) = 0$ when $(z_{1|j}, z_{2|j})$ is strictly between $\theta(\rho_1)$ and $\theta(\rho_2)$.

To ease the analysis, assume that supporting 1 is costly for M, i.e., $f_1 < 0$. F_{12} has a very simple structure when this condition holds. The assumption $f_1 < 0$ guarantees that for any given w'_2, $\partial\Delta_{12}(w_1, w'_2)/\partial w_1 > 0$ whenever $\Delta_{12}(w_1, w'_2) = 0$. This implies that a unique (w_1, w'_2) is in F_{12} and hence that F_{12} can be written as a function $w_1 = e(w_2)$ defined by $\Delta_{12}(e(w_2), w_2) = 0$. The continuity of Δ_{12} also ensures that e_{12} is continuous.

Turning to the construction of the equilibrium, suppose M supports 1 at T'. This shifts the cost ratio in 1's favor and may induce some types to drop out at T' once they see what M has done. Let (x_1, x_2) denote the lowest-payoff types remaining active after these types drop out.[3] Those types must satisfy the timing constraint $T'' - T' = \sigma_j(z_{j|1}, x_1, x_2, \rho_1, k_1, z_{1|1}, z_{2|1}) + \lambda(z_{1|1}, z_{2|1}, \rho_1, k_1)$ or

$$T'' - T' = \frac{z_{1|1} z_{2|1}}{k_1} \left[1 - \left(\frac{x_1}{z_{1|1}} \right)^{1+\rho_1} \right] + \lambda(z_{1|1}, z_{2|1}, \rho_1, k_1).$$

[3] Formally, x_j is the infimum of the set of w_j with stop times strictly later than T' conditional on M supporting 1 at T'. By contrast, r_j is the infimum of the set of types with stop times weakly later than T'.

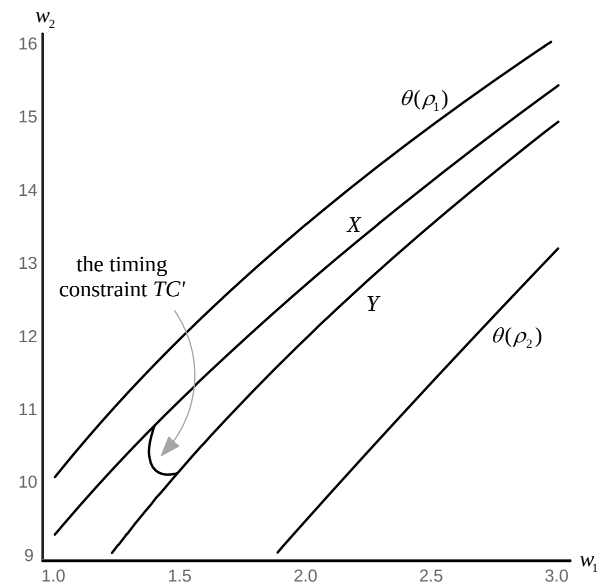

Figure 1: The equilibrium with two decision times.

Using the expression for $\sigma_j(w_j, \underline{w}_1, \underline{w}_2, \rho_0, k_0, r_1, r_2)$ and solving for x_1 gives

$$
\begin{aligned}
T'' - T' &= \frac{z_{1|1} z_{2|1}}{k_1}\left[1 - \left(\frac{x_1}{z_{1|1}}\right)^{1+\rho_1}\right] + \lambda(z_{1|1}, z_{2|1}, \rho_1, k_1) \\
x_1 &= z_{1|1}\left[\frac{z_{1|1} z_{2|1} + k_1(T' + \lambda(z_{1|1}, z_{2|1}, \rho_1, k_1) - T'')}{z_{1|1} z_{2|1}}\right]^{1/(1+\rho_1)}
\end{aligned}
$$

and similarly for x_2. Since $(z_{1|1}, z_{2|1}) \in F_{12}$, $z_{1|1} = e(z_{2|1})$. Hence, we can write x_1 and x_2 as functions of $z_{2|1}$. Of course we do not know $z_{2|1}$, so let X be the set all possible values of (x_1, x_2), i.e., $X \equiv \{(x_1(w_2), x_2(w_2)) : w_2 \in [0, \overline{w}_2]\}$. Since $z_{2|1} \in [0, \overline{w}_2]$, (x_1, x_2) must be in X which is illustrated in Figure 1.

Now let (y_1, y_2) be the lowest-payoff types remaining active at T' after M supports 2 and those types wanting to drop out have done so. Repeating the argument above, we can construct the analogue to X, which will be called Y, and is depicted in Figure 1. It follows that (y_1, y_2) must be somewhere in Y.

X and Y play roles at T' in the two-decision game analogous to the roles that $\theta(\rho_1)$ and $\theta(\rho_2)$ play at T in the one-decision game. If (z_1, z_2) are the weakest types still active at T in the one-decision game and M supports 1, play moves vertically from (z_1, z_2) up to $\theta(\rho_1)$ with types w_2 between z_2 and $\theta(\rho_1)$ dropping out as as soon M joins 1. The remaining types play a war of attrition for the rest of the game. Analogously, if (r_1, r_2) are the weakest types active at T' and M supports 1, play moves from (r_1, r_2) vertically up to X with types w_2 between r_2 and X dropping out as as soon M joins 1. The remaining types then fight a war of attrition until time T''. Similarly, if M joins 2 at (r_1, r_2), play moves horizontally to Y and so on.

An analogue of Lemma 4 also obtains. The pair (r_1, r_2), must be weakly between X and Y. If, moreover, (r_1, r_2) is on X, M must join 1 at T' ($\alpha'_1 = 1$). M must support 2 ($\alpha'_1 = 0$) if (r_1, r_2) is on Y, and M must mix if (r_1, r_2) is between X and Y.

These conditions impose restrictions on (r_1, r_2). Let $S'_j(r_1, r_2)$ denote M's payoff to supporting j at T' and define $\Delta'_{12}(r_1, r_2) \equiv S'_1(r_1, r_2) - S'_2(r_1, r_2)$. Since $\alpha'_1 = 1$ when (r_1, r_2) is on X, M must weakly prefer supporting 1 or $\Delta'_{12}(r_1, r_2) \geq 0$. When (r_1, r_2) is on Y, $\Delta'_{12}(r_1, r_2) \leq 0$. The pair (r_1, r_2) is between X and Y when $\Delta'_{12}(r_1, r_2) = 0$.

Finally, a timing constraint pins down the (r_1, r_2) that are consistent with equilibrium play and the corresponding equilibrium strategies. Types $w_j \in (\underline{w}_j, r_j)$ play a war of attrition during the initial phase of the game. This lasts until $\sigma_j(r_j, \underline{w}_1, \underline{w}_2, \rho_0, k_0, r_1, r_2)$. There follows an interval of length $\lambda'(r_1, r_2)$ during which no types drop out where α'_1 and λ'_{12} can be derived from the fact that r_j is indifferent between stopping at $T' - \lambda'$ and waiting to see what M does at T' before quitting. In symbols, (r_1, r_2) satisfies $T' = \sigma_j(r_1, \underline{w}_1, \underline{w}_2, \rho_0, r_1, r_2) +\ \lambda'_{12}(r_1, r_2)$. Call the points satisfying this timing constraint TC' as illustrated in Figure 1.

Paralleling Proposition 1, a PBE corresponds to each point (u_1, u_2) along TC' that satisfies one of the three conditions: (i) $\Delta'_{12}(u_1, u_2) \geq 0$ when $(u_1, u_2) \in X$, (ii) $\Delta'_{12}(u_1, u_2) \leq 0$ when $\Delta'_{12}(u_1, u_2) \in Y$, and (iii) $\Delta'_{12}(u_1, u_2) = 0$ when $(u_1, u_2) \notin X \cup Y$. Indexing the points along TC' by α'_1, at least one point must satisfy these conditions. If neither (i) nor (ii) holds, continuity ensures that (iii) does as $\Delta'_{12}(u_1, u_2)$ must be zero somewhere along TC'_{12}. It follows that at least one PBE exists. Indeed, a unique point satisfies these conditions in the baseline numerical example, namely, $(u_1, u_2) \approx (1.3, 9.9)$ and the equilibrium associated with this point is reported above.