# D Extension: Instrumental Voting in a Legislature (Online)

The theory we present views the voter as caring directly about his or her representative's choice(s). That is, when the voter is comparing two potential representatives, the voter directly accounts for the payoff difference between every potential choice that each of the representatives might make on the voter's behalf. Thus, from an consequentialist point of view, our theory is presuming that the representative's choice is always decisive: if the representative votes for a bill, $b$, rather than the status quo, $q$, the bill is implemented and, conversely, if the representative votes for the status quo rather than the bill, then the status quo is retained.

Of course, this is generally not the case: in a democracy, no representative is uniformly decisive with respect to all public policy decisions. Luckily, this reality is easily accommodated by our theory. Specifically, our theory is consistent with consequentialism so long as there is some positive likelihood that the representative's choice will be decisive. One complication is that this likelihood might depend on the representative's platform: a representative with a moderate platform might be more likely to cast a decisive vote than one whose platform is extreme. We show in this section that, even if this is the case, our results continue to hold.

We do this in two complementary ways. In the first, we allow the probability of a vote being pivotal to depend on the pair $(b, q)$ and in the second, we allow the probability of being decisive to depend only on the candidate's platform, $p$. In both settings, we assume that the legislator will be in a legislature with $2n$ other legislators. In the first setting, the voter does not know the platforms of the other $2n$ legislators when casting his or her vote. In the second setting, the voter knows the platforms of all $2n$ other legislators when making his or her vote choice.

## D.1 Uncertainty about Other Legislators' Platforms

The first extension we consider assumes that the voter takes into account the probability that each candidate will be pivotal in the legislature, conditional on the candidate's platform, but is uncertain about the platforms of the other legislators. Specifically, suppose that the $2n$ other legislators' platforms are independently distributed, with the probability density function describing the distribution of legislator $j$'s platform being a continuous function denoted by $g_j : \mathbf{R} \to \mathbf{R}_+$ and cumulative distribution function denoted by $G_j : \mathbf{R} \to [0,1]$. For simplicity, we will assume that $G_j = G_k$ for all legislators $j$ and $k$: the other legislators' ideal points are independently and identically distributed.

**Pivot Probability for a Given Pair $(b,q)$.** For any bill $b \in \mathbf{R}$ and status quo $q \in \mathbf{R}$, the probability that legislator $i$ is pivotal on the vote between $b$ and $q$ is the probability that exactly $n$ of the other legislators' realized platforms are less than or equal to the midpoint of $b$ and $q$:

$$\begin{aligned} \rho_i(b,q) &\equiv \Pr\left[p_{-i} : \left|\left\{j : p_j \leq \frac{b+q}{2}\right\}\right| = n\right], \\ &= \binom{2n}{n} G\left(\frac{b+q}{2}\right)^n \left(1 - G\left(\frac{b+q}{2}\right)\right)^n. \end{aligned}$$

**One's Pivot Probability is Independent of One's Platform.** Letting $f(b,q)$ denote the probability density function (pdf) of the agenda $\alpha$, the overall probability that legislator $i$ is pivotal is simply

$$\rho_i = \int \int \rho_i(b,q) f(b,q) \, \mathrm{d}b \, \mathrm{d}q.$$

Note that candidate $i$'s platform, $p_i$, does not factor into this probability. Thus, due to the assumption that the distribution of $p_j$ is independent of $p_i$, *the pivot probability for any given legislator is independent of his or her platform.* Legislator $i$'s pivot probability is a function only of the probability that the other legislators are divided so as to make legislator $i$'s vote pivotal.[32]

[32] Note that this does not depend on the presumption that the legislature uses majority rule: the conclusion holds for all counting rules.

That said, the pivot probability for any legislator *does* depend on the locations of bill, $b$, and the status quo, $q$, and therefore affects the instrumental expected utility to a voter with ideal point $v \in \mathbf{R}$ from a legislator $i$ with platform $p_i \in \mathbf{R}$:

$$EU(p_i, v) = \int \int \rho_i(b,q) u(V(b,q,p_i), v) f(b) f(q) \, \mathrm{d}b \, \mathrm{d}q.$$

The voter's net expected payoff from $p_L = v - \delta$ relative to $p_R = v + \delta$ is then

$$EU(p_L, v) - EU(p_R, v) = \int \int \rho_i(b,q) \left( u(V(b,q,p_L), v) - u(V(b,q,p_R), v) \right) f(b) f(q) \, \mathrm{d}b \, \mathrm{d}q.$$

The first derivative of legislator $i$'s pivot probability, $\rho_i(b,q)$, with respect to $\frac{b+q}{2}$ is

$$\frac{\mathrm{d}\rho_i(b,q)}{\mathrm{d}\frac{b+q}{2}} = \frac{2n!}{n!(n-1)!} g\left(\frac{b+q}{2}\right) G\left(\frac{b+q}{2}\right)^{n-1} \left(1 - G\left(\frac{b+q}{2}\right)\right)^{n-1} \left(1 - G\left(\frac{b+q}{2}\right) - G\left(\frac{b+q}{2}\right)\right),$$

so that $\rho_i(b,q)$ is increasing in $\frac{b+q}{2}$ when $G\left(\frac{b+q}{2}\right) < 1/2$, decreasing when $G\left(\frac{b+q}{2}\right) > 1/2$, and maximized when $G\left(\frac{b+q}{2}\right) = 1/2$. When the agenda, $\alpha$, is symmetric around $(\mu, \mu)$, then $\rho_i(b,q)$ is maximized when $\frac{b+q}{2} = \mu$. This leads to the following proposition.

**Proposition 5** *Suppose that the agenda, $\alpha$, is symmetric around $(\mu, \mu)$ and that the distribution of any given legislator's ideal point, $G$, is unimodal and symmetric around $\mu$. Then every voter has a taste for extremism.*

*Proof*: Let $\alpha$ be symmetric about $(\mu, \mu) \in \mathbf{R}^2$, with probability density function, $f$. Symmetry implies that for any two points $x, y \in \mathbf{R}^2$, if $|x - \mu| > |y - \mu|$ then $f(x) < f(y)$. Without loss of generality we will assume that $v < \mu$, and we will show that for any $\delta > 0$ the voter prefers platform $p_L = v - \delta$ to $p_R = v + \delta$ (the voter prefers the left candidate).

Let $p_R = v + \delta$ and $p_L = v - \delta$. Take any point $y = (y_1, y_2) \in D(p_R, v)$, the disagreement set of the voter and $R$. By the definition of this disagreement set, it must be the case that $y_2 > 2v - y_1$.

Without loss of generality, assume that $y_1 > y_2$ so that the voter prefers bill $y_2$ and $R$ prefers status quo $y_1$ (an identical argument holds for the other case). If we reflect this

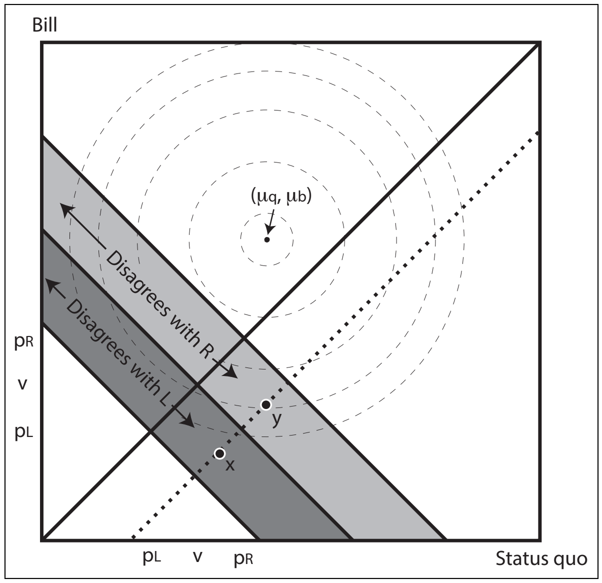


Figure 13: Illustration For Proposition 5

point on the $45^o$ line around the line $b = 2v - q$ we get the point $x = (2v - y_2, 2v - y_1)$, with $x \in D(p_L, v)$. This is pictured in Figure 13. The relevant insight is that at $y$ $R$ votes for $y_1$ whereas the voter prefers $y_2$; at $x$ $L$ votes for $x_2 = 2v - y_1$ whereas the voter prefers $x_1 = 2v - y_2$. Mirroring the structure and notation in the proof of Theorem 4, the expected disutility the voter receives in the former case from $R$'s incorrect vote for $y_1$ relative to $L$'s correct vote for $y_2$ is

$$\rho(y_1, y_2)\Delta_y(R, L, v) = \rho(y_1, y_2)(u(y_1, v) - u(y_2, v))$$

whereas the expected disutility from $L$'s incorrect vote for $x_2$ over $R$'s correct vote for $x_1$ is

$$\rho(x_1, x_2)\Delta_x(L, R, v) = \rho(x_1, x_2)(u(x_2, v) - u(x_1, v)),$$

Theorem 4 implies that the result follows whenever $\rho(y_1, y_2) = \rho(x_1, x_2) > 0$ because symmetry and strict quasiconcavity of $f$ implies that $f(y) > f(x)$. Accordingly, the result also holds if $\rho(y_1, y_2) \geq \rho(x_1, x_2)$. Given that $v < \mu$, this is the case whenever

$$\left| G\left(\frac{x_1 + x_2}{2}\right) - {}^1\!/_2 \right| \geq \left| G\left(\frac{y_1 + y_2}{2}\right) - {}^1\!/_2 \right|. \tag{8}$$

Symmetry and unimodality of $G$ around $\mu$ implies that (8) holds if

$$\begin{aligned} \left|\frac{x_1 + x_2}{2} - \mu\right| &\geq \left|\frac{y_1 + y_2}{2} - \mu\right|, \\ |v - \kappa - \mu| &\geq |v + \kappa - \mu|, \end{aligned}$$

for some $\kappa \in (0, \delta]$. Given the presumption that $v < \mu$, this reduces to

$$\begin{aligned} |v - \kappa - \mu| &\geq |v + \kappa - \mu|, \\ \kappa + \mu - v &\geq \max[\kappa + v - \mu, \mu - v - \kappa], \end{aligned}$$

so that, again given $v < \mu$, we have

$$\kappa + \mu - v > \kappa + v - \mu,$$

and $\kappa > 0$ implies

$$\kappa + \mu - v > \mu - v - \kappa.$$

Accordingly, $\rho(y_1, y_2) \geq \rho(x_1, x_2)$, and the result follows. ∎

Proposition 5 establishes that Theorem 4 is consistent with instrumental voting that acknowledges that different platforms may have different probabilities of casting a decisive vote.

## D.2 Other Legislators' Platforms Known

We now suppose that the voter knows the platforms of the other $2n$ legislators. Specifically, let $P = \{p_1, \ldots, p_{2n}\}$ denote a profile of $2n$ platforms with $p_1 \leq p_2 \leq \ldots \leq p_{2n-1} \leq p_{2n}$, representing the platforms of the legislators other than $i$ and let $\rho_i(b, q; P)$ denote the pivot probability for a vote between $b$ and $q$, given $P$. Given his or her ideal point, $v \in \mathbf{R}$, the voter's expected payoff from platform $p_i$ is then

$$EU(p_i, v) = \int \int \rho_i(b, q; P) u(V(b, q, p_i), v) f(b, q) \, \mathrm{d}b \, \mathrm{d}q.$$

Given deterministic spatial voting by the other $2n$ legislators based on their platforms, legislator $i$'s pivot probability, $\rho_i(b, q; P)$, is equal to the following:

$$\rho_i(b, q; P) = \begin{cases} 1 & \text{if } p_n \le \frac{b+q}{2} < p_{n+1}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Thus, in this setting, the pivot probability is positive only for $(b, q)$ pairs for which the midpoint falls in the median of the legislators' platforms, $[p_n, p_{n+1}]$. We refer to this interval as the *median interval of* $P$. Our first result establishes that any voter's payoff is insensitive to the candidate's platform when the platform falls outside of the median interval of $P$.

**Lemma 2** *For any* $P = \{p_1, \ldots, p_n\}$, $EU(p_i, v)$ *is constant for all* $p \le p_n$ *and* $p \ge p_{n+1}$, *with*

$$\begin{aligned} p \le p_n \quad &\Rightarrow \quad EU(p, v) = EU(p_n, v), \textit{ and} \\ p \ge p_{n+1} \quad &\Rightarrow \quad EU(p, v) = EU(p_{n+1}, v). \end{aligned}$$

*Proof*: Fix $P = \{p_1, \ldots, p_{2n}\}$ and $p_1 \le p_2 \le \ldots \le p_{2n-1} \le p_{2n}$ and a voter ideal point, $v \in \mathbf{R}$. Now consider two platforms, $p$ and $p'$, with $p < p' \le p_n$.

$$EU(p, v) - EU(p', v) = \int \int \rho_i(b, q; P)(u(V(b, q, p), v) - u(V(b, q, p'), v)) f(b, q) \, \mathrm{d}b \, \mathrm{d}q,$$

and, by Equation (9), this reduces to

$$EU(p, v) - EU(p', v) = \int_{\mathbf{R}} \int_{2p_n - q}^{2p_{n+1} - q} (u(V(b, q, p), v) - u(V(b, q, p'), v)) f(b, q) \, \mathrm{d}b \, \mathrm{d}q. \tag{10}$$

Note that $p$ and $p'$ vote identically to $p_n$ over the region of integration in Equation (10) (*i.e.* $V(b,q,p) = V(b,q,p' = V(b,q,p_n$ for all $(b,q)$ such that $\rho_i(b,q);P) > 0$). Thus,

$$\begin{aligned} EU(p,v) - EU(p',v) &= \int_{\mathbf{R}} \int_{2p_n - q}^{2p_{n+1}-q} (u(V(b,q,p_n),v) - u(V(b,q,p_n),v)) f(b,q) \, \mathrm{d}b \, \mathrm{d}q, \\ &= 0. \end{aligned}$$

Thus, platforms $p$ and $p'$, with $p < p' \leq p_n$ offer $v$ the same identical payoff. An analogous argument establishes the same conclusion for $p$ and $p'$ with $p > p' \geq p_{n+1}$, establishing the claim. ∎

Intuitively, all voters whose ideal points are outside the median interval of $P$ strictly prefer candidates whose platforms are on the same side of the median interval as the voter's ideal point ($p \leq p_n$ if $v < p_n$ and $p \geq p_{n+1}$ if $v > p_{n+1}$). Thus, if the mode of the agenda, $\mu$, is on the same side of voter's ideal point as the median interval, the voter will have a taste for extremism. This is stated formally in the following lemma.

**Lemma 3** *Suppose that $p_n < \mu$. Then any voter with $v \leq p_n$ has a taste for extremism. By symmetry, the same is true when $\mu < v$ and $v \geq p_{n+1}$.*

*Proof*: Fix $P = \{p_1, \ldots, p_n\}$ with $n$ even and $p_1 \leq p_2 \leq \ldots \leq p_{n-1} \leq p_n$, a voter ideal point, $v \leq p_n$, and an agenda $\alpha$ symmetric about $(\mu,\mu)$ with $v < \mu$. By Lemma 2,

$$\begin{aligned} p \leq p_n &\Rightarrow EU(p,v) = EU(p_n,v), \text{ and} \\ p \geq p_{n+1} &\Rightarrow EU(p,v) = EU(p_{n+1},v). \end{aligned}$$

By Theorem 2, the supposition that $v \leq p_n$ implies that $EU(p,v)$ is weakly decreasing on $[p_n, p_{n+1}]$. Thus, for any $\delta > 0$,

$$\begin{aligned} EU(v-\delta,v) &= EU(p_n,v), \text{ and} \\ EU(v+\delta,v) &< EU(p_n,v). \end{aligned}$$

Thus, by the supposition that $p_n < \mu$, the voter has a taste for extremism. An analogous argument proves the same conclusion when $\mu < v$ and $v \geq p_{n+1}$, as was to be shown. ∎

The logic behind Lemma 3 is displayed in the top panel of Figure 14. In each of the panels, the voter's expected utility in the baseline case (where the elected candidate's platform is always pivotal) is the dotted curve when this expected payoff differs from the voter's expected payoff when the candidate's platform is pivotal only for bill-status quo pairs on which $p_n$ and $p_{n+1}$ vote differently, which is displayed as a thick, solid, piecewise function. As stated in Lemma 2, this function is flat for all platforms less than $p_n$ or greater than $p_{n+1}$.

The top panel of Figure 14 displays the case covered by Lemma 3 because $v$ is located outside the median interval. The middle and bottom panels of the figure illustrate the other two cases in which $v$ is inside the median interval distinguished by whether $v$ is closer to the endpoint of the interval that is more distant from $\mu$ ($p_n$ in the figure) or the one that is closer to $\mu$ (*i.e.*, $p_{n+1}$). The next proposition establishes that, in both of these cases, the voter has a "local taste for extremism" in the sense that he or she always prefers the extreme platform when comparing two equidistant candidates whose platforms are sufficiently close to $v$ in the sense that both platforms lie within the median interval.

**Proposition 6** *If $\mu \neq v$, then any voter with ideal point $v \in [p_n, p_{n+1}]$ has a taste for extremism when comparing $p_L = v - \delta$ and $p_R = v + \delta$ for any $\delta$ satisfying the following:*

$$0 < \delta \leq \min\left[p_{n+1} - v, v - p_n\right].$$

*Proof*: Fix $P = \{p_1, \ldots, p_n\}$ with $n$ even and $p_1 \leq p_2 \leq \ldots \leq p_{n-1} \leq p_n$ and a voter ideal point, $v \in [p_n, p_{n+1}]$. Consider any

$$\delta \in (0, \min\left[p_{n+1} - v, v - p_n\right]]$$

and compare the two candidates with platforms $p_L = v - \delta$ and $p_R = v + \delta$. From Equation (10), the difference between the expected payoff from $p_L$ and the expected payoff from $p_R$

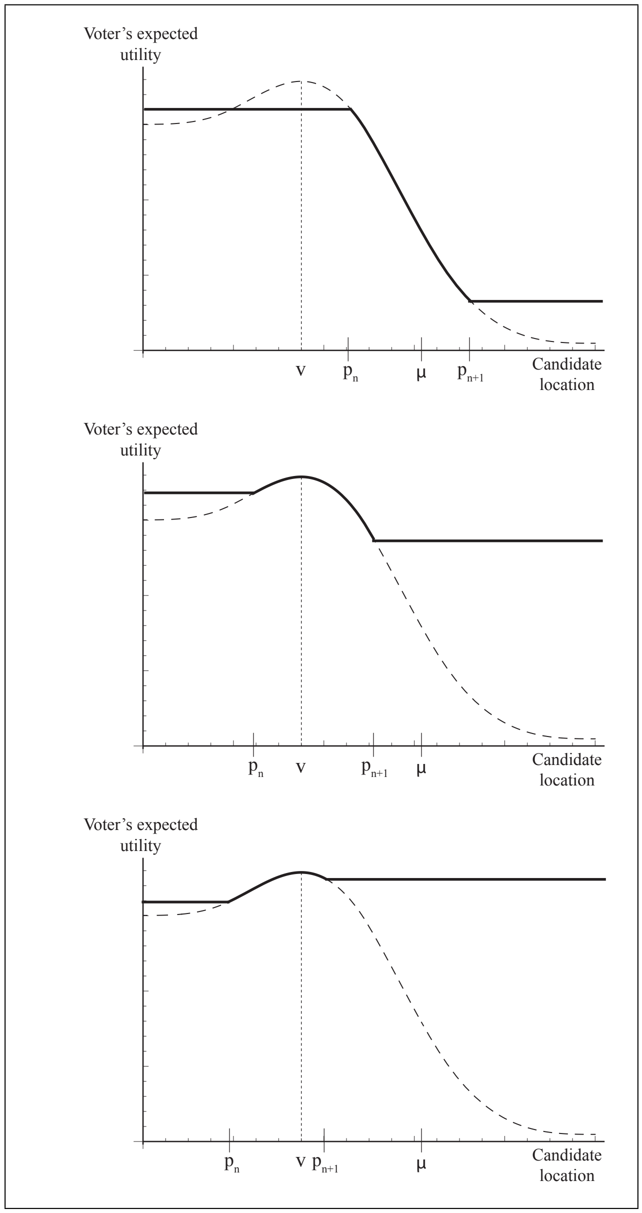


Figure 14: Instrumental Voter Preferences with Known Median Interval

is

$$EU(p_L, v) - EU(p_R, v) = \int_{\mathbf{R}} \int_{2p_n - q}^{2p_{n+1}-q} (u(V(b,q,p_L),v) - u(V(b,q,p_R),v)) f(b,q)\, \mathrm{d}b\, \mathrm{d}q.$$

This is identical to the difference in the setting considered in the body of the article. Accordingly, Theorem 4 applies and establishes the claim. ∎

Proposition 6 implies that, as long as $\mu$ is not equal to the midpoint of the median interval, then any voter whose ideal point is sufficiently close to that midpoint will have a "global" taste for extremism. The following corollary states this formally.

**Corollary 2** *If $\mu \neq \frac{p_n + p_{n+1}}{2}$, then there exists $\varepsilon > 0$ such that if $\left|v - \frac{p_n+p_{n+1}}{2}\right| < \varepsilon$, the voter has a taste for extremism.*

Finally, Figure 15 illustrates why we require that $\mu$ and the median interval both be on the same side of $v$. When the voter's ideal point is located between $\mu$ and the median interval, the voter has taste for moderation relative to the agenda ($\mu$) but extremism relative to the median interval. Whether we would refer to a preference for deviations away from the agenda or away from the median of the legislature as a "preference for extremism" is simply a matter of labeling.

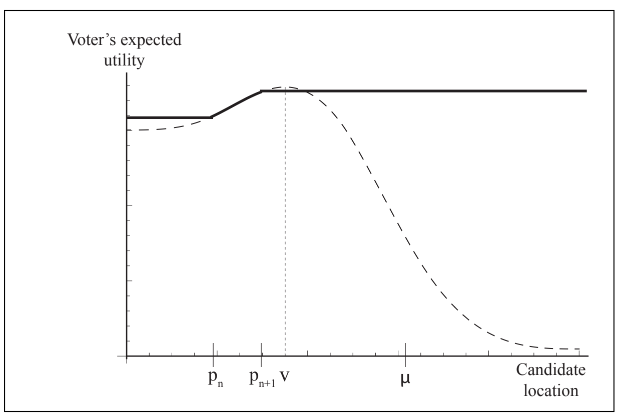


Figure 15: Instrumental Voter Preferences with Known Median Interval

# E Extension: Agenda Control (Online)

We have assumed throughout that the agenda is independent of the elected representative's platform. This is done for simplicity: as long as the representative does not *entirely* determine the agenda, our results continue to hold.

**A Simple Extension.** A general and parsimonious relaxation of this assumption is to assume that the agenda, $\alpha$, is a function of the elected representative's platform, $p$. For simplicity, we will assume that $\alpha$ is a product measure consistent with two cumulative distribution functions, $F_q$ and $F_b(\cdot;p)$: the distribution of the status quos is independent of the representative's platform, but the distribution of the bills can vary as the representative's platform changes. More specifically, we will assume that $F_b(\cdot;p)$ is consistent with a mixture distribution of the following form:

$$b \sim \begin{cases} D(p) & \text{with probability } \pi(p), \\ F_b^0 & \text{with probability } 1-\pi(p), \end{cases}$$

where $D(p)$ represents a point mass on $p$, and $\pi(p) \in [0,1]$ is an exogenous, commonly known, and continuously differentiable function, $\pi : \mathbf{R} \to [0,1]$.

This representation of agenda control is equivalent to assuming that, with probability $\pi(p)$, the representative is given complete control of the agenda and can implement his or her own platform, $p$ while, with complementary probability $1-\pi(p)$, the representative chooses between a status quo, $q$, and a bill, $b$, that are exogenously determined as in the baseline theory (*i.e.*, according to the agenda $\alpha = F_q \times F_b^0$). In this case, for a voter with ideal point $v$, the expected payoff from a representative with platform $p$ is

$$\overline{EU}(p,v;\pi) = -\pi(p)|p-v| - (1-\pi(p))EU(p;v),$$

where $EU(p;v)$ is the voter's expected utility conditional on the representative voting on the agenda as distributed by $\alpha$, given platform $p$ and the voter's ideal point $v$. This representation of agenda control is clearly stark, but its simplicity facilitates the analysis while focusing attention on the degree to which the exogenous agenda assumption can be relaxed without altering our central conclusions.

**Platform-Independent Agenda Control.** The first result establishes that, if the probability of determining the agenda is less than one and identical for all representatives, regardless of their platform $p$, then the voter's induced preference for moderation or extremism is unchanged.

**Proposition 7** *Suppose that $\pi(p)$ is a constant function: $\pi(p) = \bar{\pi} < 1$ for all platforms $p$. Then the voter's preference for moderation or extremism is independent of $\bar{\pi}$.*

*Proof*: Fix $v \leq \mu$ (the case of $v > \mu$ is symmetric), $\bar{\pi} \in [0, 1)$, and $\delta > 0$. Then the voter's net expected payoff, given ideal point $v < \mu$, from an extreme candidate with platform $p_L = v - \delta$ relative to that from a moderate candidate with platform $p_L = v + \delta$, $\Delta(v, \delta, \bar{\pi})$, is

$$\begin{aligned}
\Delta(v, \delta, \bar{\pi}) &= \overline{EU}(v - \delta, v; \pi) - \overline{EU}(v + \delta, v; \pi), \\
&= \bar{\pi}\delta + (1 - \bar{\pi})EU(v - \delta; v) - \bar{\pi}\delta - (1 - \bar{\pi})EU(v + \delta; v), \\
&= (1 - \bar{\pi})\left(EU(v - \delta; v) - EU(v + \delta; v)\right),
\end{aligned}$$

the sign of which is independent of $\bar{\pi} < 1$. This implies that the voter's taste for extremism or moderation is identical to that identified in the body of the article (where $\pi(p) = 0$ for all $p$), as was to be shown. ∎

**Platform-Dependent Agenda Control.** Considering the case where the degree of agenda control can vary with the candidate's platform, a little notation will simplify presentation. Specifically, for any ideal point $v$ and candidate divergence $\delta \geq 0$, let $\pi_v^+(\delta) \equiv \pi(v + \delta)$ and $\pi_v^-(\delta) \equiv \pi(v - \delta)$ Then the voter's net expected payoff, given ideal point $v < \mu$, from an extreme candidate with platform $p_L = v - \delta$ relative to that from a moderate candidate with platform $p_L = v + \delta$, $\Delta(v, \delta, \pi)$, is

$$\begin{aligned}
\Delta(v, \delta, \pi) &= \overline{EU}(v - \delta, v; \pi) - \overline{EU}(v + \delta, v; \pi), \\
&= (\pi_v^+(\delta) - \pi_v^-(\delta))\delta + \pi_v^+(\delta)EU(v + \delta, v; \pi) - \pi_v^+(\delta)EU(v - \delta, v; \pi) + \Delta(v, \delta, 0).
\end{aligned}$$

Our final result in this section, a corollary of Proposition 7, establishes that so long as $\pi$ is a smooth function of the candidate's platform, $p$, the voter always has a taste for extremism when comparing candidates whose platforms are close enough to his or her ideal point, $v$.

**Corollary 3** *For any agenda $\alpha$, voter ideal point, $v \neq \mu$, and any continuously differentiable function $\pi : \mathbf{R} \to [0,1]$, there exists $\hat{\delta}(v, \pi, \alpha) > 0$ such that the voter has a taste for extremism for all $\delta < \hat{\delta}(v, \pi, \alpha)$.*