Bayesian models with dominance effects for genomic evaluation of quantitative traits

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APPENDIX: Proofs

In this electronic appendix, the proofs of the numbered equations in the main text and in the printed appendix are given.

Equation

$$w_j = 1 - 2P(\text{sign}(a_j) = \text{sign}(d_j)) \tag{1}$$

Proof:

$$\begin{aligned}
P(\text{sign}(a_j) = \text{sign}(d_j)) &= P(\text{sign}(a_j) = 1, d_j > 0) + P(\text{sign}(a_j) = -1, d_j < 0) \\
&= P(\text{sign}(a_j) = 1 | d_j > 0) P(d_j > 0) \\
&+ P(\text{sign}(a_j) = -1 | d_j < 0) P(d_j < 0) \\
&= \text{pos}_j(1) P(d_j > 0) + (1 - \text{pos}_j(-1)) P(d_j < 0) \\
&= \frac{1 - w_j}{2} P(d_j > 0) + (1 - \frac{1 + w_j}{2}) P(d_j < 0) \\
&= \frac{1 - w_j}{2} P(d_j > 0) + \frac{1 - w_j}{2} P(d_j < 0) \\
&= \frac{1 - w_j}{2} (P(d_j > 0) + P(d_j < 0)) \\
&= \frac{1 - w_j}{2}.
\end{aligned}$$

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Equation

$$p(\tilde{\theta}_j|\tau_j^2,\gamma_j) = \frac{2g_j(\tilde{a}_j,\tilde{d}_j)}{\kappa_j\sqrt{2\pi\tau_j^2}}\exp\left(-\frac{\tilde{a}_j{}^2}{2\kappa_j^2\tau_j^2}\right)\psi_{\kappa_j,\tau_j^2}(\tilde{a}_j,\tilde{d}_j) \tag{2}$$

Proof:

We first consider BayesD1 - BayesD3. The joint conditional density of the putative marker effects $|a_j|$ and d_j given τ_j^2 and γ_j is

$$\begin{aligned} p(|a_j|,d_j|\tau_j^2,\gamma_j) &= p(|a_j||\tau_j^2,\gamma_j)p(d_j||a_j|,\tau_j^2,\gamma_j) \\ &= \frac{2}{\sqrt{2\pi\tau_j^2}}\exp\left(-\frac{1}{2}\left(\frac{|a_j|^2}{\tau_j^2}\right)\right)\frac{1}{\sqrt{2\pi\sigma_d^2(|a_j|,\tau_j^2)}}\exp\left(-\frac{(d_j-\mu_d(|a_j|))^2}{2\sigma_d^2(|a_j|,\tau_j^2)}\right) \end{aligned}$$

for $d_j \in I\!\!R$ and $|a_j| > 0$. Thus, the joint density of a_j, d_j given τ_j^2 and γ_j is

$$\begin{aligned} f_{\tau_j^2}((a_j,d_j)^T) &= p(a_j,d_j|\tau_j^2,\gamma_j) \\ &= p(s_j||a_j|,d_j,\tau_j^2,\gamma_j)p(|a_j|,d_j|\tau_j^2,\gamma_j) \\ &= \text{pos}_j(d_j)^{\frac{1+\text{sign}(a_j)}{2}}(1-\text{pos}_j(d_j))^{\frac{1-\text{sign}(a_j)}{2}}p(|a_j|,d_j|\tau_j^2,\gamma_j). \\ &= g_j(a_j,d_j)p(|a_j|,d_j|\tau_j^2,\gamma_j) \\ &= \frac{2g_j(a_j,d_j)}{\sqrt{2\pi\tau_j^2}}\exp\left(-\frac{1}{2}\left(\frac{|a_j|^2}{\tau_j^2}\right)\right) \\ &\cdot \frac{1}{\sqrt{2\pi\sigma_d^2(|a_j|,\tau_j^2)}}\exp\left(-\frac{(d_j-\mu_d(|a_j|))^2}{2\sigma_d^2(|a_j|,\tau_j^2)}\right), \end{aligned}$$

where s_j is the sign of a_j and

$$\begin{aligned} g_j(a_j,d_j) &= \text{pos}_j(d_j)^{\frac{1+\text{sign}(a_j)}{2}}(1-\text{pos}_j(d_j))^{\frac{1-\text{sign}(a_j)}{2}} \\ &= \text{pos}_j(d_j)\,\text{sign}(a_j) + \frac{1-\text{sign}(a_j)}{2} \\ &= \frac{1-w_j\text{sign}(d_j)}{2}\,\text{sign}(a_j) + \frac{1-\text{sign}(a_j)}{2} \\ &= \frac{\text{sign}(a_j)-w_j\text{sign}(a_j)\text{sign}(d_j)}{2} + \frac{1-\text{sign}(a_j)}{2} \\ &= \frac{1-w_j\text{sign}(a_j)\text{sign}(d_j)}{2}. \end{aligned}$$

We can write $\binom{\tilde{a}_j}{\tilde{d}_j} = \binom{\kappa_j 0}{0 \kappa_j}\binom{a_j}{d_j}$ with 2×2 matrix $\binom{\kappa_j 0}{0 \kappa_j}$. The joint density of $\tilde{a}_j$,

$\tilde{d}_j$ given τ_j^2 and γ_j is

$$\begin{aligned} p(\tilde{a}_j, \tilde{d}_j|\tau_j^2, \gamma_j) &= \frac{1}{|\det\binom{\kappa_j 0}{0\kappa_j}|} f_{\tau_j^2}\left(\begin{pmatrix}\kappa_j 0\\ 0\kappa_j\end{pmatrix}^{-1}\begin{pmatrix}\tilde{a}_j\\ \tilde{d}_j\end{pmatrix}\right) \\ &= \frac{2g_j(\tilde{a}_j, \tilde{d}_j)}{\kappa_j\sqrt{2\pi\tau_j^2}} \exp\left(-\frac{\tilde{a}_j{}^2}{2\kappa_j^2\tau_j^2}\right) \\ &\cdot \frac{1}{\kappa_j\sqrt{2\pi\sigma_d^2(|\frac{\tilde{a}_j}{\kappa_j}|, \tau_j^2)}} \exp\left(-\frac{(\frac{\tilde{d}_j}{\kappa_j} - \mu_d(|\frac{\tilde{a}_j}{\kappa_j}|))^2}{2\sigma_d^2(|\frac{\tilde{a}_j}{\kappa_j}|, \tau_j^2)}\right). \end{aligned}$$

For BayesD0 we obtain

$$p(\tilde{a}_j|\tau_j^2, \gamma_j) = \frac{2\cdot\frac{1}{2}}{\kappa_j\sqrt{2\pi\tau_j^2}} \exp\left(-\frac{1}{2}\left(\frac{\tilde{a}_j^2}{\kappa_j^2\tau_j^2}\right)\right).$$

Thus, in any case we have

$$p(\tilde{\theta}_j|\tau_j^2, \gamma_j) = \frac{2g_j(\tilde{a}_j, \tilde{d}_j)}{\kappa_j\sqrt{2\pi\tau_j^2}} \exp\left(-\frac{\tilde{a}_j{}^2}{2\kappa_j^2\tau_j^2}\right)\psi_{\kappa_j,\tau_j^2}(\tilde{a}_j, \tilde{d}_j).$$

Equation

$$\begin{aligned} E(a_j) &= -w_j E\left(|a_j|\left(1 - 2\phi\left(\frac{-\mu_d(|a_j|)}{\sigma_d(|a_j|, \tau_j^2)}\right)\right)\right) \qquad (3) \\ E(|a_j|) &= \lambda\sqrt{E(a_j^2)} \\ E(a_j^2) &= E(\tau_j^2) = s^2\frac{v}{v-2} \\ E(d_j) &= E(\mu_d(|a_j|)) \\ E(d_j^2) &= E(\sigma_d^2(|a_j|, \tau_j^2)) + E(\mu_d(|a_j|)^2) \\ E(a_j d_j) &= -w_j E(|a_j||d_j|) \\ E(|a_j||d_j|) &= E\left(|a_j|\mu_d(|a_j|)K\left(\frac{\sigma_d(|a_j|, \tau_j^2)}{\mu_d(|a_j|)}\right)\right), \end{aligned}$$

where ϕ is the distribution function of the standard normal distribution.
Proof:

We have

$$\begin{aligned}
E(a_j \big| d_j, |a_j|, \tau_j^2) &= E(\text{sign}(a_j)|a_j| \big| d_j, |a_j|, \tau_j^2) \\
&= |a_j| E(\text{sign}(a_j) \big| d_j, |a_j|, \tau_j^2) \\
&= |a_j| (P(a_j > 0 \big| d_j) - P(a_j < 0 \big| d_j)) \\
&= |a_j| (2P(a_j > 0 \big| d_j) - 1) \\
&= |a_j| (2\text{pos}_j(d_j) - 1) \\
&= |a_j| \left(2\frac{1 - w_j \text{sign}(d_j)}{2} - 1 \right) \\
&= -w_j |a_j| \text{sign}(d_j).
\end{aligned}$$

Thus,

$$\begin{aligned}
E(a_j \big| |a_j|, \tau_j^2) &= E(-w_j |a_j| \text{sign}(d_j) \big| |a_j|, \tau_j^2) \\
&= -w_j |a_j| E(\text{sign}(d_j) \big| |a_j|, \tau_j^2) \\
&= -w_j |a_j| \left(P(d_j > 0 \big| |a_j|, \tau_j^2) - P(d_j < 0 \big| |a_j|, \tau_j^2) \right) \\
&= -w_j |a_j| \left(1 - 2P(d_j < 0 \big| |a_j|, \tau_j^2) \right) \\
&= -w_j |a_j| \left(1 - 2F_{\mu_d(|a_j|), \sigma_d^2(|a_j|, \tau_j^2)}(0) \right) \\
&= -w_j |a_j| \left(1 - 2\phi \left(-\frac{\mu_d(|a_j|)}{\sigma_d(|a_j|, \tau_j^2)} \right) \right).
\end{aligned}$$

It follows that

$$E(a_j) = -w_j E \left(|a_j| \left(1 - 2\phi \left(-\frac{\mu_d(|a_j|)}{\sigma_d(|a_j|, \tau_j^2)} \right) \right) \right).$$

Since $\frac{|a_j|}{s}$ has a folded t-distribution, we have for $v > 1$ (Psarakis and Panaretos, 1990)

$$E(|a_j|) = 2s\sqrt{\frac{v}{\pi}} \frac{\Gamma\left(\frac{v+1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)(v-1)}$$

Let $u_j | \tau_j^2, \gamma_j \sim \mathcal{N}(0, \tau_j^2)$. We have

$$\begin{aligned}
E(a_j^2) &= E(E(a_j^2 | \tau_j^2, \gamma_j)) \\
&= E(E(|a_j|^2 | \tau_j^2, \gamma_j)) \\
&= E(E(u_j^2 | \tau_j^2, \gamma_j)) \\
&= E(\text{Var}(u_j | \tau_j^2, \gamma_j)) + E(E(u_j | \tau_j^2, \gamma_j)^2) \\
&= E(\tau_j^2) + E(0),
\end{aligned}$$

and

$$
\begin{aligned}
E(d_j) &= E(E(d_j||a_j|,\tau_j^2,\gamma_j)) \\
&= E(\mu_d(|a_j|)),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(d_j) &= E(\mathrm{Var}(d_j||a_j|,\tau_j^2,\gamma_j)) + \mathrm{Var}(E(d_j||a_j|,\tau_j^2,\gamma_j)) \\
&= E(\sigma_d^2(|a_j|,\tau_j^2)) + \mathrm{Var}(\mu_d(|a_j|)),
\end{aligned}
$$

$$
\begin{aligned}
E(d_j^2) &= \mathrm{Var}(d_j) + E(d_j)^2 \\
&= E(\sigma_d^2(|a_j|,\tau_j^2)) + \mathrm{Var}(\mu_d(|a_j|)) + E(\mu_d(|a_j|))^2 \\
&= E(\sigma_d^2(|a_j|,\tau_j^2)) + E(\mu_d(|a_j|)^2),
\end{aligned}
$$

Since

$$
\begin{aligned}
E(\mathrm{sign}(a_jd_j)\big||a_j|,d_j) &= P(\mathrm{sign}(a_jd_j)=1\big||a_j|,d_j) - P(\mathrm{sign}(a_jd_j)=-1\big||a_j|,d_j) \\
&= P(\mathrm{sign}(a_j)=\mathrm{sign}(d_j)\big||a_j|,d_j) - P(\mathrm{sign}(a_j)\neq\mathrm{sign}(d_j)\big||a_j|,d_j) \\
&= P(\mathrm{sign}(a_j)=1\big||a_j|,d_j)^{\frac{1+\mathrm{sign}(d_j)}{2}} P(\mathrm{sign}(a_j)=-1\big||a_j|,d_j)^{\frac{1-\mathrm{sign}(d_j)}{2}} \\
&- P(\mathrm{sign}(a_j)=1\big||a_j|,d_j)^{\frac{1-\mathrm{sign}(d_j)}{2}} P(\mathrm{sign}(a_j)=-1\big||a_j|,d_j)^{\frac{1+\mathrm{sign}(d_j)}{2}} \\
&= \mathrm{pos}_j(1)^{\frac{1+\mathrm{sign}(d_j)}{2}}(1-\mathrm{pos}_j(-1))^{\frac{1-\mathrm{sign}(d_j)}{2}} - \mathrm{pos}_j(-1)^{\frac{1-\mathrm{sign}(d_j)}{2}}(1-\mathrm{pos}_j(1))^{\frac{1+\mathrm{sign}(d_j)}{2}} \\
&= \begin{cases} \mathrm{pos}_j(1)-(1-\mathrm{pos}_j(1)) = 2\frac{1-w_j}{2}-1 = -w_j & \text{if } d_j>0 \\ (1-\mathrm{pos}_j(-1))-\mathrm{pos}_j(-1) = 1-2\frac{1+w_j}{2} = -w_j & \text{if } d_j<0 \end{cases} \\
&= -w_j
\end{aligned}
$$

we have

$$
\begin{aligned}
E(a_jd_j) &= E(\mathrm{sign}(a_jd_j)|a_j||d_j|) \\
&= E(E(\mathrm{sign}(a_jd_j)|a_j||d_j|\big||a_j|,d_j)) \\
&= E(|a_j||d_j|E(\mathrm{sign}(a_jd_j)\big||a_j|,d_j)) \\
&= -w_jE(|a_j||d_j|)
\end{aligned}
$$

Since $d_j\big||a_j|,\tau_j^2$ has a normal distribution, $|d_j|\big||a_j|,\tau_j^2$ has a folded normal distribution and we have

$$
\begin{aligned}
E(|d_j|\big||a_j|,\tau_j^2) &= \sqrt{\frac{2\sigma_d^2(|a_j|,\tau_j^2)}{\pi}}\exp\left(-\frac{\mu_d(|a_j|)^2}{2\sigma_d^2(|a_j|,\tau_j^2)}\right) + \mu_d(|a_j|)\left[1-2\phi\left(-\frac{\mu_d(|a_j|)}{\sigma_d(|a_j|,\tau_j^2)}\right)\right] \\
&= \mu_d(|a_j|)K\left(\frac{\sigma_d(|a_j|,\tau_j^2)}{\mu_d(|a_j|)}\right).
\end{aligned}
$$

Thus,

$$\begin{aligned}
E(|a_j||d_j|\big|\tau_j^2) &= E(E(|a_j||d_j|\big||a_j|,\tau_j^2)\big|\tau_j^2) \\
&= E(|a_j|E(|d_j|\big||a_j|,\tau_j^2)\big|\tau_j^2) \\
&= E\left(|a_j|\mu_d(|a_j|)K\left(\frac{\sigma_d(|a_j|,\tau_j^2)}{\mu_d(|a_j|)}\right)\big|\tau_j^2\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
E(|a_j||d_j|) &= E(E(|a_j||d_j|\big|\tau_j^2)) \\
&= E\left(E\left(|a_j|\mu_d(|a_j|)K\left(\frac{\sigma_d(|a_j|,\tau_j^2)}{\mu_d(|a_j|)}\right)\big|\tau_j^2\right)\right) \\
&= E\left(|a_j|\mu_d(|a_j|)K\left(\frac{\sigma_d(|a_j|,\tau_j^2)}{\mu_d(|a_j|)}\right)\right).
\end{aligned}$$

Equation
For BayesD3 we have

$$\begin{aligned}
E(d_j) &= sE\left(|t|\mu_\Delta(|t|)\right), \\
E(d_j^2) &= s^2\frac{\sigma_\Delta^2 v}{v-2} + s^2E\left(t^2\mu_\Delta(|t|)^2\right), \\
E(|a_j||d_j|) &= s^2E\left(t^2\mu_\Delta(|t|)\,K\left(\frac{\sigma_\Delta}{\mu_\Delta(|t|)}\right)\right),
\end{aligned}$$

where $t \sim t_v$ has a t-distribution with v degrees of freedom.
Proof:

Since $|t|$ and $\frac{|a_j|}{s}$ are identically distributed, we have

$$\begin{aligned}
E(d_j) &= E(|a_j|\mu_\Delta(\tfrac{|a_j|}{s})) \\
&= sE\left(|t|\mu_\Delta(|t|)\right), \\
E(d_j^2) &= E(a_j^2\sigma_\Delta^2) + E((|a_j|\mu_\Delta(\tfrac{|a_j|}{s}))^2) \\
&= s^2\frac{\sigma_\Delta^2 v}{v-2} + s^2E\left(t^2\mu_\Delta(|t|)^2\right), \\
E(|a_j||d_j|) &= E\left(|a_j||a_j|\mu_\Delta(\tfrac{|a_j|}{s})\,K\left(\frac{|a_j|\sigma_\Delta}{|a_j|\mu_\Delta(\frac{|a_j|}{s})}\right)\right) \\
&= s^2E\left(t^2\mu_\Delta(|t|)\,K\left(\frac{\sigma_\Delta}{\mu_\Delta(|t|)}\right)\right),
\end{aligned}$$

Equation

$$\begin{aligned} & \mathrm{Cov}(g(x_h), g(x_i)) \qquad (4) \\ = & \sum_{j=1}^{M} x_{hj} x_{ij} \left(\mathrm{Var}(\tilde{a}_j) + (4 - x_{hj} - x_{ij}) \mathrm{Cov}(\tilde{a}_j, \tilde{d}_j) + (2 - x_{ij})(2 - x_{hj}) \mathrm{Var}(\tilde{d}_j) \right) \end{aligned}$$

Proof:

Take $x_i \in \{0,1,2\}^M$ to be the genotype vector of individual $i = 1, ..., n$. That is, x_{ij} is the number of 1-alleles at marker j in individual i. We have $Z_{Aij} = x_{ij}$ and $Z_{Dij} = x_{ij}(2 - x_{ij})$. The linear regression model assumes that the genotypic value of genotype x_i is

$$g_{GV}(x_i) = \sum_{j=1}^{M} Z_{Aij}\tilde{a}_j + Z_{Dij}\tilde{d}_j = \sum_{j=1}^{M} x_{ij} \left(\tilde{a}_j + (2 - x_{ij})\tilde{d}_j \right).$$

It follows that the covariance between the genotypic values of x_h and x_i is

$$\begin{aligned} \mathrm{Cov}(g(x_h), g(x_i)) &= \mathrm{Cov}(g_{GV}(x_h), g_{GV}(x_i)) \\ &= \sum_{j=1}^{M} \sum_{k=1}^{M} \mathrm{Cov} \left(x_{hj} \left(\tilde{a}_j + (2 - x_{hj})\tilde{d}_j \right), x_{ik} \left(\tilde{a}_k + (2 - x_{ik})\tilde{d}_k \right) \right). \end{aligned}$$

Since the effects of different markers are assumed to be independent, it follows that

$$\begin{aligned} & \mathrm{Cov}(g(x_h), g(x_i)) \\ = & \sum_{j=1}^{M} \mathrm{Cov} \left(x_{hj} \left(\tilde{a}_j + (2 - x_{hj})\tilde{d}_j \right), x_{ij} \left(\tilde{a}_j + (2 - x_{ij})\tilde{d}_j \right) \right) \\ = & \sum_{j=1}^{M} x_{hj} x_{ij} \mathrm{Cov} \left(\tilde{a}_j + (2 - x_{hj})\tilde{d}_j, \tilde{a}_j + (2 - x_{ij})\tilde{d}_j \right) \\ = & \sum_{j=1}^{M} x_{hj} x_{ij} \left(\mathrm{Var}(\tilde{a}_j) + (4 - x_{hj} - x_{ij}) \mathrm{Cov}(\tilde{a}_j, \tilde{d}_j) + (2 - x_{ij})(2 - x_{hj}) \mathrm{Var}(\tilde{d}_j) \right). \end{aligned}$$

Equation

$$\begin{aligned} V_{AM} &= ME(\kappa_j^2) \left(\overline{h_\circ} E(a_j^2) - 2E(|a_j||d_j|)\tilde{\gamma}_M + \gamma_M E(d_j^2) \right) \qquad (5) \\ V_{DM} &= ME(\kappa_j^2) \overline{h_\circ^2} E(d_j^2) \\ \mathcal{I}_M &= ME(\kappa_j) \, \overline{h_\circ} E(d_j) \end{aligned}$$

Proof:

The a priori expected additive variance V_{AM} captured by markers can be split up into different components. We have $V_{AM} = V_{AM}^a + V_{AM}^{ad} + V_{AM}^d$ where

$$\begin{aligned}
V_{AM}^a &= E\left(\sum_{j\in\mathcal{M}} h_j \tilde{a}_j^2\right) \\
&= M\overline{h_\circ}E(\kappa_j^2)E(a_j^2) \\
V_{AM}^{ad} &= E\left(2\sum_{j\in\mathcal{M}} h_j(q_j-p_j)\tilde{a}_j\tilde{d}_j\right) \\
&= -2\sum_{j\in\mathcal{M}} h_j(q_j-p_j)E(\kappa_j^2)w_jE(|a_j||d_j|) \\
&= -2E(\kappa_j^2)E(|a_j||d_j|)M\tilde{\gamma}_M \\
V_{AM}^d &= E\left(\sum_{j\in\mathcal{M}} h_j(q_j-p_j)^2\tilde{d}_j^2\right) \\
&= M\gamma_M E(\kappa_j^2)E(d_j^2).
\end{aligned}$$

Moreover,

$$\begin{aligned}
V_{DM} &= E\left(\sum_{j\in\mathcal{M}} h_j^2\tilde{d}_j^2\right) = M\overline{h_\circ^2}E(\kappa_j^2)E(d_j^2), \\
\mathcal{I}_M &= E\left(\sum_{j\in\mathcal{M}} h_j\tilde{d}_j\right) = M\overline{h_\circ}E(\kappa_j)E(d_j).
\end{aligned}$$

Equation

$$s^2 = \frac{V_A}{M\overline{h_\circ}E(\kappa_j^2)}\frac{v-2}{v} \tag{6}$$

Proof:

$$V_A = V_{AM} = M\overline{h_\circ}E(\kappa_j^2)E(a_j^2) + 0 + 0$$

Equation

$$\beta|\tilde{\theta}, u, \sigma^2, \gamma, \tau^2, y \sim \mathcal{N}\left(\widehat{\beta}, \sigma^2(X^TX)^{-1}\right) \tag{7}$$

Proof:

If the posterior is considered as a function of β, then it follows with $y' = y - Zu - Z_A\tilde{a} - Z_D\tilde{d}$ that

$$\begin{aligned}
& p(\beta|\tilde{\theta}, u, \sigma^2, \gamma, \tau^2, y) \\
\propto\ & p(\tilde{\theta}, \beta, u, \sigma^2, \gamma, \tau^2|y) \\
\propto\ & p(y|\tilde{\theta}, \xi) \\
\propto\ & \exp\left(-\frac{(y' - X\beta)^T(y' - X\beta)}{2\sigma^2}\right) \\
=\ & \exp\left(-\frac{y'^T y' - 2\beta^T X^T y' + \beta^T X^T X\beta}{2\sigma^2}\right) \\
=\ & \exp\left(-\frac{y'^T y' - 2\beta^T (X^T X)\widehat{\beta} + \beta^T X^T X\beta}{2\sigma^2}\right) \\
\propto\ & \exp\left(-\frac{\widehat{\beta}^T X^T X\widehat{\beta} - 2\beta^T (X^T X)\widehat{\beta} + \beta^T X^T X\beta}{2\sigma^2}\right) \\
=\ & \exp\left(-\frac{(\beta - \widehat{\beta})^T X^T X(\beta - \widehat{\beta})}{2\sigma^2}\right).
\end{aligned}$$

Thus,

$$\beta|\tilde{\theta}, u, \sigma^2, \gamma, \tau^2, y \sim \mathcal{N}\left(\widehat{\beta}, \sigma^2(X^TX)^{-1}\right),$$

where

$$\widehat{\beta} = (X^TX)^{-1}X^T(y - Zu - Z_A\Gamma a - Z_D\Gamma d).$$

Equation

$$u|\tilde{\theta}, \beta, \sigma^2, \gamma, \tau^2, y \ \sim\ \mathcal{N}_p(\overline{u}, \sigma^2(Z^TZ + \sigma^2\Sigma^{-1})^{-1}) \tag{8}$$

Proof:

Since

$$p(u) \ \propto\ \exp\left(-\frac{u^T\Sigma^{-1}u}{2}\right),$$

we have

$$
\begin{aligned}
p(u|\tilde{\theta},\beta,\sigma^2,\gamma,\tau^2,y) &\propto p(\tilde{\theta},\beta,u,\sigma^2,\gamma,\tau^2|y)\\
&\propto p(y|\tilde{\theta},\xi)p(u)\\
&\propto \exp\left(-\frac{(\tilde{y}-Zu)^T(\tilde{y}-Zu)}{2\sigma^2}\right)\exp\left(-\frac{u^T\Sigma^{-1}u}{2}\right)\\
&\propto \exp\left(-\frac{u^TZ^TZu-2u^TZ^T\tilde{y}+u^T\sigma^2\Sigma^{-1}u}{2\sigma^2}\right)\\
&= \exp\left(-\frac{u^T(Z^TZ+\sigma^2\Sigma^{-1})u-2u^TZ^T\tilde{y}}{2\sigma^2}\right)\\
&= \exp\left(-\frac{u^T(Z^TZ+\sigma^2\Sigma^{-1})u-2u^T(Z^TZ+\sigma^2\Sigma^{-1})\overline{u}}{2\sigma^2}\right)\\
&\propto \exp\left(-\frac{(u-\overline{u})^T(Z^TZ+\sigma^2\Sigma^{-1})(u-\overline{u})}{2\sigma^2}\right),
\end{aligned}
$$

where

$$
\tilde{y} = y - X\beta - Z_A\tilde{a} - Z_D\tilde{d}.
$$

Thus,

$$
u|\tilde{\theta},\beta,\sigma^2,\gamma,\tau^2,y \sim \mathcal{N}_p(\overline{u},\sigma^2(Z^TZ+\sigma^2\Sigma^{-1})^{-1}).
$$

Equation

$$p(\tilde{a}_j|\tilde{a}_{-j}, \tilde{d}, \xi, \gamma, \tau^2, y) \quad \propto \quad f(\tilde{a}_j) g_j(\tilde{a}_j, \tilde{d}_j)\psi(\tilde{a}_j) \tag{9}$$

Proof:

If the posterior is considered as a function of $\tilde{a}_j$, then it follows

$$\begin{aligned}
& p(\tilde{a}_j|\tilde{a}_{-j}, \tilde{d}, \beta, u, \sigma^2, \gamma, \tau^2, y) \\
\propto\ & p(\tilde{a}, \tilde{d}, \beta, u, \sigma^2, \gamma, \tau^2|y) \\
\propto\ & p(y|\tilde{\theta}, \xi) p(\tilde{\theta}_j|\tau_j^2, \gamma_j) \\
\propto\ & p(y|\tilde{\theta}, \xi) g_j(\tilde{a}_j, \tilde{d}_j)\psi(\tilde{a}_j) \exp\left(-\frac{\tilde{a}_j^2}{2\kappa_j^2\tau_j^2}\right) \\
\propto\ & \exp\left(-\frac{1}{2}\left(\frac{(y' - \tilde{a}_j Z_{\mathbb{A}(j)})^T (y' - \tilde{a}_j Z_{\mathbb{A}(j)})}{\sigma^2} + \frac{\tilde{a}_j^2}{\kappa_j^2\tau_j^2}\right)\right) g_j(\tilde{a}_j, \tilde{d}_j)\psi(\tilde{a}_j) \\
\propto\ & \exp\left(-\frac{\tilde{a}_j^2 Z_{\mathbb{A}(j)}^T Z_{\mathbb{A}(j)} - 2\tilde{a}_j y'^T Z_{\mathbb{A}(j)} + \tilde{a}_j^2 \frac{\sigma^2}{\kappa_j^2\tau_j^2}}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j)\psi(\tilde{a}_j) \\
=\ & \exp\left(-\frac{\tilde{a}_j^2 (Z_{\mathbb{A}(j)}^T Z_{\mathbb{A}(j)} + \frac{\sigma^2}{\kappa_j^2\tau_j^2}) - 2\tilde{a}_j y'^T Z_{\mathbb{A}(j)}}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j)\psi(\tilde{a}_j) \\
\propto\ & f(\tilde{a}_j) g_j(\tilde{a}_j, \tilde{d}_j)\psi(\tilde{a}_j).
\end{aligned}$$

Equation

$$p(\tilde{d}_j|\tilde{a}, \tilde{d}_{-j}, \xi, \gamma, \tau^2, y) \quad \propto \quad \tilde{f}(\tilde{d}_j) g_j(\tilde{a}_j, \tilde{d}_j) \tag{10}$$

Proof:

If the posterior is considered as a function of $\tilde{d}_j$, then it follows

$$\begin{aligned}
& p(\tilde{d}_j|\tilde{a}, \tilde{d}_{-j}, \beta, u, \sigma^2, \gamma, \tau^2, y) \\
\propto\ & p(\tilde{a}, \tilde{d}, \beta, u, \sigma^2, \gamma, \tau^2|y) \\
\propto\ & p(y|\tilde{\theta}, \xi) p(\tilde{a}_j, \tilde{d}_j|\tau_j^2, \gamma_j) \\
\propto\ & \exp\left(-\frac{(y' - Z_{D(j)}\tilde{d}_j)^T(y' - Z_{D(j)}\tilde{d}_j)}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j)\psi_{\kappa_j,\tau_j^2}(\tilde{a}_j, \tilde{d}_j) \\
\propto\ & \exp\left(-\frac{(y' - Z_{D(j)}\tilde{d}_j)^T(y' - Z_{D(j)}\tilde{d}_j)}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j) \exp\left(-\frac{\left(\frac{\tilde{d}_j}{\kappa_j} - \mu_d\right)^2}{2\sigma_d^2}\right) \\
\propto\ & \exp\left(-\frac{\tilde{d}_j^2 Z_{D(j)}^T Z_{D(j)} - 2\tilde{d}_j y'^T Z_{D(j)} + \frac{\sigma^2}{\sigma_d^2}\left(\frac{\tilde{d}_j}{\kappa_j} - \mu_d\right)^2}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j) \\
\propto\ & \exp\left(-\frac{\tilde{d}_j^2 Z_{D(j)}^T Z_{D(j)} - 2\tilde{d}_j y'^T Z_{D(j)} + \tilde{d}_j^2 \frac{\sigma^2}{\sigma_d^2\kappa_j^2} - 2\tilde{d}_j \frac{\sigma^2\mu_d}{\kappa_j\sigma_d^2}}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j) \\
=\ & \exp\left(-\frac{\tilde{d}_j^2 (Z_{D(j)}^T Z_{D(j)} + \frac{\sigma^2}{\sigma_d^2\kappa_j^2}) - 2\tilde{d}_j (y'^T Z_{D(j)} + \frac{\sigma^2\mu_d}{\kappa_j\sigma_d^2})}{2\sigma^2}\right) g_j(\tilde{a}_j, \tilde{d}_j) \\
\propto\ & \tilde{f}(\tilde{d}_j) g_j(\tilde{a}_j, \tilde{d}_j).
\end{aligned}$$

Equation

$$\gamma_j|\tilde{\theta}, \xi, \gamma_{-j}, \tau^2, y \quad \sim \quad \mathcal{B}\left(1, \frac{\omega_1 p_{LD}}{\omega_1 p_{LD} + \omega_0(1 - p_{LD})}\right), \tag{11}$$

where

$$\begin{aligned}
\omega_0 &= \frac{1}{\epsilon}\exp\left(-\frac{\tilde{a}_j^{\,2}}{2\epsilon^2\tau_j^2}\right)\psi_{\epsilon,\tau_j^2}(\tilde{a}_j, \tilde{d}_j), \\
\omega_1 &= \exp\left(-\frac{\tilde{a}_j^{\,2}}{2\tau_j^2}\right)\psi_{1,\tau_j^2}(\tilde{a}_j, \tilde{d}_j).
\end{aligned}$$

Proof:

All markers have equal prior probability for having a non-negligible effect, so $\gamma_1, ..., \gamma_M \sim \mathcal{B}(1, p_{\mathrm{LD}})$ are Bernoulli distributed. Thus,

$$p(\gamma_j) = p_{LD}^{\gamma_j}(1 - p_{LD})^{(1-\gamma_j)}.$$

If the posterior is considered as a function of γ_j, then it follows

$$\begin{aligned}
& p(\gamma_j|\tilde{\theta}, \xi, \gamma_{-j}, \tau^2, y) \\
\propto \;& p(\tilde{\theta}, \xi, \gamma, \tau^2|y) \\
\propto \;& p(\tilde{\theta}_j|\tau_j^2, \gamma_j)p(\gamma_j) \\
= \;& \frac{2g_j(\tilde{a}_j, \tilde{d}_j)}{\kappa_j\sqrt{2\pi\tau_j^2}} \exp\left(-\frac{\tilde{a}_j{}^2}{2\kappa_j^2\tau_j^2}\right) \psi_{\kappa_j,\tau_j^2}(\tilde{a}_j, \tilde{d}_j) p_{LD}^{\gamma_j}(1-p_{LD})^{(1-\gamma_j)} \\
\propto \;& \frac{1}{\kappa_j} \exp\left(-\frac{\tilde{a}_j{}^2}{2\kappa_j^2\tau_j^2}\right) \psi_{\kappa_j,\tau_j^2}(\tilde{a}_j, \tilde{d}_j) p_{LD}^{\gamma_j}(1-p_{LD})^{(1-\gamma_j)} \\
= \;& \begin{cases} \omega_0(1-p_{LD}) & \text{if } \gamma_j = 0 \\ \omega_1 p_{LD} & \text{if } \gamma_j = 1 \end{cases},
\end{aligned}$$

where

$$\begin{aligned}
\omega_0 &= \frac{1}{\epsilon} \exp\left(-\frac{\tilde{a}_j{}^2}{2\epsilon^2\tau_j^2}\right) \psi_{\epsilon,\tau_j^2}(\tilde{a}_j, \tilde{d}_j), \\
\omega_1 &= \exp\left(-\frac{\tilde{a}_j{}^2}{2\tau_j^2}\right) \psi_{1,\tau_j^2}(\tilde{a}_j, \tilde{d}_j).
\end{aligned}$$

Thus,

$$P(\gamma_j = 1|\tilde{\theta}, \xi, \gamma_{-j}, \tau^2, y) = \frac{\omega_1 p_{LD}}{\omega_1 p_{LD} + \omega_0(1-p_{LD})}.$$

Equation

$$\tau_j^2|\tilde{\theta}, \xi, \gamma, \tau_{-j}^2, y \sim \text{Inv}-\chi^2\left(v+2, \frac{\frac{\tilde{a}_j^2}{\kappa_j^2} + \left(\frac{\frac{\tilde{d}_j}{\kappa_j} - \mu_D}{s_D}\right)^2 + vs^2}{v+2}\right) \tag{12}$$

Proof:

The prior density of τ_j^2 is

$$p(\tau_j^2) \propto (\tau_j^2)^{-\frac{v}{2}-1} \exp\left(-\frac{vs^2}{2\tau_j^2}\right).$$

For BayesD1 we have

$$\begin{aligned}
& p(\tau_j^2|\tilde{\theta},\xi,\gamma,\tau_{-j}^2,y) \\
\propto\ & p(\tilde{\theta},\xi,\gamma,\tau^2|y) \\
\propto\ & p(\tilde{a}_j,\tilde{d}_j|\tau_j^2,\gamma_j)p(\tau_j^2) \\
\propto\ & \frac{1}{\tau_j^2}\exp\left(-\frac{1}{2}\left(\frac{\tilde{a}_j^2}{\kappa_j^2\tau_j^2}+\frac{(\frac{\tilde{d}_j}{\kappa_j}-\mu_D)^2}{s_D^2\tau_j^2}\right)\right)(\tau_j^2)^{-\frac{v}{2}-1}\exp\left(-\frac{vs^2}{2\tau_j^2}\right) \\
=\ & (\tau_j^2)^{-\frac{v}{2}-1-1}\exp\left(-\frac{1}{2}\left(\frac{\tilde{a}_j^2}{\kappa_j^2\tau_j^2}+\frac{(\frac{\tilde{d}_j}{\kappa_j}-\mu_D)^2}{s_D^2\tau_j^2}+\frac{vs^2}{\tau_j^2}\right)\right) \\
=\ & (\tau_j^2)^{-\frac{v+2}{2}-1}\exp\left(-\frac{\frac{\tilde{a}_j^2}{\kappa_j^2}+\frac{(\frac{\tilde{d}_j}{\kappa_j}-\mu_D)^2}{s_D^2}+vs^2}{2\tau_j^2}\right).
\end{aligned}$$

Thus,

$$\tau_j^2|\tilde{\theta},\xi,\gamma,\tau_{-j}^2,y\sim \text{Inv}-\chi^2\left(v+2,\frac{\frac{\tilde{a}_j^2}{\kappa_j^2}+\frac{(\frac{\tilde{d}_j}{\kappa_j}-\mu_D)^2}{s_D^2}+vs^2}{v+2}\right).$$

Equation

$$\tau_j^2|\tilde{\theta},\xi,\gamma,\tau_{-j}^2,y\sim \text{Inv}-\chi^2\left(v+1,\frac{\frac{\tilde{a}_j^2}{\kappa_j^2}+vs^2}{v+1}\right) \tag{13}$$

Proof:

We have

$$\begin{aligned}
& p(\tau_j^2|\tilde{\theta},\xi,\gamma,\tau_{-j}^2,y) \\
\propto\ & p(\tilde{\theta},\xi,\gamma,\tau^2|y) \\
\propto\ & p(\tilde{\theta}_j|\tau_j^2,\gamma_j)p(\tau_j^2) \\
\propto\ & \frac{1}{\tau_j}\exp\left(-\frac{\tilde{a}_j^2}{2\kappa_j^2\tau_j^2}\right)(\tau_j^2)^{-\frac{v}{2}-1}\exp\left(-\frac{vs^2}{2\tau_j^2}\right) \\
=\ & (\tau_j^2)^{-\frac{v+1}{2}-1}\exp\left(-\frac{\frac{\tilde{a}_j^2}{\kappa_j^2}+vs^2}{2\tau_j^2}\right).
\end{aligned}$$

Thus,

$$\tau_j^2|\tilde{\theta}, \xi, \gamma, \tau_{-j}^2, y \sim \text{Inv}-\chi^2\left(v+1, \frac{\frac{\tilde{a}_j^2}{\kappa_j^2}+vs^2}{v+1}\right).$$

Equation

$$\sigma^2|\tilde{\theta}, \beta, u, \gamma, \tau^2, y \quad \sim \quad \text{Inv}-\chi^2\left(n+v^*, s'^2\right), \tag{14}$$

where

$$s'^2 \quad = \quad \frac{(y - X\beta - Zu - Z_A\tilde{a} - Z_D\tilde{d})^T(y - X\beta - Zu - Z_A\tilde{a} - Z_D\tilde{d}) + v^* s^{*2}}{n+v^*}$$

Proof:

Regardless of the choice of v^* and s^{*2} the prior density is

$$p(\sigma^2) \quad \propto \quad (\sigma^2)^{-\frac{v^*}{2}-1}\exp\left(-\frac{v^* s^{*2}}{2\sigma^2}\right).$$

If the posterior is considered as a function of σ^2, then

$$\begin{aligned}
& p(\sigma^2|\tilde{\theta}, \beta, u, \gamma, \tau^2, y) \\
\propto \quad & p(\tilde{\theta}, \beta, u, \sigma^2, \gamma, \tau^2|y) \\
\propto \quad & p(y|\tilde{\theta}, \xi)p(\sigma^2) \\
\propto \quad & (\sigma^2)^{-\frac{n+v^*}{2}-1} \\
\cdot \quad & \exp\left(-\frac{(y - X\beta - Zu - Z_A\tilde{a} - Z_D\tilde{d})^T(y - X\beta - Zu - Z_A\tilde{a} - Z_D\tilde{d}) + v^* s^{*2}}{2\sigma^2}.\right)
\end{aligned}$$

Equation

$$\begin{aligned}
\mu_D \quad &= \quad \frac{\mathcal{I}}{M\overline{h_\circ}E(\kappa_j)} \\
s^2 \quad &= \quad \frac{V_A - \frac{\gamma_M}{h_\circ^2}V_D}{M\overline{h_\circ}E(\kappa_j^2)}\frac{v-2}{v} \\
s_D^2 \quad &= \quad \frac{\frac{V_D}{M\overline{h_\circ^2}E(\kappa_j^2)} - \mu_D^2}{s^2}\frac{v-2}{v}
\end{aligned} \tag{15}$$

Proof:

Since

$$\mathcal{I} = \mathcal{I}_M = M\overline{h_\circ}E(\kappa_j)\mu_D$$

we have

$$\mu_D = \frac{\mathcal{I}}{M\overline{h_\circ}E(\kappa_j)}.$$

Since $V_D = V_{DM} = M\overline{h_\circ^2}E(\kappa_j^2)E(d_j^2)$ we have

$$\begin{aligned} V_A = V_{AM} &= M\overline{h_\circ}E(\kappa_j^2)E(a_j^2) + 0 + M\gamma_M E(\kappa_j^2)E(d_j^2) \\ &= M\overline{h_\circ}E(\kappa_j^2)E(a_j^2) + M\gamma_M \frac{V_D}{M\overline{h_\circ^2}}. \end{aligned}$$

Thus,

$$s^2 \frac{v}{v-2} = E(a_j^2) = \frac{V_A - V_D \frac{\gamma_M}{\overline{h_\circ^2}}}{M\overline{h_\circ}E(\kappa_j^2)}.$$

It follows that

$$\frac{V_D}{M\overline{h_\circ^2}} = E(\kappa_j^2)E(d_j^2) = E(\kappa_j^2)(s_D^2 E(a_j^2) + \mu_D^2).$$

Thus,

$$s_D^2 = \frac{\frac{V_D}{M\overline{h_\circ^2}E(\kappa_j^2)} - \mu_D^2}{s^2} \frac{v-2}{v}.$$

Equation

$$\frac{\sigma_\Delta^2}{\mu_\Delta^2} = \frac{V_D}{\mathcal{I}^2} \frac{M\overline{h_\circ}^2 E(\kappa_j)^2 \lambda^2}{\overline{h_\circ^2}E(\kappa_j^2)} - 1, \tag{16}$$

and

$$\mu_\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{17}$$

where

$$\begin{aligned} a &= V_A - \frac{\gamma_M V_D}{\overline{h_\circ^2}}, \\ b &= \frac{2E(\kappa_j^2)\mathcal{I}^2 K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\tilde{\gamma}_M}{\lambda^2 M\overline{h_\circ}^2 E(\kappa_j)^2}, \\ c &= \frac{-\mathcal{I}^2 E(\kappa_j^2)}{\lambda^2 M\overline{h_\circ}E(\kappa_j)^2}. \end{aligned}$$

Proof:

Since

$$\begin{aligned}
\mathcal{I} &= \mathcal{I}_M \\
&= M\overline{h_\circ}E(\kappa_j)E(d_j) \\
&= M\overline{h_\circ}E(\kappa_j)\mu_\Delta E(|a_j|) \\
&= M\overline{h_\circ}E(\kappa_j)\mu_\Delta\lambda\sqrt{E(a_j^2)} \\
&= M\overline{h_\circ}E(\kappa_j)\mu_\Delta\lambda s\sqrt{\frac{v}{v-2}},
\end{aligned}$$

and

$$\begin{aligned}
V_D &= V_{DM} \\
&= M\overline{h_\circ^2}E(\kappa_j^2)E(d_j^2) \\
&= M\overline{h_\circ^2}E(\kappa_j^2)E(a_j^2)(\sigma_\Delta^2+\mu_\Delta^2) \\
&= M\overline{h_\circ^2}E(\kappa_j^2)s^2\frac{v}{v-2}(\sigma_\Delta^2+\mu_\Delta^2),
\end{aligned}$$

we have

$$\begin{aligned}
\frac{V_D}{\mathcal{I}^2} &= \frac{M\overline{h_\circ^2}E(\kappa_j^2)s^2\frac{v}{v-2}(\sigma_\Delta^2+\mu_\Delta^2)}{M^2\overline{h_\circ}^2E(\kappa_j)^2\mu_\Delta^2\lambda^2s^2\frac{v}{v-2}} \\
&= \frac{\overline{h_\circ^2}E(\kappa_j^2)}{M\overline{h_\circ}^2E(\kappa_j)^2\lambda^2}\left(\frac{\sigma_\Delta^2}{\mu_\Delta^2}+1\right).
\end{aligned}$$

Thus,

$$\frac{\sigma_\Delta^2}{\mu_\Delta^2} = \frac{V_D}{\mathcal{I}^2}\frac{M\overline{h_\circ}^2E(\kappa_j)^2\lambda^2}{\overline{h_\circ^2}E(\kappa_j^2)} - 1.$$

Since

$$\begin{aligned}
M\overline{h_\circ}E(a_j^2) &= \frac{M\overline{h_\circ}E(|a_j|)^2}{\lambda^2} \\
&= \frac{M\overline{h_\circ}}{\lambda^2}\frac{\mathcal{I}^2}{M^2\overline{h_\circ}^2E(\kappa_j)^2\mu_\Delta^2} \\
&= \frac{\mathcal{I}^2}{\lambda^2M\overline{h_\circ}E(\kappa_j)^2\mu_\Delta^2}
\end{aligned}$$

we have

$$
\begin{aligned}
V_A &= V_{AM} \\
&= M\overline{h_\circ}E(a_j^2)E(\kappa_j^2) - 2ME(\kappa_j^2)E(|a_j||d_j|)\tilde{\gamma}_M + ME(\kappa_j^2)\gamma_M E(d_j^2) \\
&= M\overline{h_\circ}E(a_j^2)E(\kappa_j^2) - 2ME(\kappa_j^2)E(a_j^2)\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\tilde{\gamma}_M + ME(\kappa_j^2)\gamma_M E(a_j^2)(\sigma_\Delta^2 + \mu_\Delta^2) \\
&= \frac{\mathcal{I}^2 E(\kappa_j^2)}{\lambda^2 M\overline{h_\circ}E(\kappa_j)^2\mu_\Delta^2} - \frac{2ME(\kappa_j^2)\mathcal{I}^2\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\tilde{\gamma}_M}{\lambda^2 M^2\overline{h_\circ}^2 E(\kappa_j)^2\mu_\Delta^2} + \frac{\gamma_M V_D}{\overline{h_\circ^2}} \\
&= \frac{\mathcal{I}^2 E(\kappa_j^2)}{\lambda^2 M\overline{h_\circ}E(\kappa_j)^2\mu_\Delta^2} - \frac{2E(\kappa_j^2)\mathcal{I}^2 K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\tilde{\gamma}_M}{\lambda^2 M\overline{h_\circ}^2 E(\kappa_j)^2\mu_\Delta} + \frac{\gamma_M V_D}{\overline{h_\circ^2}}.
\end{aligned}
$$

Thus,

$$
\left(V_A - \frac{\gamma_M V_D}{\overline{h_\circ^2}}\right)\mu_\Delta^2 + \frac{2E(\kappa_j^2)\mathcal{I}^2 K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\tilde{\gamma}_M}{\lambda^2 M\overline{h_\circ}^2 E(\kappa_j)^2}\mu_\Delta = \frac{\mathcal{I}^2 E(\kappa_j^2)}{\lambda^2 M\overline{h_\circ}E(\kappa_j)^2}.
$$

It follows that

$$
\mu_\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
$$

where

$$
\begin{aligned}
a &= V_A - \frac{\gamma_M V_D}{\overline{h_\circ^2}}, \\
b &= \frac{2E(\kappa_j^2)\mathcal{I}^2 K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\tilde{\gamma}_M}{\lambda^2 M\overline{h_\circ}^2 E(\kappa_j)^2}, \\
c &= \frac{-\mathcal{I}^2 E(\kappa_j^2)}{\lambda^2 M\overline{h_\circ}E(\kappa_j)^2}.
\end{aligned}
$$

Equation

$$
\begin{aligned}
\overline{h_\circ}\frac{v}{v-2} &= \frac{V_A\overline{h_\circ^2} - V_D\gamma_M}{\mathcal{I}^2}\frac{E(\kappa_j)^2\,\overline{h_\circ}^2}{E(\kappa_j^2)\overline{h_\circ^2}}ME(|t|\mu_\Delta(|t|))^2 \\
&+ 2\tilde{\gamma}_M E\left(t^2\mu_\Delta(|t|)\,K\left(\frac{\sigma_\Delta}{\mu_\Delta(|t|)}\right)\right),
\end{aligned} \tag{18}
$$

where

$$
\sigma_\Delta^2 = \left(\frac{V_D}{\mathcal{I}^2}\frac{E(\kappa_j)^2\,\overline{h_\circ}^2}{E(\kappa_j^2)\overline{h_\circ^2}}ME(|t|\mu_\Delta(|t|))^2 - E\left(t^2\mu_\Delta(|t|)^2\right)\right)\frac{v-2}{v}.
$$

Proof:

For brevity we write $\mu_\Delta = \mu_\Delta(|t|)$. From conditions $V_D = V_{DM}$ and $\mathcal{I} = \mathcal{I}_M$ we obtain

$$\frac{\mathcal{I}^2}{M^2 E(\kappa_j)^2\, \overline{h_\circ}^2\, E(|t|\mu_\Delta)^2} = s^2 = \frac{V_D}{ME(\kappa_j^2)\overline{h_\circ^2}\left(\frac{\sigma_\Delta^2 v}{v-2} + E\left(t^2\mu_\Delta^2\right)\right)}.$$

Thus,

$$\mathcal{I}^2 ME(\kappa_j^2)\overline{h_\circ^2}\frac{v}{v-2}\sigma_\Delta^2 + \mathcal{I}^2 ME(\kappa_j^2)\overline{h_\circ^2}E\left(t^2\mu_\Delta^2\right) = M^2E(\kappa_j)^2\,\overline{h_\circ}^2\,E(|t|\mu_\Delta)^2 V_D$$

It follows that

$$\begin{aligned}
\sigma_\Delta^2 &= \frac{ME(\kappa_j)^2\,\overline{h_\circ}^2 E(|t|\mu_\Delta)^2 V_D - \mathcal{I}^2 E(\kappa_j^2)\overline{h_\circ^2}E\left(t^2\mu_\Delta^2\right)}{\mathcal{I}^2 E(\kappa_j^2)\overline{h_\circ^2}}\frac{v-2}{v} \\
&= \left(\frac{V_D}{\mathcal{I}^2}\frac{E(\kappa_j)^2\,\overline{h_\circ}^2}{E(\kappa_j^2)\overline{h_\circ^2}}ME(|t|\mu_\Delta)^2 - E\left(t^2\mu_\Delta^2\right)\right)\frac{v-2}{v}
\end{aligned}$$

From conditions $V_A = V_{AM}$ and $V_D = V_{DM}$ we obtain

$$\begin{aligned}
&\frac{V_A}{\overline{h_\circ}\frac{v}{v-2} - 2E\left(t^2\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\right)\tilde{\gamma}_M + \gamma_M\left(\frac{\sigma_\Delta^2 v}{v-2} + E\left(t^2\mu_\Delta^2\right)\right)} \\
=\ & s^2 ME(\kappa_j^2) \\
=\ & \frac{V_D}{\overline{h_\circ^2}\left(\frac{\sigma_\Delta^2 v}{v-2} + E\left(t^2\mu_\Delta^2\right)\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& V_A\overline{h_\circ^2}\left(\frac{\sigma_\Delta^2 v}{v-2} + E\left(t^2\mu_\Delta^2\right)\right) \\
=\ & V_D\overline{h_\circ}\frac{v}{v-2} - 2V_D E\left(t^2\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\right)\tilde{\gamma}_M + V_D\gamma_M\left(\frac{\sigma_\Delta^2 v}{v-2} + E\left(t^2\mu_\Delta^2\right)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
V_D\overline{h_\circ}\frac{v}{v-2} &= \left(V_A\overline{h_\circ^2} - V_D\gamma_M\right)\left(\frac{\sigma_\Delta^2 v}{v-2} + E\left(t^2\mu_\Delta^2\right)\right) + 2V_D E\left(t^2\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\right)\tilde{\gamma}_M \\
&= \left(V_A\overline{h_\circ^2} - V_D\gamma_M\right)\frac{V_D}{\mathcal{I}^2}\frac{E(\kappa_j)^2\,\overline{h_\circ}^2}{E(\kappa_j^2)\overline{h_\circ^2}}ME(|t|\mu_\Delta)^2 + 2V_D E\left(t^2\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\right)\tilde{\gamma}_M.
\end{aligned}$$

Thus,

$$\overline{h_\circ}\frac{v}{v-2} = \frac{V_A\overline{h_\circ^2} - V_D\gamma_M}{\mathcal{I}^2}\frac{E(\kappa_j)^2\,\overline{h_\circ}^2}{E(\kappa_j^2)\overline{h_\circ^2}}ME(|t|\mu_\Delta)^2 + 2E\left(t^2\mu_\Delta K\left(\frac{\sigma_\Delta}{\mu_\Delta}\right)\right)\tilde{\gamma}_M.$$