On-line Appendix for “Cooperation, Conflict, and the Costs of Anarchy”

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On-line Appendix

This appendix contains a fuller statement of assumptions on the contest success functions, proofs of (non-trivial) propositions, and analysis of some extensions or variants on the model discussed or mentioned in the text.

Note: Shortly before publication I found a mistake in Proposition 5. The statement of the proposition was corrected (it applies as stated only for the Powell model with sequential moves), and this version of the appendix contains an different proof from previous versions (anything dated before May 16, 2018), and remarks that address the issues that arise in, and comparative statics for, the simultaneous move model.

1 Assumptions about $q(a_1, a_2)$ and $p^i(a_i, a_j)$

A1. $q(a_1, a_2) \in [0, 1]$ satisfies the following conditions.[1]

1. q is symmetric, so that $q(a_1, a_2) = 1 - q(a_2, a_1)$, and continuous and differentiable whenever both arguments are positive.
2. An unarmed or nearly unarmed state gets no part of γ if the other state is armed: $q(a, 0) = 1$ for $a > 0$ and $\lim_{b \to 0} q(a, b) = 1$ for $a > 0$.
3. More arms increase a state's share of γ: $q_1(a_1, a_2) > 0$ for $a_2 > 0$.
4. The return from an additional increment of arms is decreasing in the level of arms: $q_1(a, a) < q_1(b, b)$ if $a > b$, and $\lim_{a \to 0} q_1(a, a) = \infty$.
5. The return from additional spending at equal arms levels is decreasing: $q_{11}(a, a) < 0$.

A2. $p^i(a_1, a_2; m) \in [0, 1]$ satisfies the following conditions.

1. p^i is continuous and differentiable whenever both arguments are positive, and is symmetric in the sense that $p^1(a, b; m) = p^2(a, b; m)$, $(a, b) \in [0, 1]^2$. That is, chances of winning as the attacker depend only on the force levels and a common military technology. Note that it is *not* necessarily the case that $p^1(a, b) = 1 - p^2(b, a)$; the probability of winning as attacker can differ from the probability winning as defender, for the same force levels.

[1] Throughout, subscript notation is used for derivatives: for example, $q_1(a_1, a_2)$ is the derivative of q with respect to a_1, and q_{12} is the cross-partial with respect to a_1 and a_2.

2. A unarmed or nearly unarmed state can be taken over by an attacker with any positive amount of arms: $\lim_{b \to 0} p^i(a, b) = 1$ for $a > 0$.

3. More arms increase a state's chance of winning: $p_1^i(a_i, a_j) > 0$ for $a_j > 0$.

4. The return from an additional increment of arms is decreasing in the level of arms: $p_1^i(a, a) < p_1^i(b, b)$ if $a > b$.

5. The attacker's chance of winning is increasing in offensive advantage (m) for given force levels, and decreasing in defensive advantage, with limits at 1 and 0: $p_m^i(a_i, a_j; m) > 0$ and $\lim_{m \to \infty} p^i(a_i, a_j; m) = 1$ and $\lim_{m \to 0} p^i(a_i, a_j; m) = 0$

2 Proofs of Propositions

Observation 1. *Consider utility functions $u_i(a_i, a_j) = 1 - a_i + \gamma q(a_i, a_j)$ and the one-shot game in which the states simultaneously choose arms levels, with no option for war. A1 implies that there exists a symmetric Nash equilibrium of the game in which both states choose an arms level $a^{NE} > 0$.*

Proof. A1(2) implies that $(0, 0)$ cannot be a Nash equilibrium, since a state can get certain victory by deviating to an arbitrarily small positive amount of arms. Next, in a Nash equilibrium the first-order conditions either hold exactly for (a_1, a_2) such that $\gamma q_1(a_1, a_2) = 1$ for state 1 and $\gamma q_1(a_2, a_1) = 1$ for state 2, or with a corner solution for one or both if $\gamma q_1(1, a_j) > 1$. By A1(3) and A1(4), $q_1(a, a)$ is positive but decreasing in a for $a > 0$, and since for small enough a $q_1(a, a)$ is very large, there exists a unique $a^{NE} > 0$ such that both states FOC's are satisfied. A1(5) also implies that the second-order condition is satisfied at such a point.

Observation 2. *Given A1, a^{NE} is strictly increasing in γ for interior solutions.*

Proof. Implicitly differentiate the FOC $\gamma q_1(a, a) = 1$ in γ to get $q_1 + \gamma(q_{11} + q_{12})a'(\gamma) = 0$. A1(4) implies that the derivative of $q_1(a, a)$ with respect to a is $q_{11} + q_{12} < 0$, so the term in parentheses is negative. $a'(\gamma) > 0$ then follows because $q_1 > 0$ (A1(3)).

Proposition 2. *If there is a smallest a^* that solves the war constraint (5) with equality, and $a^* \leq a^{NE}$, then for large enough δ the game has peaceful equilibria in which, on the path of play, the states choose $\hat{a} \in [a^*, \bar{a}]$ in every period, where $\bar{a} \leq a^{NE}$ is determined by specific parameters. Thus in the most cooperative equilibrium the states choose a^* in all periods on the path. Further, if defensive advantage is sufficiently large (that is, m is small enough), such an a^* definitely exists and $a^* < a^{NE}$.*

Proof of Proposition 2 . The war constraint is satisfied with equality by any a such that

$$1 - a + \gamma/2 = \max_{a'}(1 - a')(1 - \delta) + p^i(a', a)(\gamma - c + \delta(1 + \mu)).$$

By the argument in footnote 26,there exists a $\underline{\delta}$ such that for all $\delta \in [\underline{\delta}, 1)$, $a' = 1$ solves the maximum problem here, so that we have for $\delta > \underline{\delta}$

$$1 - a + \gamma/2 = p^i(1, a)(\gamma - c + \delta(1 + \mu)). \tag{1}$$

Since $1 + \gamma/2 < \gamma - c + 1 + \mu$ by assumption (else the war constraint does not bind at $(0, 0)$ and so there is no problem due to anarchy), there exists a δ' such that for all $\delta \in (\delta', 1]$, $1 + \gamma/2 < w \equiv \gamma - c + \delta(1 + \mu)$. Redefine $\underline{\delta}$ as the larger of δ' and $\underline{\delta}$ defined above. In what follows results are for δ's at least as large as this $\underline{\delta}$.

$1 + \gamma/2 < w$ implies that the LHS of (1) is less than the RHS when $a = 0$ since $p^i(1, 0) = 1$. Both the LHS and RHS are decreasing in a so that if the two lines intersect, there is a smallest $a^* > 0$ that satisfies (1), at which point the RHS cuts or touches the LHS value from above. Thus there must also exist an $a' \geq a^*$ such that LHS $\geq$ RHS for all $a \in (a^*, a')$, with strict inequality if $a' > a^*$.

Suppose (as assumed in the Proposition) that this $a^* \leq a^{NE}$, and let $\bar{a} = \min\{a^{NE}, a'\}$. We now propose a strategy that will support a peaceful subgame-perfect equilibrium in which the states choose $\hat{a} \in [a^*, \bar{a}]$ in each period and neither state attacks on the path of play, for large enough δ. We begin with the simplest case where $\bar{a} = a^{NE}$ and then comment on modifications necessary if $\bar{a} < a^{NE}$.

σ: There are two phases. In the *normal phase*, both states choose $\hat{a}$, but attack iff there are deviations in a period to (a_i, a_j) such that

$$\delta(1 - \hat{a} + \gamma/2) < p^i(a_i, a_j)w. \tag{2}$$

If either or both states deviate to a smaller $a_i < \hat{a}$ such that (2) is not satisfied, this deviation is ignored and the normal phase continues. If both states deviate to higher $a_i > \hat{a}$, these deviations are ignored and the normal phase continues. If one state deviates to $a_i > \hat{a}$, then play enters the *punishment phase.*

In the punishment phase, both states choose a^{NE} in each period regardless of prior history, but attack iff there are deviations in a period to (a_i, a_j) such that

$$\delta(1 - a^{NE} + \gamma/2) < p^i(a_i, a_j)w. \tag{3}$$

To see that this strategy can constitute a SGP equilibrium, begin with the punishment phase. Because $a^{NE} \leq \bar{a}$, the war constraint is satisfied at this arms level and so neither has an incentive to deviate to $a_i = 1$ and then attack. Nor does either have an incentive to deviate to $a_i \neq a^{NE}$ and then not attack, since by the definition of a^{NE} there is no other a_i that yields a higher payoff in the stage game with no war (i.e., $1-a_i+\gamma q(a_i, a^{NE})$) than a^{NE}. Off the path, (3) governs whether attacks are optimal given (a_i, a_j).

In the normal phase, the war constraint is satisfied by construction so neither wishes to deviate to $a_i = 1$ and then attack. Condition (2) governs optimal choices for attack given (a_i, a_j) and the expectation of the normal phase continuing if no attack occurs. Deviations down to $a_i < \hat{a}$ only lower i's payoff in a period given that $\hat{a} < a^{NE}$. An upward deviation to $a_i > \hat{a}$ yields

$$(1-\delta)(1-a_i+\gamma q(a_i,\hat{a}))+\delta(1-a^{NE}+\gamma/2),$$

which for large enough δ is certainly less than $1-\hat{a}+\gamma/2$. This proves that σ forms a peaceful SGP equilibrium of the game for large enough δ for the case of $\bar{a} = a^{NE}$.

If $\bar{a} < a^{NE}$, then it is no longer feasible to use the "grim trigger" strategy of resorting to (a^{NE}, a^{NE}) as the punishment threat, since these levels do not satisfy the war constraint in this case.[2] Alternative off-path punishment strategies can then be used, such as a finite number of periods T at (a^{NE}, a^{NE}), where T is chosen to be large enough to deter upward deviation from $\hat{a}$ in the normal phase but small enough that the states want to get back to the peaceful path at $\hat{a}$ rather than go to war in the punishment phase. For large enough δ, getting back to $\hat{a}$ rather than going to war at (a^{NE}, a^{NE}) is worth waiting for regardless of how large T is (if finite), and there will also be a large enough T such that the deviation from $\hat{a}$ for a one-period gain is not worthwhile.

The final part of the Proposition asserts that for large enough defensive advantage, meaning small enough m, there exists an a^* that satisfies the war constraint and $a^* < a^{NE}$. First, existence: Rewrite (1) as $1+\gamma/2 = f(a,m) \equiv a + p^i(1,a;m)w$. $1+\gamma/2 < f(0) = w$ by assumption. By A2(5), for any $a' \in (0,1]$ there exists an $m(a')$ such that $1+\gamma/2 > f(a', m(a'))$. Since $f(a,m)$ is continuous, there must exist an $a \in (0,a')$ such that (1) holds, and that is the smallest such a.

Second, $a^* < a^{NE}$ for small enough m: Simply take $a^{NE} = a'$ in the argument above. QED.

Proposition 3 *Maximum feasible international cooperation (smaller a^*) is (1) decreasing in the value the states derive from controlling the other's territory (μ); (2) increasing in defensive advantage (smaller m) and war costs (larger c); and (3) may increase or decrease with the value of the international issues in dispute (or the gains from trade) between the two states, γ.*

Proof of Proposition 3. We will prove the claims for a slightly more general version of the model, in which states can have risk-neutral or risk averse preferences over outcomes, and the functional form of the benefits of successful war is left open. Let $u(y)$, with $u(0) = 0$, be a weakly concave, increasing utility function that represents state preferences over outcomes. Let $w(z)$ be a state's

[2] Alternatively, for this case we could use the threat of war with both choosing $a_i = 1$ and attack off the path to support a peaceful equilibrium.

payoff for winning a war, where z is a vector of factors such as μ, c, and γ. In the case in the text $w = \gamma - c + \delta(1 + \mu)$. For this more general case, a^* solves

$$u(1 - a + \gamma/2) = p^i(1, a)u(w(z)). \tag{4}$$

Differentiate both sides in any factor z that is positively related to w (the value of winning a war) but not γ and we have, at $a = a^*$,

$$\begin{aligned} -u'(1 - a + \gamma/2)a'(z) &= p_2^i(1, a)a'(z)u(w(z)) + p^i(1, a)u'(w(z))w'(z) \\ 0 &= a'(z)[u'(1 - a + \gamma/2) + p_2^i(1, a)u(w(z))] + p^i(1, a)u'(w(z))w'(z). \end{aligned}$$

The term in brackets is negative because at a^*, the RHS of (4) cuts the LHS from above, and the term is the difference between the LHS and RHS slopes at this point.[3] Therefore, since $p^i(1, a)u'(w)w'(z) > 0$, $a'(z) > 0$ and thus increasing c or decreasing μ implies decreasing a^*.

Similarly, differentiating both sides in m at $a = a^*$ yields

$$\begin{aligned} -u'(1 - a + \gamma/2)a'(m) &= u(w)[p_2^i(1, a; m)a'(m) + p_m^i(1, a; m)] \\ 0 &= a'(m)[u'(1 - a + \gamma/2) + p_2^i(1, a)u(w)] + u(w)p_m^i(1, a; m), \end{aligned}$$

which implies that $a'(m) > 0$ since the term in brackets is negative and the other term is positive by A2(5).

Differenting in γ yields

$$\begin{aligned} u'(1 - a + \gamma/2)[-a'(\gamma) + 1/2] &= u(w)p_2^i(1, a)a'(\gamma) + p^i(1, a)u'(w)w'(\gamma) \\ u'(1 - a + \gamma/2)/2 - p^i(1, a)u'(w)w'(g) &= a'(\gamma)[u'(1 - a + \gamma/2) + u(w)p_2^i(1, a)] \end{aligned}$$

Again the term in brackets is negative so that here $a'(\gamma)$ has the opposite sign of the LHS. However, the LHS can be positive or negative. Even in the risk neutral case of $u(y) = y$ with $w = \gamma - c + \delta(1 + \mu)$, it can be that LHS $= 1/2 - p^i(1, a) > 0$, because with $m < 1$ it can happen that $p^i(1, a^*) < 1/2$. QED.

Proposition 4. *Consider the model with any contest success function $p^i(a_i, a_j; m)$, where m indexes offensive advantage in the sense that p^i is increasing in m for positive arms levels. In a peaceful equilibrium with arms levels a^*, the probability that a break-out attack would succeed is* decreasing *in offensive advantage.*

Proof of Proposition 4. See footnote 47 in text.

[3] For comparative statics we are restricting attention to situations where varying a parameter locally does not eliminate a peaceful equilibrium, and so we can ignore the case of a tangency.

2.1 Asymmetric capabilities and preferences

This section provides a fuller definition and analysis of the baseline model allowing states to differ in key parameters, and in particular total resources (or state size).

Let total resources available to state i be $r_i > 0$, and let $\mu_i \in [0, 1]$ and $c_i > 0$ reflect the value i puts on control of j's territory and i's costs for war, respectively. In the risk-neutral case of the baseline model, $w_i = \gamma - c_i + \delta(r_i + \mu_i r_j)$ is now i's value for winning a war against j.

We are interested in comparative statics about i's military burden, which is now $b_i \equiv a_i/r_i$.

Force levels (a_1, a_2) satisfy the war constraints when, for $i = 1, 2$,

$$r_i - a_i + \gamma q(a_i, a_j) \geq p^i(r_i, a_j)w_i. \tag{5}$$

For comparative statics on relative resources, $R = r_1/r_2$, we encounter a new issue. Varying r_1 or r_2 by itself with the model as posed will change not only R but also the ratio of resources to costs of war, thus varying the costs of war relative to the potential benefits (which include $\mu_j r_j$). We need to keep costs proportional to resources to identify the distinct impact of relative (potential) power, R. In other words, we would like a formulation such that equilibrium military burdens (b_1, b_2) depend only on R and not on r_1 and r_2 separately from R: states with resources $(20, 15)$ should have the same military burdens (in the most efficient peaceful equilibrium), as states with resources $(4, 3)$.[4]

The same concerns apply for γ. If we increase r_i and nothing else, we are making state j care less about the issues relative to successful conquest, by implicitly assuming that γ does not vary with relative resources of the states. It is not clear (to me anyway) what should be assumed about how the stakes on issues should vary with r_i and r_j. In addition, general or closed-form results can be impossible to derive in the non-symmetric case due to differences in a_1 and a_2 affecting states' values for a peaceful status quo through $q(a_1, a_2)$. So in what follows I will revert to "the Powell model" in which $\gamma = 0$. To avoid the "downward deviations" problem, we can assume a sequential extensive form as in Powell (1993).[5]

It seems most natural to suppose a big state pays a smaller cost for a war against a small state than against another big state, and likewise that a small state pays a larger cost in a war against a big state than against another small state. So let i's costs for war against j be $c_i r_j$. The war constraints then become

$$r_i - a_i \geq p^i(r_i, a_j)(r_i + \mu_i r_j - c_i r_j).$$

[4] To be clear, this is not an empirical claim but a statement about what is needed to isolate the impact of relative resources on military burdens in the model, holding all other parameters equal.

[5] I have computed numerical solutions for the model with $\gamma > 0$ and find, not surprisingly, that the qualitative results (such as in Figure 4) are not affected.

Using the simple ratio form $p^i(a_i, a_j; m) = a_i/(a_i + a_j/m)$ and rearranging, we have

$$a_1(a_2) = r_1 \frac{a_2/m}{r_1 + a_2/m} - \frac{r_1}{r_1 + a_2/m} r_2(\mu_1 - c_1).$$

Dividing through by r_1 to get the military burden $b_1 = a_1/r_1$, and simplifying yields

$$b_1(b_2) = \frac{b_2 - m(\mu_1 - c_1)}{b_2 + mR}, \text{ and likewise } b_2(b_1) = \frac{b_1 - m(\mu_2 - c_2)}{b_1 + m/R}$$

Let $g_i = \mu_i - c_i$. This system can be solved explicitly for the smallest intersection of the two curves (if it exists), as

$$b_1^* = \frac{B - \sqrt{B^2 - 4(mR + 1)C}}{2(mR + 1)},$$

where $B = 1 - m^2 - m(g_1 - g_2)$ and $C = m^2 g_1/R + m g_2$. b_2^* is the same with the subscripts on g_i reversed and R switched to $1/R$. These expressions can then be used to produce the results illustrated in Figure 5.

For $g_1 = g_2$, it is easy to show the derivative of b_2^* with respect to R is positive for $R \geq 1$, which means that greater resource inequality associates with an increasing military burden for the smaller state. It is also straightforward to show that for R within a neighborhood of 1, $db_1^*/dR < 0$ and $db_2^*/dR > 0$.

Proposition 5. *Consider a version of the model in which there is no issue competition and states arms sequentially, as in Powell's formulation. Let $w_i(r_i, r_j, \mu_i, c_i)$ be state i's value for winning a war, and suppose that w_i is increasing in μ_i and decreasing in c_i. Then for any military technology, increasing the value i puts on control of j's territory, or reducing i's costs for fighting j, implies greater equilibrium military burdens for* both *states.*

Proof. Conditions for the minimal arms peaceful equilibrium (a_1, a_2) in the general case are

$$\begin{aligned} r_1 - a_1 + \gamma q(a_1, a_2) &= p^1(r_1, a_2) w_1(r_1, r_2, \gamma, \mu_1, c_1) \\ r_2 - a_2 + \gamma q(a_2, a_1) &= p^2(r_2, a_1) w_2(r_1, r_2, \gamma, \mu_2, c_2). \end{aligned}$$

The case considered in the Proposition is for the Powell (1993, 1999) model, with sequential arming, which here is the special case of $\gamma = 0$. In what follows I do the analysis for factors that affect w_i but not w_j (such as μ_i and c_i) for the general case and then consider $\gamma = 0$.

The first step is to get expressions for da_i/da_j when the equations above hold. Differentiating in a_j,

$$-\frac{da_1}{da_2} + \gamma q_1 \frac{da_1}{da_2} + \gamma q_2 = p_2(r_2, a_2) w_1 \text{ and } -\frac{da_2}{da_1} + \gamma q_1 \frac{da_2}{da_1} + \gamma q_2 = p_2(r_1, a_1) w_2, \text{ so}$$
$$\frac{da_1}{da_2} = \frac{1}{\gamma q_1 - 1}\left(p_2(r_1, a_2) w_1 - \gamma q_2\right) \text{ and } \frac{da_2}{da_1} = \frac{1}{\gamma q_1 - 1}\left(p_2(r_2, a_1) w_2 - \gamma q_2\right).$$

Some observations:

1. For $\gamma = 0$, $da_i/da_j > 0$ since $p_2 < 0$.
2. In the Powell model (with $\gamma = 0$), $da_1/da_2 = -p_2(r_1, a_2)w_1$ and $da_2/da_1 = -p_2(r_2, a_1)w_2$. Since we are considering the intersection of these two curves at the lower tip of the "lens" (the minimal arms peaceful equilibrium),

$$\frac{da_1}{da_2} > \frac{1}{da_2/da_1}$$

   so defined, which implies that

$$p_2(r_1, a_2) p_2(r_2, a_1) w_1 w_2 > 1.$$

3. Notice also that by the condition for a pure-strategy equilibrium in Proposition 3, in the simultaneous move game, $\gamma q_1 - 1 > 0$ at equilibrium arms levels. This implies that the sign of da_i/da_j depends whether $p_2(r_i, a_j)w_i - \gamma q_2$ is greater or less than zero, which can go either way (since both p_2 and q_2 are negative).

Next we differentiate the equilibrium conditions in x_1, any factor with $dw_1/dx_1 > 0$ and $dw_2/dx_1 = 0$.

$$-a_1' + \gamma q_1 a_1' + \gamma q_2 a_2' = p_2(r_1, a_2) a_2' w_1 + p(r_1, a_2) w_1'$$
$$-a_2' - \gamma q_1 a_1' - \gamma q_2 a_2' = p_2(r_2, a_1) a_1' w_2.$$

Solving for a_1' and a_2',

$$a_1'\left(1 - \frac{p_2(r_1, a_2) w_1 - \gamma q_2}{\gamma q_1 - 1} \frac{p_2(r_2, a_1) w_2 - \gamma q_2}{\gamma q_1 - 1}\right) = \frac{p(r_1, a_2) w_1'}{\gamma q_1 - 1} \quad \text{and } a_2' = a_1' \frac{p_2(r_1, a_2) w_2 + \gamma q_1}{\gamma q_1 - 1}.$$

In the Powell model $\gamma = 0$, so the left expression becomes

$$a_1'(1 - p_2(r_1, a_2) p_2(r_2, a_1) w_1 w_2) = -p(r_1, a_2) w_1'.$$

Observation 2 above and $w_1' > 0$ thus imply that $a_1' > 0$, which implies $a_2' > 0$ as well, proving the proposition for the Powell model. For the simultaneous move game with $\gamma q_1 - 1 > 0$, da_1/da_2 and

da_2/da_1 can take either positive or negative signs at efficient arms levels that satisfy the conditions for a peaceful equilibrium, so that comparative statics on w_1 are indeterminate.

Remarks. For the simultaneous-move game with issue competition and a pure-strategy equilibrium that satisfies the conditions for Proposition 2, we can gain insight by considering the functions that define the "lens" as in Powell (1993, 1999). The lens is the set of arms allocations (a_1, a_2) such that (5) holds. Let $a_1(a_2)$ be the arms level for state 1 such that

$$1 - a_1 + \gamma q(a_1, a_2) = p^1(1, a_2)w_1.$$

That is, $a_1(a_2)$ defines a lens boundary. In the Powell model where $\gamma = 0$, a small increase in a_2 reduces the war payoff on the RHS, so equality can only be maintained by increasing a_1 on the LHS. Thus in the Powell model, $a_1'(a_2) > 0$, and likewise for $a_2(a_1)$.

Now consider what happens to $a_1(a_2)$ if we increase a_2 slightly in our game with $\gamma > 0$ and the pure-strategy equilibrium constraint that $\gamma q_1 > 1$. The RHS decreases as before. Fixing a_1, increasing a_2 also decreases the LHS at a rate of $-\gamma q_2 = \gamma q_1$ (recall that q is symmetric). So, whether we need to increase or decrease a_1 to regain the equality depends on how sensitive γq is relative to $p^1 w_1$. If γq is very sensitive by comparison to $p^1 w_1$, then a small increase in a_2 causes a large drop in 1's peace payoff due to greater losses on international issues, by comparison to not much reduction in the RHS payoff for all-out war. To restore equality, a_1 has to increase so as to gain back value in disputes or contests on the issues (the LHS is increasing in a_1 since $\gamma q_1 > 1$). By contrast, if γq is not very sensitive relative to $p^1 w_1$, then a_1 may have to *decrease* to restore equality, since the constraint that $\gamma q_1 > 1$ means that the LHS is increasing in a_1 for given a_2 at a pure-strategy equilibrium.

To illustrate, consider the case of maximally sensitive $q(a_1, a_2)$, where $q(a, a) = 1/2$ and $q(a_i, a_j) = 1$ if $a_i > a_j$. The only candidates for pure-strategy equilibria in the simultaneous-move game then have $a_1 = a_2$, since otherwise reducing one's arms level has no effect on one's issue payoff. Potential equilibria are thus defined by (a, a) such that

$$1 - a + \gamma/2 \geq p^1(1, a)w_1 \text{ and}$$
$$1 - a + \gamma/2 \geq p^2(1, a)w_2,$$

only one of which can hold with equality unless $w_1 = w_2$. The binding constraint is thus for the state i with the larger w_i, and the peaceful equilibrium with the lowest costs of anarchy is given by the a^* that solves

$$1 - a + \gamma/2 = p^i(1, a)w_i.$$

Comparative statics: a^* is increasing in w_i, and weakly increasing in w_j.

When γq is less sensitive, it is possible to find examples in which da_2/dw_1 is greater than or less than zero, while da_1/dw_1 is less than or greater than zero. (The only impossibility is both negative.) The claim in the text of the published article that da_2/dw_1 is necessarily positive is wrong – it can be negative.

Figure 1 illustrates several of these cases. The correspondences $a_1(a_2)$ and $a_2(a_1)$ (as defined above) are shown for the model with

$$q(a_1, a_2) = \frac{a_1^n}{a_1^n + a_2^n} \text{ and } p(a_1, a_2) = \frac{a_1}{a_1 + a_2/m}.$$

The parameter $n > 0$ in the q function controls how sensitive issue resolutions are to changes in relative military power, with larger n making for greater sensitivity. Other parameters used in all four figures are $m = .1$, $\mu_i = .5$, and $c_i = .1$.

The upper-left plot shows the bottom of the lens for the Powell model, where $\gamma = 0$. To the right, we set $\gamma = .5$ and $n = 1$, so this is a case with relatively low sensitivity. The green line is $a_1(a_2)$ when μ_1 has been increased to .55, while μ_2 remains at .5. In this example, making state 1 more greedy increases state 2's equilibrium military effort while very slightly reducing state 2's.

The lower-left plot increases sensitivity of the issues function further; here $n = 3$. Now greater greed for state 1 implies increased armament for state 1, but slightly less arms for state 2.

Finally, with high sensitivity (the lower-right figure), we return to the comparative statics of the Powell model, in that greater greed for state 1 increases both state's equilibrium military effort. There is a slight difference in that in $\gamma = 0$ case, the more status-quo state's arms respond more to increased greed by the rival, the reverse holds in the model with $\gamma > 0$ and high sensitivity of issue resolutions to relative military capabilities.[6]

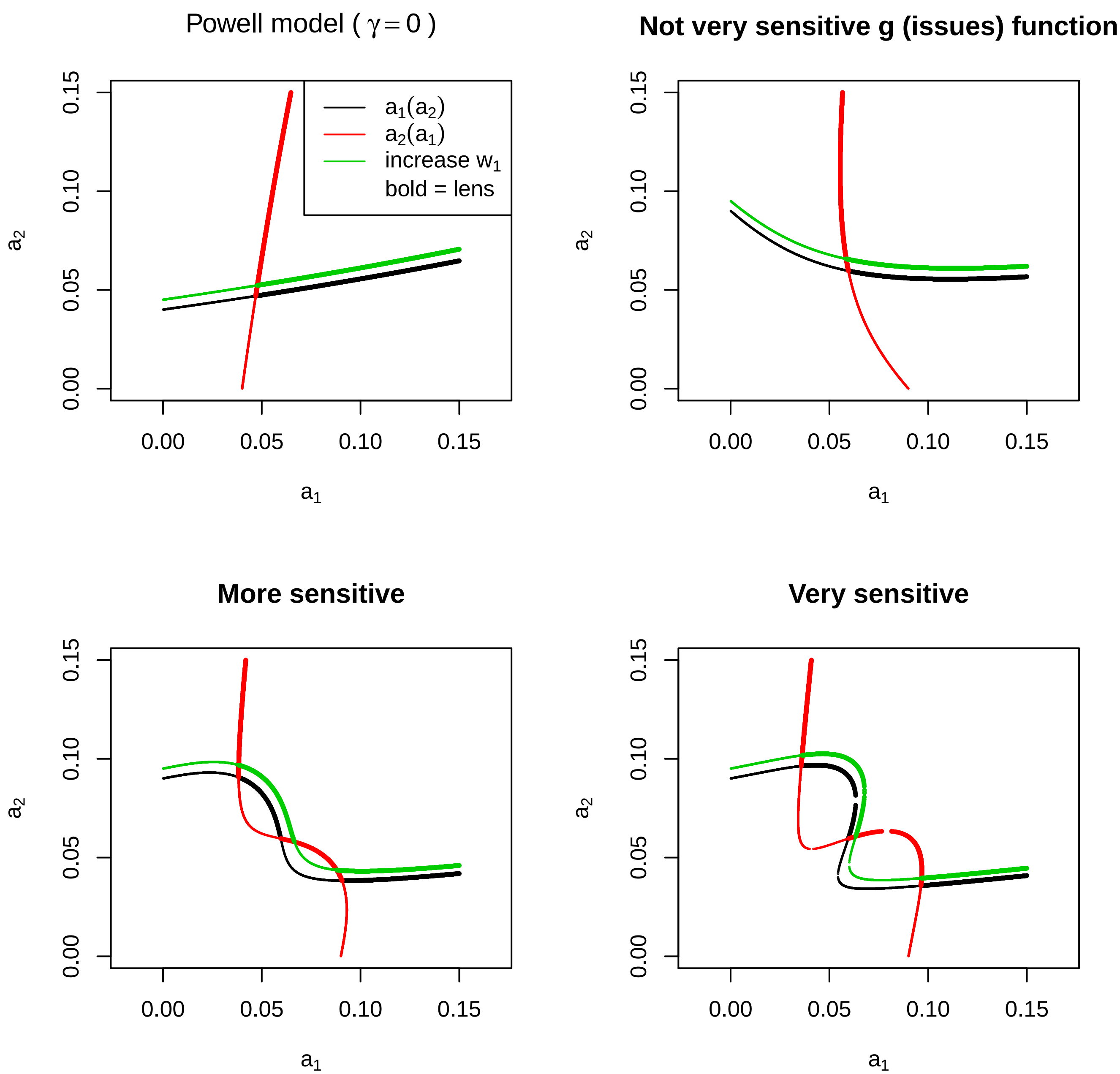
3 Explicit protocols for bargaining over the issues

Major powers can use their militaries to invade and try to take over some or all of another major power, and they can also use them in more limited aims disputes where something less than the disposition of home-state territory is at issue. For example, in the vast majority of militarized interstate disputes in the Correlates of War MIDs data set, uses of force are attempted faits accomplis or attempts at coercing or preventing some relatively small change by threatening military engagement of some sort.[7] For simplicity, the baseline model assumed that relative military strength represented by arms levels a_1 and a_2 translates into how the issues γ are divided up via a function $q(a_i, a_j)$ that is increasing in the first argument and decreasing in the second. Here are two explicit and often-used bargaining models that imply this reduced form.

[6] In the cases shown where there are multiple intersections, only the "center" one supports a pure-strategy equilibrium.

[7] Altman 2016 finds that almost all disputes that resulted in territorial changes since 1918 involved faits accomplis rather than concessions to explicit threats.

Figure 1: Arms allocations with asymmetric greed or costs for war



First, suppose that in each period, after arms levels are chosen (and if neither attacked), each state has an equal chance of finding itself in a position to attempt a fait accompli on division of the issue γ, meaning that the randomly selected state chooses how to divide γ. Call the share it gives itself $x_i^t \in [0, 1]$ where i is the lucky state in period t. State j then decides whether to accept this division or use force to try to get control of the issue. If j accepts, then issue payoffs in this period are $(\gamma x_i^t, \gamma(1-x_i^t))$. If j uses force, the winner of the military contest gets γ and the loser gets 0 (on the issue) in this period. Finally, let $q(a_i, a_j)$ be the probability that i wins the military contest and let the cost of a military contest be $k > 0$ for both states. (Notice that the model now explicitly distinguishes between a big war that puts sovereignty at risk, and a limited skirmish over the issue or territory γ.)

Given (a_i, a_j) the most i can demand before j prefers to use force is $x^i(a_i, a_j)$ such that $\gamma(1 - x^i(a_i, a_j)) = \max\{0, \gamma(1 - q(a_i, a_j)) - k\}$, or $x^i(a_i, a_j) = \min\{q(a_i, a_j) + k/\gamma, 1\}$. Assume small enough k that the solution is interior. Then a state's expected payoffs (in a peaceful period) when choosing arms levels are

$$1 - a_i + \frac{1}{2}(\gamma q(a_i, a_j) + k) + \frac{1}{2}(\gamma q(a_i, a_j) - k) = 1 - a_i + \gamma q(a_i, a_j) \qquad (6)$$

which is exactly as in the reduced-form baseline model.[8]

Suppose next that we imagine γ as (1) something that was divided in previous periods and so has a preexisting status quo, and (2) cannot be changed unless both parties agree. These assumptions rule out faits accomplis. If both parties strictly preferred this division to fighting in the previous period, then this will still be the case today (other things equal). Thus a small reduction in one state's arms level will not be enough to give the other state a credible threat to use force in bargaining over γ, which implies in turn that we may lose pure-strategy equilibrium (since small deviations down come with no cost on γ or risk of attack).

If all issues had these characteristics, *and* if states have no way to raise military risk from an initial position of preferring the status quo, then the interstate bargaining solution proposed for arms stability here would not work. But neither premise seems plausible.

A simple way to relax the second one – that is, to allow states to threaten military conflict from any status quo – is to suppose that in each period after arms levels are chosen, bargaining over γ yields a result given by the Nash bargaining solution, using as the disagreement point the same costly military lottery above. Now the division of γ is the x (for state i) that maximizes $(x - (q(a_i, a_j) -$

[8] Boundary cases are not very enlightening so I will just summarize how they work. If $k > \gamma$ then a demand of $x^i = 1$ is accepted for any arms levels, which means that the expected issue resolution is $\gamma/2$ regardless of arms choices and we are effectively in the Jackson and Morelli situation. If $k \in (\gamma/2, \gamma)$, then at equal arms levels i's optimal demand is 1, but unilateral deviations can still affect one's expected issue payoff if they are large enough: a large enough increase gets state j an offer greater than zero, and a large enough decrease means that i won't be able to demand 1. Here payoffs are non-monotonic in a_i and there will be a threshold k such that below this, the equilibrium is in pure strategies.

$k/\gamma))(1 - x - (1 - q(a_i, a_j) - k/\gamma))$, which is $x = q(a_i, a_j)$, which leads to the same ex ante payoff expression (6).

In the analysis so far I have neglected the fact that this is a repeated game. In other words, the preceding shows how the equilibrium given in Proposition 2 can be supported in models with explicit protocols (or mechanisms) for bargaining over γ, and strategies in these that are not history dependent. We can also ask about what can be supported using more complicated punishment strategies.

With history-dependent strategies, the specifics of the bargaining protocol can matter much less. For patient enough players, we will be able to support the pure-strategy equilibrium outcomes in Proposition 3 by having the states expect to choose whatever strategies are needed in the bargaining game such that a division $q(a_i, a_j)$ results, where $q(a_i, a_j)$ can now be *any* symmetric function with properties that imply deviations down are sufficiently unattractive at a^*. (The states just need to expect that failure to implement the expected agreement according to $q(a_i, a_j)$ would lead to a switch to the one-period Nash profile.) In other words, q doesn't in principle need to be tied to specific military technologies or assumptions about who gets to move when on the ground or at the table. It could instead reflect a diplomatic cultural understanding, perhaps a taken-for-granted commonsense, that relative military strength should determine who gets what in negotiations, at least in some types of situations.

In the approach taken above, we distinguish between two kinds of militarized conflicts – an all-out war between the two states, and bargaining over particular issues that might in principle be decided by the use of force short of all-out war. An alternative would be to suppose that there is only one kind of conflict, the all-out war, so that the disagreement point in bargaining over the issues is likewise a costly and final lottery that will eliminate one of the two states. I think this is much less realistic, but it is worth commenting on what changes.

Suppose that after arms choices in each period, if neither attacks then one state i is randomly given a fait accompli opportunity concerning division of γ. State j then chooses whether to accept or go to all-out war, with odds of winning $p^j(a_j, a_i; m)$.

For this case it can be shown that for large enough δ, (a) there is no peaceful Markov Perfect equilibrium, and (b) history-dependent strategies (like those discussed above) can be used to support a peaceful equilibrium with outcomes as in Proposition 3. The problem for Markov Perfect equilibrium is that now a small deviation upwards from (say) $\hat{a}$ by state i can have a very large effect on the present value of war payoffs in the bargaining game when δ is close to 1. So a state j faced with a one-time demand of $x^1 = 1$ after i deviated slightly upwards to $a_i = \hat{a} + \epsilon$ will certainly want to accept. (The demand is 'one-time' given that we are trying to construct an MPE, so by assumption future choices will not change as a result of i's deviation.) So each state will always want to deviate upwards from any symmetric choice of a's, and $a = 1$ cannot be peaceful if condition (2) in the text holds.[9]

Allowing for history-dependent strategies, we can induce restraint in the bargaining game so that states don't grab as much as they could get (in the fait accompli) because they expect this will lead to higher equilibrium arms levels starting in the next period.

[9] For δ bounded below 1 and a military technology that is sufficiently defensive dominant, we will still be able to get peaceful pure-strategy equilibrium in this situation.

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