Addendum for the paper "*The Liberal Ethics of Non-Interference*"

May 25, 2017

Abstract

This Addendum provides the demonstration of Theorem 1.

1 Formal proof of the impossibility result

We focus on societies with a finite set $\mathcal{N} = \{1, ..., N\}$ of agents with generic element i. Let X be the set of social states. Individual preferences are denoted $\succcurlyeq_i$ for $i \in \mathcal{N}$ and are assumed to be orderings, that is reflexive, transitive and complete binary relations on X. A social preference relation is denoted $\succcurlyeq$ and it is also assumed to be an ordering on X.

Let $\mathcal{R}$ (resp., $\mathcal{R}_{++}$) denote the set of (resp., positive) real numbers. We suppose that X can be written as $X = T \times (M_1 \times ... \times M_N)$, where $M_i \subseteq \mathcal{R}_{++}$ is the set of possible quantities of a special commodity called 'money' that agent i may own, while T is an arbitrary nonempty set of social states described entirely except for the money allocation. Let $\mathbf{m} = (m_1, ..., m_N)$ be a vector in $M_1 \times ... \times M_N$ which describes the amount of money that each agent $i \in \mathcal{N}$ obtains. A social state $x \in X$ is thus a tuple $x = (t, \mathbf{m})$ where $t \in T$. For every $\mathbf{m} \in M_1 \times ... \times M_N$ and every $i \in \mathcal{N}$, let $\mathbf{m}_{-i}$ denote the amount of money that all agents *except* i receive, so that we can write $\mathbf{m} = (\mathbf{m}_{-i}, m_i)$. For every $x \in X$, and for every $\overline{m}_i \in \mathcal{R}$, we write

$$x \circledast_i \overline{m}_i \equiv (t, (m_i + \overline{m}_i, \mathbf{m}_{-i})) \in X$$

to denote the modification of alternative x consisting of giving (or taking away) $\overline{m}_i$ extra units of money to agent i, such that the modified alternative is still in the set of social states X. For every $x \in X$, and for every $\overline{\mathbf{m}} = (\overline{m}_1, ..., \overline{m}_N) \in \mathcal{R}^N$, we also write

$$x \circledast \overline{\mathbf{m}} \equiv (t, (\mathbf{m} + \overline{\mathbf{m}})) \in X$$

to denote the modification of alternative x consisting of giving (or taking away) $\overline{m}_1$ units of money to agent 1, $\overline{m}_2$ units of money to agent 2, and so on, such that the modified alternative is still in the set of social states X. Observe that for a given $\overline{\mathbf{m}} = (\overline{m}_1, ..., \overline{m}_N) \in \mathcal{R}^N$, $x \circledast \overline{\mathbf{m}} = (((x \circledast_1 \overline{m}_1) \circledast_2 \overline{m}_2) ...) \circledast_N \overline{m}_N$.

The following assumptions together define a *rich* economic environment:

Richness (R):

- *Indifference to others' money (R1).* For all $x \in X$, all $i \in \mathcal{N}$, and all $\overline{m}_i \in \mathcal{R}$ such that $x \circledast_i \overline{m}_i \in X$: $x \circledast_i \overline{m}_i \sim_j x$ for all $j \in \mathcal{N} \setminus \{i\}$.

- *Desirability of own money (R2).* For all $x \in X$, all $i \in \mathcal{N}$, and all $\overline{m}_i \in \mathcal{R}$ such that $x \circledast_i \overline{m}_i \in X$: $x \circledast_i \overline{m}_i \succcurlyeq_i x \Leftrightarrow \overline{m}_i \geq 0$.

- *Divisibility of own money (R3).* For all $i \in \mathcal{N}$ and all $x, y, z \in X$ such that $x \succ_i y \succ_i z$ there exist $\overline{m}_i, \overline{m}'_i \in \mathcal{R}$ such that $x \circledast_i \overline{m}_i \in X$, $z \circledast_i \overline{m}'_i \in X$, $y \succ_i x \circledast_i \overline{m}_i \succ_i z$, and $x \succ_i z \circledast_i \overline{m}'_i \succ_i y$.

R1 entails a society in which there is neither 'money envy' nor 'money sympathy': therefore our impossibility result is not due to any externality of this sort. R2 imposes a mild monotonicity assumption on individual preferences. R3 implies some sort of Archimedean continuity of individual preferences: it states that it is always possible to vary the amount of money possessed by an individual in such a way as to alter the ranking of any physical alternatives. This assumption is satisfied if, for example, money becomes progressively more valuable as its scarcity increases.

At a more general level, our *Richness* assumption plays a conceptually similar role to the standard '*Diversity*' assumptions used in the literature on single-profile Arrovian impossibility results[1] in that it guarantees a sufficiently large space of alternatives for a given profile of individual preference relations. Our economic environment is similar to that analysed by Kaplow and Shavell in their study of nonwelfarist social welfare functions.[2]

Finally, our three basic axioms for $\succcurlyeq$ can be formally stated as follows:

Weak Pareto (WP): For all $x, y \in X$, if $x \succ_i y$ for all $i \in \mathcal{N}$, then $x \succ y$.

Non-Dictatorship (ND): For all $i \in \mathcal{N}$, there exist $x, y \in X$ such that $x \succ_i y$ and $y \succcurlyeq x$.

Non-Interference (NI): Let $x, y \in X$ be such that $x \succ y$, and let $x', y' \in X$ be such that, for some $i \in \mathcal{N}$,

$$\begin{aligned} x' &\succcurlyeq_i x \Leftrightarrow y' \succcurlyeq_i y \\ &\textit{not}\,(x' \sim_i x) \\ x' &\sim_j x \text{ and } y' \sim_j y \text{ for all } j \in \mathcal{N} \setminus \{i\}. \end{aligned}$$

Then $y' \not\succ x'$ whenever $x' \succ_i y'$.

In order to prove Theorem 1, we first prove a simple technical Lemma.

[1] See, for example, Feldman and Serrano 2008.

[2] Kaplow and Shavell 2001.

Lemma 1 *: Assume R. Let a social welfare ordering $\succcurlyeq$ satisfy WP. Consider $x, y \in X$, $x \neq y$, such that $x \succ_i y$ for some $i \in \mathcal{N}$. Then:*

(i) if $y \succcurlyeq x$, then there exist $x', y' \in X$, such that $x' \succ_i y'$ and $y' \succ x'$;

(ii) if $y \succ x$, then for any $j \in \mathcal{N} \setminus \{i\}$, there exist $x', y' \in X$, such that $x' \succ_i y'$, $y' \succ_j x'$ and $y' \succ x'$.

Proof. 1. By R2 and R3, for all $k \in \mathcal{N}$ and all $x,y \in X$ such that $x \succ_k y$, there exist $\overline{m}_k, \overline{m}'_k \in \mathcal{R}$ such that $x \circledast_k \overline{m}_k \in X$, $y \circledast_k \overline{m}'_k \in X$ and $x \succ_k x \circledast_k \overline{m}_k \succ_k y$ and $x \succ_k y \circledast_k \overline{m}'_k \succ_k y$.

Part (i). 2. Suppose $x, y \in X$, $x \neq y$, are such that $x \succ_i y$ and $y \succcurlyeq x$. If $y \succ x$ then the result follows trivially. So, suppose $y \sim x$.

3. By step 1, there exists $\overline{m}_i \in \mathcal{R}$ such that $x \circledast_i \overline{m}_i \in X$, $x \succ_i x \circledast_i \overline{m}_i \succ_i y$. By R1, $x \circledast_i \overline{m}_i \sim_j x$, for all $j \in \mathcal{N} \setminus \{i\}$. Consider next any $j \in \mathcal{N} \setminus \{i\}$. By R2, there exists $\overline{m}_j \in \mathcal{R}$ such that $\overline{m}_j < 0$, $(x \circledast_i \overline{m}_i) \circledast_j \overline{m}_j \in X$ and $x \circledast_i \overline{m}_i \succ_j (x \circledast_i \overline{m}_i) \circledast_j \overline{m}_j$ and by the transitivity of $\succcurlyeq_j$, $x \succ_j (x \circledast_i \overline{m}_i) \circledast_j \overline{m}_j$. By R1, $x \circledast_i \overline{m}_i \sim_l (x \circledast_i \overline{m}_i) \circledast_j \overline{m}_j$, for all $l \in \mathcal{N} \setminus \{j\}$. Proceeding recursively for all $k \in \mathcal{N}$, it is possible to construct a social state $x \circledast \overline{\mathbf{m}} \in X$ such that $x \succ_k x \circledast \overline{\mathbf{m}}$ for all $k \in \mathcal{N}$. By WP and the transitivity of $\succcurlyeq$, $y \succ x \circledast \overline{\mathbf{m}}$. The desired result then follows noting that $x \circledast \overline{\mathbf{m}} \succ_i y$.

Part (ii). 4. Suppose $x, y \in X$, $x \neq y$, are such that $x \succ_i y$ and $y \succ x$. If $y \succ_k x$, for all $k \in \mathcal{N} \setminus \{i\}$, then the result follows trivially. So, suppose $x \succcurlyeq_j y$, for some $j \in \mathcal{N} \setminus \{i\}$.

5. Consider $i \in \mathcal{N}$. By step 1, there exists $\overline{m}_i \in \mathcal{R}$ such that $x \succ_i y \circledast_i \overline{m}_i \succ_i y$. By R1, $y \circledast_i \overline{m}_i \sim_k y$, for all $k \in \mathcal{N} \setminus \{i\}$. Consider next $j \in \mathcal{N}$. By the transitivity of $\succcurlyeq_j$, $x \succcurlyeq_j (y \circledast_i \overline{m}_i)$. Then, by R2 and R3, there exists $\overline{m}_j \in \mathcal{R}$ such that $\overline{m}_j > 0$, $(y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j \in X$ and $(y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j \succ_j x$. By R1, $y \circledast_i \overline{m}_i \sim_k (y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j$, for all $k \in \mathcal{N} \setminus \{j\}$. Finally, consider $k \in \mathcal{N} \setminus \{i, j\}$. By R2, there exists $\overline{m}_k \in \mathcal{R}$ such that $\overline{m}_k > 0$, $((y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j) \circledast_k \overline{m}_k \in X$ and $((y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j) \circledast_k \overline{m}_k \succ_k (y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j$ and by the transitivity of $\succcurlyeq_k$, $((y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j) \circledast_k \overline{m}_k \succ_k y$. By R1, $((y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j) \circledast_k \overline{m}_k \sim_l (y \circledast_i \overline{m}_i) \circledast_j \overline{m}_j$, for all $l \in \mathcal{N} \setminus \{k\}$. Proceeding recursively for all $k \in \mathcal{N}$, it is possible to construct a social state $y \circledast \overline{\mathbf{m}} \in X$ such that $y \circledast \overline{\mathbf{m}} \succ_k y$ for all $k \in \mathcal{N}$. By WP and the transitivity of $\succcurlyeq$, $y \circledast \overline{\mathbf{m}} \succ x$. The desired result then follows noting that $x \succ_i y \circledast \overline{\mathbf{m}}$ and $y \circledast \overline{\mathbf{m}} \succ_j x$. ■

We can now prove our main result.

Theorem 1*: Assume R. There is no social preference ordering $\succcurlyeq$ that satisfies WP, ND, and NI.*

Proof. 1. By R2 and R3, for all $i \in \mathcal{N}$ and all $x,y \in X$ such that $x \succ_i y$, there exist $\overline{m}_i, \overline{m}'_i \in \mathcal{R}$ such that $x \circledast_i \overline{m}_i \in X$, $y \circledast_i \overline{m}'_i \in X$ and $x \succ_i x \circledast_i \overline{m}_i \succ_i y$ and $x \succ_i y \circledast_i \overline{m}'_i \succ_i y$.

2. Fix $i, j \in \mathcal{N}$, $j \neq i$. By ND and Lemma 1(i), there exist $p, q \in X$ such that $p \succ_i q$, and $q \succ p$. By Lemma 1(ii), we suppose $q \succ_j p$, without loss of generality. By ND and Lemma 1(i), there are $x, y \in X$ such that $y \succ_j x$ and $x \succ y$. Again, by Lemma 1(ii), x, y can be chosen such that $x \succ_i y$, without

loss of generality. Further, without loss of generality, let $j = 1$, and $i = 2$. We want to show that, starting from p, q, x, y, it is possible to construct social states $\widetilde{p}, \widetilde{q}, \widetilde{x}, \widetilde{y} \in X$ such that $\widetilde{q} \succ \widetilde{p}$, $\widetilde{x} \succ \widetilde{y}$, but $\widetilde{p} \succ \widetilde{x}$, $\widetilde{y} \succ \widetilde{q}$, yielding the desired contradiction.

3. In particular, for any social states $z, z' \in X$, and for any $k \leq N$, we write $z \succ_{(k)} z'$ (respectively, $z \sim_{(k)} z'$) to mean that z is strictly preferred (respectively, indifferent) to z' by the first k agents in $\mathcal{N}$. We shall proceed by induction to show that starting from p, q, x, y, for any $k \geq 2$, it is possible to construct social states $p^{k-1}, q^{k-1} \in X$ such that $q^{k-1} \succ p^{k-1}$ and to find two social states $x^{k-1}, y^{k-1} \in X$ such that $x^{k-1} \succ y^{k-1}$, $p^{k-1} \succ_{(k)} x^{k-1}$, and $y^{k-1} \succ_{(k)} q^{k-1}$. The desired contradiction is then obtained by WP, and the transitivity of $\succcurlyeq$, for $k = N$.

4. ($k = 2$.) We show that, starting from p, q, x, y, there exist social states $p^1, q^1, x^1, y^1 \in X$ such that $q^1 \succ p^1$, $x^1 \succ y^1$, $p^1 \succ_{(2)} x^1$, and $y^1 \succ_{(2)} q^1$. There are in principle a number of cases to consider. Consider the configuration $x \succcurlyeq_1 p$, $p \succ_2 x$, $y \succ_1 q$, and $q \succcurlyeq_2 y$.

By step 1, there exist $\overline{m}_2, \overline{m}'_2 \in \mathcal{R}$ such that $x \circledast_2 \overline{m}_2 \in X$, $y \circledast_2 \overline{m}'_2 \in X$, $p \succ_2 x \circledast_2 \overline{m}_2 \succ_2 x$ and $y \circledast_2 \overline{m}'_2 \succ_2 q \succcurlyeq_2 y$. Furthermore, noting that by construction $p \succ_2 q$ and $x \succ_2 y$, by step 1, $\overline{m}_2, \overline{m}'_2 \in \mathcal{R}$ can be chosen such that $x \circledast_2 \overline{m}_2 \succ_2 y \circledast_2 \overline{m}'_2$. By R1, $x \circledast_2 \overline{m}_2 \sim_i x$ and $y \circledast_2 \overline{m}'_2 \sim_i y$, for all $i \in \mathcal{N} \setminus \{2\}$. Therefore, by NI, it follows that $x \circledast_2 \overline{m}_2 \succcurlyeq y \circledast_2 \overline{m}'_2$.

Further, by step 1, there exist $\overline{m}_1, \overline{m}'_1 \in \mathcal{R}$ such that $p \circledast_1 \overline{m}_1 \in X$, $q \circledast_1 \overline{m}'_1 \in X$, $p \circledast_1 \overline{m}_1 \succ_1 x \circledast_2 \overline{m}_2 \sim_1 x \succcurlyeq_1 p$ and $y \circledast_2 \overline{m}'_2 \sim_1 y \succ_1 q \circledast_1 \overline{m}'_1 \succ_1 q$. Furthermore, noting that by construction $q \succ_1 p$ and $y \succ_1 x$, by step 1, $\overline{m}_1, \overline{m}'_1 \in \mathcal{R}$ can be chosen such that $q \circledast_1 \overline{m}'_1 \succ_1 p \circledast_1 \overline{m}_1$. By R1, $p \circledast_1 \overline{m}_1 \sim_i p$ and $q \circledast_1 \overline{m}'_1 \sim_i q$, for all $i \in \mathcal{N} \setminus \{1\}$. Therefore, by NI, it follows that $q \circledast_1 \overline{m}'_1 \succcurlyeq p \circledast_1 \overline{m}_1$.

If $x \circledast_2 \overline{m}_2 \sim y \circledast_2 \overline{m}'_2$, then consider the social state $(x \circledast_2 \overline{m}_2) \circledast \overline{\mathbf{m}}'' \in X$ with $\overline{m}''_k > 0$ for all $k \in \mathcal{N}$. By step 1 and R1, $\overline{m}''_1, \overline{m}''_2 \in \mathcal{R}$ can be chosen such that $p \circledast_1 \overline{m}_1 \succ_l (x \circledast_2 \overline{m}_2) \circledast \overline{\mathbf{m}}'' \succ_l (x \circledast_2 \overline{m}_2)$, $l = 1, 2$. By WP and transitivity, it follows that $(x \circledast_2 \overline{m}_2) \circledast \overline{\mathbf{m}}'' \succ y \circledast_2 \overline{m}'_2$. A similar argument holds if $q \circledast_1 \overline{m}'_1 \sim p \circledast_1 \overline{m}_1$. Therefore, without loss of generality, we can assume $x \circledast_2 \overline{m}_2 \succ y \circledast_2 \overline{m}'_2$ and $q \circledast_1 \overline{m}'_1 \succ p \circledast_1 \overline{m}_1$. It is easily checked that any initial configuration of the social states p, q, x, y can be similarly transformed using NI (and WP), which proves our claim for $k = 2$, by setting $p^1 = p \circledast_1 \overline{m}_1, q^1 = q \circledast_1 \overline{m}'_1$, $x^1 = x \circledast_2 \overline{m}_2$, and $y^1 = y \circledast_2 \overline{m}'_2$. Furthermore, if $N = 2$, by WP, it follows that $p^1 \succ x^1$ and $y^1 \succ q^1$, yielding the desired contradiction. So suppose that $N > 2$.

5. (Induction step.)

5.1. Suppose that there exist p^{k-2}, q^{k-2}, $x^{k-2}, y^{k-2} \in X$ such that $q^{k-2} \succ p^{k-2}$, $x^{k-2} \succ y^{k-2}$, $p^{k-2} \succ_{(k-1)} x^{k-2}$, and $y^{k-2} \succ_{(k-1)} q^{k-2}$. If $p^{k-2} \succ_k x^{k-2}$, and $y^{k-2} \succ_k q^{k-2}$, the result follows immediately. So suppose $x^{k-2} \succcurlyeq_k p^{k-2}$.

By R2, for any $\overline{m}_k, \overline{m}'_k \in \mathcal{R}_{++}$, $p^{k-2} \circledast_k \overline{m}_k \in X$, $q^{k-2} \circledast_k \overline{m}'_k \in X$, $p^{k-2} \circledast_k \overline{m}_k \succ_k p^{k-2}$, and $q^{k-2} \circledast_k \overline{m}'_k \succ_k q^{k-2}$. Noting that $x^{k-2} \succcurlyeq_k p^{k-2}$, by step 1, $\overline{m}_k, \overline{m}'_k \in \mathcal{R}_{++}$ can be chosen such that $q^{k-2} \circledast_k \overline{m}'_k \succ_k p^{k-2} \circledast_k \overline{m}_k \succ_k x^{k-2}$. By R1, $p^{k-2} \circledast_k \overline{m}_k \sim_i p^{k-2}$ and $q^{k-2} \circledast_k \overline{m}'_k \sim_i q^{k-2}$, for all $i \in \mathcal{N} \setminus \{k\}$. Therefore, by NI, it follows that $q^{k-2} \circledast_k \overline{m}'_k \succcurlyeq p^{k-2} \circledast_k \overline{m}_k$.

If $q^{k-2} \circledast_k \overline{m}'_k \sim p^{k-2} \circledast_k \overline{m}_k$, then consider the social state $\left(p^{k-2} \circledast_k \overline{m}_k\right) \circledast \overline{\mathbf{m}}'' \in X$, with $\overline{m}''_i < 0$ for all $i \in \mathcal{N}$. By step 1, and R1, $\overline{m}''_1, \overline{m}''_2, ..., \overline{m}''_k \in \mathcal{R}$ can be chosen such that $p^{k-2} \circledast_k \overline{m}_k \succ_j \left(p^{k-2} \circledast_k \overline{m}_k\right) \circledast \overline{\mathbf{m}}'' \succ_j x^{k-2}$, all $j \leq k$. By R2, WP and the transitivity of $\succcurlyeq$, it follows that $q^{k-2} \circledast_k \overline{m}'_k \succ \left(p^{k-2} \circledast_k \overline{m}_k\right) \circledast \overline{\mathbf{m}}''$. Therefore, without loss of generality, we can assume $q^{k-2} \circledast_k \overline{m}'_k \succ p^{k-2} \circledast_k \overline{m}_k$. If $y^{k-2} \succ_k q^{k-2} \circledast_k \overline{m}'_k$, the desired result is obtained. Hence, suppose $q^{k-2} \circledast_k \overline{m}'_k \succcurlyeq_k y^{k-2}$. Let ${}_kp^{k-2} \equiv p^{k-2} \circledast_k \overline{m}_k$ and ${}_kq^{k-2} \equiv q^{k-2} \circledast_k \overline{m}'_k$.

5.2. The reasoning in step 5.1 can be repeated for all $i > k$ to give a sequence of social states $\left\{{}_ip^{k-2}, {}_iq^{k-2}\right\}_{i=k}^{N}$ in X such that for all i, ${}_ip^{k-2} \succ_i {}_{i-1}p^{k-2}$, ${}_iq^{k-2} \succ_i {}_{i-1}q^{k-2}$, ${}_iq^{k-2} \succ_i {}_ip^{k-2} \succ_i x^{k-2}$, and by R1 ${}_ip^{k-2} \sim_j {}_{i-1}p_i^{k-2}$, ${}_iq^{k-2} \sim_j {}_{i-1}q^{k-2}$, all $j \in \mathcal{N} \setminus \{i\}$, where ${}_{k-1}p^{k-2} \equiv p^{k-2}$ and ${}_{k-1}q^{k-2} \equiv q^{k-2}$. Therefore, ${}_Nq^{k-2} \succ {}_Np^{k-2}$. Let $q'^{k-2} \equiv {}_Nq^{k-2}$and $p'^{k-2} \equiv {}_Np^{k-2}$.

5.3. By WP $p'^{k-2} \succ x^{k-2}$: if $y^{k-2} \succcurlyeq q'^{k-2}$ then a contradiction immediately follows. Therefore assume $q'^{k-2} \succ y^{k-2}$ and noting that $y^{k-2} \succ_{(k-1)} q'^{k-2} \sim_{(k-1)} q^{k-2}$, by Lemma 1, we suppose $q'^{k-2} \succ_k y^{k-2}$, without loss of generality. In order to simplify the notation, let $q^{k-1} \equiv q'^{k-2}$ and $p^{k-1} \equiv y^{k-2}$, and recall that $p^{k-1} \succ_{k-1} q^{k-1}$, $q^{k-1} \succ_k p^{k-1}$, and $q^{k-1} \succ p^{k-1}$.

By ND and Lemma 1, there exist $x^{k-1}, y^{k-1} \in X$ such that $x^{k-1} \succ_{k-1} y^{k-1}$, $y^{k-1} \succ_k x^{k-1}$, and $x^{k-1} \succ y^{k-1}$. Following the same reasoning as in step 4 above, without loss of generality, we suppose that $p^{k-1} \succ_{k-1} x^{k-1}$, $p^{k-1} \succ_k x^{k-1}$, $y^{k-1} \succ_{k-1} q^{k-1}$, and $y^{k-1} \succ_k q^{k-1}$. If $p^{k-1} \succ_{(k-2)} x^{k-1}$ and $y^{k-1} \succ_{(k-2)} q^{k-1}$, then $p^{k-1} \succ_{(k)} x^{k-1}$, and $y^{k-1} \succ_{(k)} q^{k-1}$, and the desired result holds.

5.4. If, for some $i \leq k-2$, either $x^{k-1} \succcurlyeq_i p^{k-1}$, or $q^{k-1} \succcurlyeq_i y^{k-1}$, or both, holds, then let $m = \max\left\{i \leq k-2 : \text{either } x^{k-1} \succcurlyeq_i p^{k-1}, \text{ or } q^{k-1} \succcurlyeq_i y^{k-1}, \text{ or both}\right\}$ and note that since $p^{k-1} \succ_{(k-1)} q^{k-1}$ by construction, then by applying (repeatedly, if necessary) NI, WP, and R1, it is possible to construct social states ${}_mx^{k-1}, {}_my^{k-1} \in X$ such that $p^{k-1} \succ_m {}_mx^{k-1} \succ_m {}_my^{k-1} \succ_m q^{k-1}$, ${}_mx^{k-1} \sim_i x^{k-1}$, ${}_my^{k-1} \sim_i y^{k-1}$, all $i \in \mathcal{N} \setminus \{m\}$, and ${}_mx^{k-1} \succcurlyeq {}_my^{k-1}$. If ${}_mx^{k-1} \sim {}_my^{k-1}$, then consider the social state ${}_my^{k-1} \circledast \overline{\mathbf{m}} \in X$ with $\overline{m}_l < 0$ for all $l \in \mathcal{N}$. By step 1 and R1, $\overline{m}_m, \overline{m}_{m+1}, ..., \overline{m}_k \in \mathcal{R}$ can be chosen such that ${}_my^{k-1} \succ_i {}_my^{k-1} \circledast \overline{\mathbf{m}} \succ_i q^{k-1}$, all $m \leq i \leq k$. By WP and the transitivity of $\succcurlyeq$, it follows that ${}_mx^{k-1} \succ {}_my^{k-1} \circledast \overline{\mathbf{m}}$. Therefore, without loss of generality, we can assume ${}_mx^{k-1} \succ {}_my^{k-1}$.

5.5. This reasoning can be iterated for all $i < m$, to give a sequence of social states $\left\{{}_{m-i}x^{k-1}, {}_{m-i}y^{k-1}\right\}_{i=0}^{m-1}$ in X such that, for all i, $p^{k-1} \succ_{m-i} {}_{m-i}x^{k-1} \succ_{m-i} {}_{m-i}y^{k-1} \succ_{m-i} q^{k-1}$, and by R1 ${}_{m-i}x^{k-1} \sim_j {}_{m-i+1}x^{k-1}$, ${}_{m-i}y^{k-1} \sim_j {}_{m-i+1}y^{k-1}$, all $j \in \mathcal{N} \setminus \{m-i\}$, and ${}_{m-i}x^{k-1} \succ {}_{m-i}y^{k-1}$, where ${}_{m+1}x^{k-1} \equiv x^{k-1}$ and ${}_{m+1}y^{k-1} \equiv y^{k-1}$. Therefore, by letting $x'^{k-1} \equiv {}_1x^{k-1}$ and $y'^{k-1} \equiv {}_1y^{k-1}$, we have $x'^{k-1} \succ y'^{k-1}$ and noting that from the previous argument it follows that $p^{k-1} \succ_{(k)} x'^{k-1}$, $y'^{k-1} \succ_{(k)} q^{k-1}$, the desired result is obtained.

6. For any finite N, if $k = N$, define the social states $\widetilde{p} \equiv p^{k-1}$, $\widetilde{q} \equiv q^{k-1}$, $\widetilde{x} \equiv x'^{k-1}$, and $\widetilde{y} \equiv y'^{k-1}$. By construction $\widetilde{q} \succ \widetilde{p}$ and $\widetilde{x} \succ \widetilde{y}$, whereas given

step 5 above, it follows by WP that $\widetilde{p} \succ \widetilde{x}$, and $\widetilde{y} \succ \widetilde{q}$, which yields the desired contradiction. ■

References

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