Applied Probability Trust (January 19, 2017)

SUPPLEMENTARY MATERIAL OF "OPTIMAL SCALING OF THE RANDOM WALK METROPOLIS ALGORITHM UNDER L^p MEAN DIFFERENTIABILITY"

1. Proof of Theorem 4

The proof of this theorem follows the same steps as the the proof of Theorem 2. Note that ξ_θ and ξ_0, given by (12), are well defined on $\mathcal{I} \cap \{x \in \mathbb{R} \mid x + \mathrm{r}\theta \in \mathcal{I}\}$. Let the function $\upsilon : \mathbb{R}^2 \to \mathbb{R}$ be defined for $x, \theta \in \mathbb{R}$ by

$$\upsilon(x,\theta) = \mathbb{1}_{\mathcal{I}}(x + \mathrm{r}\theta)\mathbb{1}_{\mathcal{I}}(x + (1-\mathrm{r})\theta)\,. \tag{S1}$$

Lemma S1. *Assume* $\boldsymbol{G}$*1 holds. Then, there exists* $C > 0$ *such that for all* $\theta \in \mathbb{R}$,

$$\left(\int_{\mathcal{I}} \left(\{\xi_\theta(x) - \xi_0(x)\}\,\upsilon(x,\theta) + \theta\dot{V}(x)\xi_0(x)/2\right)^2 \mathrm{d}x\right)^{1/2} \leq C|\theta|^\beta\,.$$

Proof. The proof follows as Lemma 1 and is omitted.

Lemma S2. *Assume that* $\boldsymbol{G}$*1 holds. Let* X *be a random variable distributed according to* π *and* Z *be a standard Gaussian random variable independent of* X*. Define*

$$\mathcal{D}_{\mathcal{I}} = \{X + \mathrm{r}\ell d^{-1/2}Z \in \mathcal{I}\} \cap \{X + (1-\mathrm{r})\ell d^{-1/2}Z \in \mathcal{I}\}\,.$$

Then,

(i) $\lim_{d\to+\infty} d\, \left\|\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z) + \ell Z\dot{V}(X)/(2\sqrt{d})\right\|_2^2 = 0.$

(ii) Let p *be given by* $\boldsymbol{G}$*1(i). Then,*

$$\lim_{d\to+\infty}\sqrt{d}\left\|\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\left\{V(X) - V(X+\ell Z/\sqrt{d})\right\} + \ell Z\dot{V}(X)/\sqrt{d}\right\|_p = 0\,.$$

(iii) $\lim_{d\to\infty} d \left\| \mathbb{1}_{\mathcal{D}_\mathcal{I}} \left(\log(1+\zeta_d(X,Z)) - \zeta^d(X,Z) + [\zeta^d]^2(X,Z)/2\right) \right\|_1 = 0,$

where ζ^d *is given by* (19).

Proof. Note by definition of ζ^d and ξ_θ (12), for $x \in \mathcal{I}$ and $x + \mathrm{r}\ell d^{-1/2} z \in \mathcal{I}$,

$$\zeta^d(x,z) = \xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1\,. \tag{S2}$$

Using Lemma S1,

$$\begin{aligned}
&\left\| \mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z) + \ell Z \dot{V}(X)/(2\sqrt{d}) \right\|_2^2 \\
&\quad = \mathbb{E}\left[\int_\mathcal{I} \left(\upsilon(x, \ell Z d^{-1/2}) \left\{ \xi_{\ell Z d^{-1/2}}(x) - \xi_0(x) \right\} + \ell Z \dot{V}(x) \xi_0(x)/(2\sqrt{d}) \right)^2 \mathrm{d}x \right] \\
&\quad \le C \ell^{2\beta} d^{-\beta} \mathbb{E}\left[|Z|^{2\beta} \right] .
\end{aligned}$$

The proof of (i) is completed using $\beta > 1$. For (ii), write for all $x \in \mathcal{I}$ and $x + \ell z d^{-1/2} z \in \mathcal{I}$, $\Delta V(x,z) = V(x) - V(x + \ell z d^{-1/2})$. By **G**1(i)

$$\begin{aligned}
&\left\| \mathbb{1}_{\mathcal{D}_\mathcal{I}} \Delta V(X,Z) + \ell Z \dot{V}(X)/\sqrt{d} \right\|_p^p \\
&\quad = \mathbb{E}\left[\int_\mathcal{I} \left(\upsilon(x, \ell Z d^{-1/2}) \Delta V(X,Z) + \ell Z \dot{V}(x)/\sqrt{d} \right)^p \pi(x) \mathrm{d}x \right] \\
&\quad \le C \ell^{\beta p} d^{-\beta p/2} \mathbb{E}\left[|Z|^{\beta p} \right]
\end{aligned}$$

and the proof of (ii) follows from $\beta > 1$. For (iii), note that for all $x > 0$, $u \in [0,x]$, $|(x-u)(1+u)^{-1}| \le |x|$, and the same inequality holds for $x \in (-1,0]$ and $u \in [x,0]$. Then by (21) and (22), for all $x > -1$,

$$\left| \log(1+x) - x + x^2/2 \right| = |R(x)| \le x^2 \left| \log(1+x) \right| .$$

Then by (S2), for $x \in \mathcal{I}$ and $x + \ell d^{-1/2} z \in \mathcal{I}$,

$$\begin{aligned}
&\left| \log(1+\zeta_d(x,z)) - \zeta^d(x,z) + [\zeta^d]^2(x,z)/2 \right| \\
&\quad \le \left(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1 \right)^2 \left| \log(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x)) \right| , \\
&\quad \le \left(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1 \right)^2 \left| V(x + \ell z d^{-1/2}) - V(x) \right| /2\,.
\end{aligned}$$

Since for all $x \in \mathbb{R}$, $|\exp(x) - 1| \le |x|(\exp(x)+1)$, this yields,

$$\begin{aligned}
&\left|\log(1+\zeta_d(x,z)) - \zeta^d(x,z) + [\zeta^d]^2(x,z)/2\right| \\
&\qquad \le \left|V(x+\ell z d^{-1/2}) - V(x)\right|^3 \left(\exp\left(V(x) - V(x+\ell z d^{-1/2})\right) + 1\right)/4\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathcal{I}} \upsilon(x, \ell z d^{-1/2}) \left|\log(1+\zeta_d(x,z)) - \zeta^d(x,z) + [\zeta^d]^2(x,z)/2\right| \pi(x)\mathrm{d}x \\
&\qquad\qquad\qquad\qquad \le (I_1 + I_2)/4\,,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_{\mathcal{I}} \upsilon(x, \ell z d^{-1/2}) \left|V(x+\ell z d^{-1/2}) - V(x)\right|^3 \pi(x)\mathrm{d}x \\
I_2 &= \int_{\mathcal{I}} \upsilon(x, \ell z d^{-1/2}) \left|V(x+\ell z d^{-1/2}) - V(x)\right|^3 \pi(x+\ell z d^{-1/2})\mathrm{d}x\,.
\end{aligned}$$

By Hölder's inequality, a change of variable and using **G**1(i),

$$I_1 + I_2 \le C\left(\left|\ell z d^{-1/2}\right|^3 \left(\int_{\mathcal{I}} \left|\dot{V}(x)\right|^4 \pi(x)\mathrm{d}x\right)^{3/4} + \left|\ell z d^{-1/2}\right|^{3\beta}\right)\,.$$

The proof follows from **G**1(ii) and $\beta > 1$.

For ease of notation, write for all $d \ge 1$ and $i, j \in \{1, \ldots, d\}$,

$$\begin{aligned}
\mathcal{D}^d_{\mathcal{I},j} &= \left\{X^d_j + \mathrm{r}\ell d^{-1/2} Z^d_j \in \mathcal{I}\right\} \cap \left\{X^d_j + (1-\mathrm{r})\ell d^{-1/2} Z^d_j \in \mathcal{I}\right\}\,, \\
\mathcal{D}^d_{\mathcal{I},i:j} &= \bigcap_{k=i}^{j} \mathcal{D}^d_{\mathcal{I},k}\,.
\end{aligned} \tag{S3}$$

Lemma S3. *Assume that* ***G****1 holds. For all* $d \ge 1$*, let* X^d *be distributed according to* π^d*, and* Z^d *be d-dimensional Gaussian random variable independent of* X^d*. Then,* $\lim_{d\to+\infty} \mathrm{J}^d_{\mathcal{I}} = 0$ *where*

$$\mathrm{J}^d_{\mathcal{I}} = \left\|\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}} \sum_{i=2}^{d} \left\{\left(\Delta V^d_i + \frac{\ell Z^d_i}{\sqrt{d}}\dot{V}(X^d_i)\right) - 2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},i}} \zeta^d(X^d_i, Z^d_i)\right] + \frac{\ell^2}{4d}\dot{V}^2(X^d_i)\right\}\right\|_1\,.$$

Proof. The proof follows the same lines as the proof of Lemma 3 and is omitted.

Define for all $d \geq 1$,

$$\mathrm{E}^d_{\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \left| \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1:d}} 1 \wedge \exp\left\{\sum_{i=1}^d \Delta V^d_i\right\} - 1 \wedge \exp\left\{-\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^d b^d_{\mathcal{I}}(X^d_i, Z^d_i)\right\}\right|\right] ,$$

where ΔV^d_i is given by (5), for all $x \in \mathcal{I}$, $z \in \mathbb{R}$,

$$b^d_{\mathcal{I}}(x,z) = -\frac{\ell z}{\sqrt{d}} \dot{V}(x) + 2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \zeta^d(X^d_1, Z^d_1)\right] - \frac{\ell^2}{4d} \dot{V}^2(x) , \tag{S4}$$

and ζ^d is given by (19).

Proposition S1. *Assume **G**1 holds. Let X^d be a random variable distributed according to π^d and Z^d be a zero-mean standard Gaussian random variable, independent of X. Then* $\lim_{d\to+\infty} \mathrm{E}^d_{\mathcal{I}} = 0$.

Proof. Let $\Lambda^d = -\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^d \Delta V^d_i$. By the triangle inequality, $\mathrm{E}^d \leq \mathrm{E}^d_1 + \mathrm{E}^d_2 + \mathrm{E}^d_3$ where

$$\mathrm{E}^d_{1,\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1:d}} \left| 1 \wedge \exp\left\{\sum_{i=1}^d \Delta V^d_i\right\} - 1 \wedge \exp\left\{\Lambda^d\right\}\right|\right] ,$$

$$\mathrm{E}^d_{2,\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}} \left| 1 \wedge \exp\left\{\Lambda^d\right\} - 1 \wedge \exp\left\{-\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^d b^d(X^d_i, Z^d_i)\right\}\right|\right] ,$$

$$\mathrm{E}^d_{3,\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\left(\mathcal{D}^d_{\mathcal{I},2:d}\right)^c} 1 \wedge \exp\left\{-\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^d b^d(X^d_i, Z^d_i)\right\}\right] ,$$

Since $t \mapsto 1 \wedge \mathrm{e}^t$ is 1-Lipschitz, by the Cauchy-Schwarz inequality we get

$$\mathrm{E}^d_{1,\mathcal{I}} \leq \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \left|\Delta V^d_1 + \ell d^{-1/2} Z^d_1 \dot{V}(X^d_1)\right|\right] \leq \|Z^d_1\|_4^2 \left\|\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \Delta V^d_1 + \ell d^{-1/2} Z^d_1 \dot{V}(X^d_1)\right\|_2 .$$

By Lemma 2(ii), $\mathrm{E}^d_{1,\mathcal{I}}$ goes to 0 as d goes to $+\infty$. Using again that $t \mapsto 1 \wedge \mathrm{e}^t$ is 1-Lipschitz and Lemma S3, $\mathrm{E}^d_{2,\mathcal{I}}$ goes to 0 as well. Note that, as Z^d_1 and $\mathbb{1}_{\left(\mathcal{D}^d_{\mathcal{I},2:d}\right)^c}$ are independent, by (15),

$$\mathrm{E}^d_{3,\mathcal{I}} \leq d\mathbb{P}\left(\left\{\mathcal{D}^d_{\mathcal{I},1}\right\}^c\right) \leq C d^{1-\gamma/2} .$$

Therefore, $\mathrm{E}^d_{3,\mathcal{I}}$ goes to 0 as d goes to $+\infty$ by **G**1(iii).

Lemma S4. *Assume* ***G****1 holds. For all* $d \in \mathbb{N}^*$*, let* X^d *be a random variable distributed according to* π^d *and* Z^d *be a standard Gaussian random variable in* $\mathbb{R}^d$*, independent of* X*. Then,*

$$\lim_{d \to +\infty} 2d\,\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \zeta^d(X^d_1, Z^d_1)\right] = -\frac{\ell^2}{4} I \,,$$

where I *is defined in* (6) *and* ζ^d *in* (19).

Proof. Noting that for all $\theta \in \mathbb{R}$,

$$\int_{\mathcal{I}} \mathbb{1}_{\mathcal{I}}(x+\mathrm{r}\theta)\mathbb{1}_{\mathcal{I}}(x+(1-\mathrm{r})\theta)\pi(x+\theta)\mathrm{d}x = \int_{\mathcal{I}} \mathbb{1}_{\mathcal{I}}(x+(\mathrm{r}-1)\theta)\mathbb{1}_{\mathcal{I}}(x-\mathrm{r}\theta)\pi(x)\mathrm{d}x \,.$$

the proof follows the same steps as the the proof of Lemma 4 and is omitted.

Proof of Theorem 4. The proof follows the same lines as the proof of Theorem 2 and is therefore omitted.

2. Proof of tightness

Lemma S5. *Assume* ***G****1 holds. Then, the sequence* $(\mu_d)_{d \geq 1}$ *is tight in* $\mathbf{W}$*.*

As for the proof of Lemma 5, the proof follows from Lemma S6.

Lemma S6. *Assume* ***G****1. Then, there exists* $C > 0$ *such that, for all* $0 \leq k_1 < k_2$*,*

$$\mathbb{E}\left[\left(X^d_{k_2,1} - X^d_{k_1,1}\right)^4\right] \leq C \sum_{p=2}^{4} \frac{(k_2-k_1)^p}{d^p} \,.$$

Proof. We use the same decomposition of $\mathbb{E}[(X^d_{k_2,1} - X^d_{k_1,1})^4]$ as in the proof of Lemma 6 so that we only need to upper bound the following term:

$$d^{-2}\mathbb{E}\left[\left(\sum_{k=k_1+1}^{k_2} Z^d_{k,1}\mathbb{1}_{(\mathcal{A}^d_k)^c}\right)^4\right] = d^{-2}\sum \mathbb{E}\left[\prod_{i=1}^4 Z^d_{m_i,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_i}\right)^c}\right] ,$$

where the sum is over all the quadruplets $(m_p)_{p=1}^4$ satisfying $m_p \in \{k_1 + 1, \ldots, k_2\}$, $p = 1, \ldots, 4$. Let $(m_1, m_2, m_3, m_4) \in \{k_1 + 1, \ldots, k_2\}^4$ and $(\tilde{X}^d_k)_{k\geq 0}$ be defined as:

$$\tilde{X}^d_0 = X^d_0 \quad \text{and} \quad \tilde{X}^d_{k+1} = \tilde{X}^d_k + \mathbb{1}_{k\notin\{m_1-1,m_2-1,m_3-1,m_4-1\}}\ell d^{-1/2} Z^d_{k+1}\mathbb{1}_{\tilde{\mathcal{A}}^d_{k+1}} ,$$

where for all $k \geq 0$ and all $1 \leq i \leq d$,

$$\tilde{\mathcal{A}}^d_{k+1} = \left\{U_{k+1} \leq \exp\left(\sum_{i=1}^d \Delta\tilde{V}^d_{k,i}\right)\right\}$$
$$\Delta\tilde{V}^d_{k,i} = V\left(\tilde{X}^d_{k,i}\right) - V\left(\tilde{X}^d_{k,i} + \ell d^{-1/2} Z^d_{k+1,i}\right) .$$

Define, for all $k_1 + 1 \leq k \leq k_2$, $1 \leq i, j \leq d$,

$$\tilde{\mathcal{D}}^{d,k}_{\mathcal{I},j} = \left\{\tilde{X}^d_{k,j} + \mathrm{r}\ell d^{-1/2} Z^d_{k+1,j} \in \mathcal{I}\right\} \cap \left\{\tilde{X}^d_{k,j} + (1-\mathrm{r})\ell d^{-1/2} Z^d_{k+1,j} \in \mathcal{I}\right\} ,$$
$$\tilde{\mathcal{D}}^{d,k}_{\mathcal{I},i:j} = \bigcap_{\ell=i}^{j} \tilde{\mathcal{D}}^{d,k}_{\mathcal{I},\ell} .$$

Note that by convention $V(x) = -\infty$ for all $x \notin \mathcal{I}$, $\tilde{\mathcal{A}}^d_{k+1} \subset \tilde{\mathcal{D}}^{d,k}_{\mathcal{I},1:d}$ so that $\left(\tilde{\mathcal{A}}^d_{k+1}\right)^c$ may be written $\left(\tilde{\mathcal{A}}^d_{k+1}\right)^c = \left(\tilde{\mathcal{D}}^{d,k}_{\mathcal{I},1:d}\right)^c \bigcup \left(\left(\tilde{\mathcal{A}}^d_{k+1}\right)^c \cap \tilde{\mathcal{D}}^{d,k}_{\mathcal{I},1:d}\right)$. Let $\mathcal{F}$ be the σ-field generated by $\left(\tilde{X}^d_k\right)_{k\geq 0}$. Consider the case $\#\{m_1, \ldots, m_4\} = 4$. The case $\#\{m_1, \ldots, m_4\} = 3$ is dealt with similarly and the two other cases follow the same lines as the proof of Lemma S6. As $\left\{\left(U_{m_j}, Z^d_{m_j,1}, \cdots, Z^d_{m_j,d}\right)\right\}_{1\leq j\leq 4}$ are independent conditionally to $\mathcal{F}$,

$$\mathbb{E}\left[\prod_{j=1}^4 Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\middle|\mathcal{F}\right]$$
$$= \prod_{j=1}^4 \left\{\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}\right)^c} Z^d_{m_j,1}\middle|\mathcal{F}\right] + \mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}}\mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c} Z^d_{m_j,1}\middle|\mathcal{F}\right]\right\} .$$

As U_{m_j} is independent of $(Z^d_{m_j,1}, \cdots, Z^d_{m_j,d})$ conditionally to $\mathcal{F}$, the second term may be written:

$$\mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}} \mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c} Z^d_{m_j,1} \middle| \mathcal{F}\right] = \mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}} Z^d_{m_j,1}\left(1-\exp\left\{\sum_{i=1}^d \Delta\tilde{V}^d_{m_j-1,i}\right\}\right)_+ \middle| \mathcal{F}\right] .$$

Since the function $x \mapsto (1-\mathrm{e}^x)_+$ is 1-Lipschitz, on $\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}$

$$\left|\left(1-\exp\left\{\sum_{i=1}^d \Delta\tilde{V}^d_{m_j-1,i}\right\}\right)_+ - \Theta_{m_j}\right| \leq \left|\Delta\tilde{V}^d_{m_j-1,1} + \ell d^{-1/2}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1}\right| ,$$

where $\Theta_{m_j} = (1-\exp\{-\ell d^{-1/2}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1} + \sum_{i=2}^d \Delta\tilde{V}^d_{m_j-1,i}\})_+$. Then,

$$\left|\mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}} Z^d_{m_j,1}\left(1-\exp\left\{\sum_{i=1}^d \Delta\tilde{V}^d_{m_j-1,i}\right\}\right)_+ \middle| \mathcal{F}\right]\right| \leq A^d_{m_j} + B^d_{m_j} ,$$

where

$$A^d_{m_j} = \mathbb{E}\left[\left|Z^d_{m_j,1}\right|\left|\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1}}\Delta\tilde{V}^d_{m_j-1,1} + \ell d^{-1/2}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1}\right|\middle|\mathcal{F}\right] ,$$
$$B^d_{m_j} = \left|\mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},2:d}} Z^d_{m_j,1}\Theta_{m_j}\middle|\mathcal{F}\right]\right| .$$

By Jensen inequality,

$$\left|\mathbb{E}\left[\prod_{j=1}^4 Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\right]\right| \leq \mathbb{E}\left[\prod_{j=1}^4\left\{\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}\right)^c}|Z^d_{m_j,1}|\middle|\mathcal{F}\right] + A^d_{m_j} + B^d_{m_j}\right\}\right] ,$$
$$\leq C\mathbb{E}\left[\sum_{j=1}^4 \mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}\right)^c}|Z^d_{m_j,1}|^4\middle|\mathcal{F}\right] + \left(A^d_{m_j}\right)^4 + \left(B^d_{m_j}\right)^4\right] ,$$

By **G** 1(iii) and Holder's inequality applied with $\alpha = 1/(1-2/\gamma) > 1$, for all $1 \leq j \leq 4$,

$$\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1:d}\right)^c}|Z^d_{m_j,1}|^4\right] \leq \mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},1}\right)^c}|Z^d_{m_j,1}|^4\right] + \sum_{i=2}^d \mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_{j-1}}_{\mathcal{I},i}\right)^c}\right] ,$$
$$\leq \mathbb{E}\left[|Z^d_{m_j,1}|^{4\alpha/(\alpha-1)}\right]^{(\alpha-1)/\alpha} d^{-\gamma/(2\alpha)} + d^{1-\gamma/2} ,$$
$$\leq Cd^{1-\gamma/2} .$$

By Lemma S2(ii) and the Holder's inequality, there exists $C > 0$ such that $\mathbb{E}\left[\left(A^d_{m_j}\right)^4\right] \leq Cd^{-2}$. On the other hand, by [1, Lemma 6] since $Z^d_{m_j,1}$ is independent of $\mathcal{F}$,

$$B^d_{m_j} = \left|\mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},2:d}} \ell d^{-1/2}\dot{V}(\tilde{X}^d_{m_j-1,1}) \times \mathcal{G}\left(\ell^2 d^{-1}\dot{V}(\tilde{X}^d_{m_j-1,1})^2, -2\sum_{i=2}^d \Delta\tilde{V}^d_{m_j-1,i}\right)\middle|\mathcal{F}\right]\right| ,$$

where the function $\mathcal{G}$ is defined in (24). By **G** 1(ii) and since $\mathcal{G}$ is bounded, $\mathbb{E}[(B^d_{m_j})^4] \leq Cd^{-2}$. Since $\gamma \geq 6$ in **G** 1(iii), $|\mathbb{E}[\prod_{j=1}^4 Z^d_{m_j,1}\mathbb{1}_{(\mathcal{A}^d_{m_j})^c}]| \leq Cd^{-2}$, showing that

$$\sum_{(m_1,m_2,m_3,m_4)\in\mathcal{I}_4} \left|\mathbb{E}\left[\prod_{i=1}^4 Z^d_{m_i,1}\mathbb{1}_{(\mathcal{A}^d_{m_i})^c}\right]\right| \leq Cd^{-2}\binom{k_2-k_1}{4} . \tag{S5}$$

3. Proof of Theorem 5

Lemma S7. *Assume **G** 1 holds. Let X^d be distributed according to π^d and Z^d be a d-dimensional standard Gaussian random variable, independent of X^d. Then, $\lim_{d\to+\infty} \mathrm{E}^d = 0$, where*

$$\mathrm{E}^d = \mathbb{E}\left[\left|\dot{V}(X^d_1)\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left\{\mathcal{G}\left(\ell^2\dot{V}(X^d_1)^2/d, 2\bar{Y}_d\right) - \mathcal{G}\left(\ell^2\dot{V}(X^d_1)^2/d, 2\bar{X}_d\right)\right\}\right|\right] ,$$

where $\bar{Y}_d = \sum_{i=2}^d \Delta V^d_i$, ΔV^d_i and $\mathcal{D}^d_{\mathcal{I},2:d}$ are given by (5) *and* (S3) *and $\bar{X}_d = \sum_{i=2}^d b^d_{\mathcal{I},i}$, $b^d_{\mathcal{I},i} = b^d_{\mathcal{I}}(X^d_i, Z^d_i)$ with $b^d_{\mathcal{I}}$ given by* (S4).

Proof. Set for all $d \geq 1$, $\bar{Y}_d = \sum_{i=2}^d \Delta V^d_i$ and $\bar{X}_d = \sum_{i=2}^d b^d_{\mathcal{I},i}$. By definition of $b^d_{\mathcal{I}}$ (S4), $\bar{X}_d$ may be expressed as $\bar{X}_d = \sigma_d\bar{S}_d + \mu_d$, where

$$\mu_d = 2(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}}\zeta^d(X^d_1, Z^d_1)\right] - \frac{\ell^2(d-1)}{4d}\mathbb{E}\left[\dot{V}(X^d_1)^2\right] ,$$

$$\sigma^2_d = \ell^2\mathbb{E}\left[\dot{V}(X^d_1)^2\right] + \frac{\ell^4}{16d}\mathbb{E}\left[\left(\dot{V}(X^d_1)^2 - \mathbb{E}\left[\dot{V}(X^d_1)^2\right]\right)^2\right] ,$$

$$\bar{S}_d = (\sqrt{d}\sigma_d)^{-1}\sum_{i=2}^d \beta^d_i ,$$

$$\beta^d_i = -\ell Z^d_i\dot{V}(X^d_i) - \frac{\ell^2}{4\sqrt{d}}\left(\dot{V}(X^d_i)^2 - \mathbb{E}\left[\dot{V}(X^d_i)^2\right]\right) .$$

By **G** 1(ii) the Berry-Essen Theorem [2, Theorem 5.7] can be applied to $\bar{S}_d$. Then, there exists a universal constant C such that for all $d > 0$,

$$\sup_{x\in\mathbb{R}} \left| \mathbb{P}\left[\sqrt{\frac{d}{d-1}}\bar{S}_d \leq x\right] - \Phi(x)\right| \leq C/\sqrt{d}\,.$$

It follows, with $\tilde{\sigma}_d^2 = (d-1)\sigma_d^2/d$, that

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}\left[\bar{X}_d \leq x\right] - \Phi((x-\mu_d)/\tilde{\sigma}_d)\right| \leq C/\sqrt{d}\,.$$

By this result and (35), Lemma 7 can be applied to obtain a constant $C \geq 0$, independent of d, such that:

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}} \left|\mathcal{G}\left(\ell^2 \dot{V}(X_1^d)^2/d, 2\bar{Y}_d\right) - \mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{X}_d\right)\right| X_1^d\right]\\
&\leq C\Bigg(\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right] + d^{-1/2} + \sqrt{2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right](2\pi\tilde{\sigma}_d^2)^{-1/2}}\\
&\qquad\qquad + \sqrt{\ell|\dot{V}(X_1^d)|/(2\pi d^{1/2}\tilde{\sigma}_d^2)}\Bigg)\,.
\end{aligned}$$

Using this result, we have

$$\begin{aligned}
\mathrm{E}^d &\leq C\Bigg\{\ell^{1/2}\mathbb{E}\left[|\dot{V}(X_1^d)|^{3/2}\right](2\pi d^{1/2}\tilde{\sigma}_d^2)^{-1/2} + \mathbb{E}\left[|\dot{V}(X_1^d)|\right] \qquad \text{(S6)}\\
&\times\left(\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right] + d^{-1/2} + \sqrt{2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right](2\pi\tilde{\sigma}_d^2)^{-1/2}}\right)\Bigg\}\,.
\end{aligned}$$

By Lemma S3, $\mathbb{E}[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}|\bar{X}_d - \bar{Y}_d|]$ goes to 0 as d goes to infinity, and by **G**1(ii) $\lim_{d\to+\infty}\tilde{\sigma}_d^2 = \ell^2\mathbb{E}\left[\dot{V}(X)^2\right]$. Combining these results with (S6), it follows that E^d goes to 0 when d goes to infinity.

For all $n \geq 0$, define $\mathcal{F}_n^d = \sigma(\{X_k^d, k \leq n\})$ and for all $\phi \in C_c^\infty(\mathbb{R},\mathbb{R})$,

$$\begin{aligned}
M_n^d(\phi) &= \frac{\ell}{\sqrt{d}}\sum_{k=0}^{n-1}\phi'(X_{k,1}^d)\left\{Z_{k+1,1}^d\mathbb{1}_{\mathcal{A}^d_{k+1}} - \mathbb{E}\left[Z^d_{k+1,1}\mathbb{1}_{\mathcal{A}^d_{k+1}}\,\middle|\,\mathcal{F}_k^d\right]\right\}\\
&\quad + \frac{\ell^2}{2d}\sum_{k=0}^{n-1}\phi''(X_{k,1}^d)\left\{(Z_{k+1,1}^d)^2\mathbb{1}_{\mathcal{A}^d_{k+1}} - \mathbb{E}\left[(Z^d_{k+1,1})^2\mathbb{1}_{\mathcal{A}^d_{k+1}}\,\middle|\,\mathcal{F}_k^d\right]\right\}\,. \qquad \text{(S7)}
\end{aligned}$$

Proposition S2. *Assume* ***G** 1 and **G** 2 hold. Then, for all $s \leq t$ and all $\phi \in C_c^\infty(\mathbb{R}, \mathbb{R})$,*

$$\lim_{d \to +\infty} \mathbb{E}\left[\left|\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) - \int_s^t \mathrm{L}\phi(Y_{r,1}^d)\mathrm{d}r - \left(M_{\lceil dt \rceil}^d(\phi) - M_{\lceil ds \rceil}^d(\phi)\right)\right|\right] = 0\,.$$

Proof. Using the same decomposition as in the proof of Proposition 4, we only need to prove that for all $1 \leq i \leq 5$, $\lim_{d \to +\infty} \mathbb{E}[|T_i^d|] = 0$, where

$$T_1^d = \int_s^t \phi'(X_{\lfloor dr \rfloor,1}^d)\left(\ell\sqrt{d}\,\mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dr \rceil}^d} \,\middle|\, \mathcal{F}_{\lfloor dr \rfloor}^d\right] + \frac{h(\ell)}{2}\dot{V}(X_{\lfloor dr \rfloor,1}^d)\right)\mathrm{d}r\,,$$
$$T_2^d = \int_s^t \phi''(X_{\lfloor dr \rfloor,1}^d)\left(\frac{\ell^2}{2}\,\mathbb{E}\left[(Z_{\lceil dr \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil dr \rceil}^d} \,\middle|\mathcal{F}_{\lfloor dr \rfloor}^d\right] - \frac{h(\ell)}{2}\right)\mathrm{d}r\,,$$
$$T_3^d = \int_s^t \left(\mathrm{L}\phi(Y_{\lfloor dr \rfloor/d,1}^d) - \mathrm{L}\phi(Y_{r,1}^d)\right)\mathrm{d}r\,,$$
$$T_4^d = \frac{\ell(\lceil dt \rceil - dt)}{\sqrt{d}}\phi'(X_{\lfloor dt \rfloor,1}^d)\left(Z_{\lceil dt \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} - \mathbb{E}\left[Z_{\lceil dt \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} \,\middle|\mathcal{F}_{\lfloor dt \rfloor}^d\right]\right)$$
$$+ \frac{\ell^2(\lceil dt \rceil - dt)}{2d}\phi''(X_{\lfloor dt \rfloor,1}^d)\left((Z_{\lceil dt \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} - \mathbb{E}\left[(Z_{\lceil dt \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} \,\middle|\mathcal{F}_{\lfloor dt \rfloor}^d\right]\right)\,,$$
$$T_5^d = \frac{\ell(\lceil ds \rceil - ds)}{\sqrt{d}}\phi'(X_{\lfloor ds \rfloor,1}^d)\left(Z_{\lceil ds \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} - \mathbb{E}\left[Z_{\lceil ds \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} \,\middle|\mathcal{F}_{\lfloor ds \rfloor}^d\right]\right)$$
$$+ \frac{\ell^2(\lceil ds \rceil - ds)}{2d}\phi''(X_{\lfloor ds \rfloor,1}^d)\left((Z_{\lceil ds \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} - \mathbb{E}\left[(Z_{\lceil ds \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} \,\middle|\mathcal{F}_{\lfloor ds \rfloor}^d\right]\right)\,.$$

First, as ϕ' and ϕ'' are bounded, $\mathbb{E}\left[\left|T_4^d\right| + \left|T_5^d\right|\right] \leq Cd^{-1/2}$. Denote for all $r \in [s,t]$ and $d \geq 1$,

$$\Delta V_{r,i}^d = V\left(X_{\lfloor dr \rfloor,i}^d\right) - V\left(X_{\lfloor dr \rfloor,i}^d + \ell d^{-1/2} Z_{\lceil dr \rceil,i}^d\right)$$
$$\Xi_r^d = 1 \wedge \exp\left\{-\ell Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d)/\sqrt{d} + \sum_{i=2}^d b_{\mathcal{I},i}^{d,\lfloor dr \rfloor}\right\}\,,$$

where for all $k, i \geq 0$, $b_{\mathcal{I},i}^{d,k} = b_{\mathcal{I}}^d(X_{k,i}^d, Z_{k+1,i}^d)$, and for all $x, z \in \mathbb{R}$, $b_{\mathcal{I}}^d(x,y)$ is given by (S4). For all $k \geq 0$, $1 \leq i, j \leq d$, define

$$\mathcal{D}_{\mathcal{I},j}^{d,k} = \left\{X_{k,j}^d + \mathrm{r}\ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I}\right\} \cap \left\{X_{k,j}^d + (1-\mathrm{r})\ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I}\right\}$$
$$\mathcal{D}_{\mathcal{I},i:j}^{d,k} = \bigcap_{\ell=i}^{j} \mathcal{D}_{\mathcal{I},\ell}^{d,k}\,.$$

By the triangle inequality,

$$|T_1| \le \int_s^t \left|\phi'(X^d_{\lfloor dr\rfloor,1})\right| (A_{1,r} + A_{2,r} + A_{3,r} + A_{4,r})\mathrm{d}r\,, \tag{S8}$$

where

$$\begin{aligned}
\Pi^d_r &= 1 \wedge \exp\left\{-\ell d^{-1/2} Z^d_{\lceil dr\rceil,1}\dot{V}(X^d_{\lfloor dr\rfloor,1}) + \sum_{i=2}^d \Delta V^d_{r,i}\right\}\,,\\
A_{1,r} &= \left|\ell\sqrt{d}\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\left(\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}} - \mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1:d}}\Pi^d_r\right)\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]\right|\,,\\
A_{2,r} &= \left|\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1:d}}\left(\Pi^d_r - \Xi^d_r\right)\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]\right|\,,\\
A_{3,r} &= \left|\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\mathbb{1}_{\left(\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1:d}\right)^c}\Xi^d_r\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]\right|\,,\\
A_{4,r} &= \left|\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\Xi^d_r\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right] + \dot{V}(X^d_{\lfloor dr\rfloor,1})h(\ell)/2\right|\,.
\end{aligned}$$

Since $t \mapsto 1 \wedge \exp(t)$ is 1-Lipschitz,

$$\begin{aligned}
\mathbb{E}\left[\left|A^d_{1,r}\right|\right] &\le \ell\sqrt{d}\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1:d}}\left|Z^d_{\lceil dr\rceil,1}\right|\left|\Delta V^d_{r,1} - \ell d^{-1/2}Z^d_{\lceil dr\rceil,1}\dot{V}(X^d_{\lfloor dr\rfloor,1})\right|\right]\,,\\
&\le \ell\sqrt{d}\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1}}\left|Z^d_{\lceil dr\rceil,1}\right|\left|\Delta V^d_{r,1} - \ell d^{-1/2}Z^d_{\lceil dr\rceil,1}\dot{V}(X^d_{\lfloor dr\rfloor,1})\right|\right]\,,\\
&\le \ell\sqrt{d}\mathbb{E}\left[\left|Z^d_{\lceil dr\rceil,1}\right|\left|\mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1}}\Delta V^d_{r,1} - \ell d^{-1/2}Z^d_{\lceil dr\rceil,1}\dot{V}(X^d_{\lfloor dr\rfloor,1})\right|\right]
\end{aligned}$$

and $\mathbb{E}[|A^d_{1,r}|]$ goes to 0 as $d \to +\infty$ for almost all r by Lemma S2(ii). So by the Fubini theorem, the first term in (S8) goes to 0 as $d \to +\infty$. For $A^d_{2,r}$, note that

$$A_{2,r} \le \left|\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},2:d}}\left(\Pi^d_r - \Xi^d_r\right)\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]\right|\,.$$

Then, by [1, Lemma 6],

$$\begin{aligned}
\mathbb{E}\left[\left|A^d_{2,r}\right|\right] \le \mathbb{E}\Bigg[\Bigg|\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})\mathbb{1}_{\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},2:d}}\Bigg\{&\mathcal{G}\left(\frac{\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}, 2\sum_{i=2}^d \Delta V^d_{r,i}\right)\\
&-\mathcal{G}\left(\frac{\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}, 2\sum_{i=2}^d b^{d,\lfloor dr\rfloor}_{\mathcal{I},i}\right)\Bigg\}\Bigg|\Bigg]\,,
\end{aligned}$$

where $\mathcal{G}$ is defined in (24). By Lemma S7, this expectation goes to zero when d goes to infinity. Then by the Fubini theorem and the Lebesgue dominated convergence theorem, the second term of (S8) goes 0 as $d \to +\infty$. On the other hand, by **G**1(iii) and Holder's inequality applied with $\alpha = 1/(1-2/\gamma) > 1$, for all $1 \le j \le 4$,

$$\mathbb{E}\left[\left|A^d_{3,r}\right|\right] \le \ell\sqrt{d}\left(\mathbb{E}\left[\left|Z^d_{\lceil dr \rceil,1}\right| \mathbb{1}_{(\mathcal{D}^{d,\lfloor dr \rfloor}_{\mathcal{I},1})^c}\right] + \sum_{i=2}^d \mathbb{E}\left[\mathbb{1}_{(\mathcal{D}^{d,\lfloor dr \rfloor}_{\mathcal{I},i})^c}\right]\right) ,$$
$$\le \ell\sqrt{d}\left(\mathbb{E}\left[|Z^d_{m_j,1}|^{\alpha/(\alpha-1)}\right]^{(\alpha-1)/\alpha} d^{-\gamma/(2\alpha)} + d^{1-\gamma/2}\right) \le C d^{3/2-\gamma/2}$$

and $\mathbb{E}[|A^d_{3,r}|]$ goes to 0 as $d \to +\infty$ for almost all r. Define

$$\bar{V}_{d,1} = \sum_{i=1}^d \dot{V}(X^d_{\lfloor dr \rfloor,i})^2 \quad \text{and} \quad \bar{V}_{d,2} = \bar{V}_{d,1} - \dot{V}(X^d_{\lfloor dr \rfloor,1})^2 .$$

For the last term, by [1, Lemma 6]:

$$\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr \rceil,1}\Xi^d_r \,\middle|\, \mathcal{F}^d_{\lfloor dr \rfloor}\right] = -\ell^2 \dot{V}(X^d_{\lfloor dr \rfloor,1})$$
$$\times\,\mathcal{G}\left(\frac{\ell^2}{d}\bar{V}_{d,1}, \left\{\frac{\ell^2}{2d}\bar{V}_{d,2} - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_\mathcal{I}}\zeta^d(X,Z)\right]\right\}\right) , \quad \text{(S9)}$$

where $\mathcal{D}_\mathcal{I} = \left\{X + \ell d^{-1/2} Z \in \mathcal{I}\right\}$, X is distributed according to π and Z is a standard Gaussian random variable independent of X. As $\mathcal{G}$ is continuous on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0,0\}$ (see [1, Lemma 2]), by **G**1(ii), Lemma S4 and the law of large numbers, almost surely,

$$\lim_{d\to+\infty} \ell^2 \mathcal{G}\left(\ell^2\bar{V}_{d,1}/d, \ell^2\bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_\mathcal{I}}\zeta^d(X,Z)\right]\right)$$
$$= \ell^2\mathcal{G}\left(\ell^2\mathbb{E}[\dot{V}(X)^2], \ell^2\mathbb{E}[\dot{V}(X)^2]\right) = h(\ell)/2 , \quad \text{(S10)}$$

where $h(\ell)$ is defined in (11). Therefore by Fubini's Theorem, (S9) and Lebesgue's dominated convergence theorem, the last term of (S8) goes to 0 as d goes to infinity. The proof for T^d_2 follows the same lines. By the triangle inequality,

$$\left|T^d_2\right| \le \left|\int_s^t \phi''(X^d_{\lfloor dr \rfloor,1})(\ell^2/2)\;\mathbb{E}\left[(Z^d_{\lceil dr \rceil,1})^2\left(\mathbb{1}_{\mathcal{A}^d_{\lceil dr \rceil}} - \Xi^d_r\right)\middle|\mathcal{F}^d_{\lfloor dr \rfloor}\right]\mathrm{d}r\right|$$
$$+\left|\int_s^t \phi''(X^d_{\lfloor dr \rfloor,1})\left((\ell^2/2)\;\mathbb{E}\left[(Z^d_{\lceil dr \rceil,1})^2\Xi^d_r\middle|\mathcal{F}^d_{\lfloor dr \rfloor}\right] - h(\ell)/2\right)\mathrm{d}r\right| . \quad \text{(S11)}$$

By Fubini's Theorem, Lebesgue's dominated convergence theorem and Proposition S1, the expectation of the first term goes to zero when d goes to infinity.

For the second term, by [1, Lemma 6 (A.5)],

$$(\ell^2/2)\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2 1 \wedge \exp\left\{-\frac{\ell Z^d_{\lceil dr\rceil,1}}{\sqrt{d}}\dot{V}(X^d_{\lfloor dr\rfloor,1}) + \sum_{i=2}^{d} b^{d,\lfloor dr\rfloor}_{\mathcal{I},i}\right\}\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]$$
$$= (B_1 + B_2 - B_3)/2\,, \quad \text{(S12)}$$

where

$$B_1 = \ell^2\Gamma\left(\ell^2\bar{V}_{d,1}/d, \ell^2\bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)\right]\right)\,,$$
$$B_2 = \frac{\ell^4\dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}\mathcal{G}\left(\ell^2\bar{V}_{d,1}/d, \ell^2\bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)\right]\right)\,,$$
$$B_3 = \frac{\ell^4\dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}\left(2\pi\ell^2\bar{V}_{d,1}/d\right)^{-1/2}$$
$$\times \exp\left\{-\frac{\left[-2(d-1)\mathbb{E}[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)] + (\ell^2/(4d))\bar{V}_{d,2}\right]^2}{2\ell^2\bar{V}_{d,1}/d}\right\}\,,$$

where Γ is defined in (25). As Γ is continuous on $\mathbb{R}_+ \times \mathbb{R}\setminus\{0,0\}$ (see [1, Lemma 2]), by $\mathbf{G}1$(ii), Lemma S4 and the law of large numbers, almost surely,

$$\lim_{d\to+\infty}\ell^2\Gamma\left(\ell^2\bar{V}_{d,1}/d, \left\{\ell^2\bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)\right]\right\}\right)$$
$$= \ell^2\Gamma\left(\ell^2\mathbb{E}[\dot{V}(X)^2], \ell^2\mathbb{E}[\dot{V}(X)^2]\right) = h(\ell)\,. \quad \text{(S13)}$$

By Lemma S4, by $\mathbf{G}1$(ii) and the law of large numbers, almost surely,

$$\lim_{d\to+\infty}\exp\left\{-\frac{\left[-2(d-1)\mathbb{E}[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)] + (\ell^2/(4d))\bar{V}_{d,2}\right]^2}{2\ell^2\bar{V}_{d,1}/d}\right\}$$
$$= \exp\left\{-\frac{\ell^2}{8}\mathbb{E}[\dot{V}(X)^2]\right\}\,.$$

Then, as $\mathcal{G}$ is bounded on $\mathbb{R}_+ \times \mathbb{R}$,

$$\lim_{d\to+\infty}\mathbb{E}\left[\left|\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})\,(B_2 - B_3)\,\mathrm{d}r\right|\right] = 0\,. \quad \text{(S14)}$$

Therefore, by Fubini's Theorem, (S12), (S13), (S14) and Lebesgue's dominated convergence theorem, the second term of (S11) goes to 0 as d goes to infinity. The proof for T^d_3 follows exactly the same lines as the proof of Proposition 4.

Proof of Theorem 5. Using Lemma S5, Proposition 1 and Proposition S2, the proof follows the same lines as the proof of Theorem 3.

4. Detailed computations for the Gamma distribution

This section provides the explicit computations to check **G**1(i) in Example 2. The result is proved for $\theta < 0$ (the proof for $\theta > 0$ follows the same lines). For all $\theta \in \mathbb{R}$ using $\mathrm{a}_1 > 6$,

$$\begin{aligned}\int_{\mathbb{R}_+^*} |\mathcal{E}_1|^5 \, \pi_\gamma(x)\mathrm{d}x &\leq C|\theta|^5 \int_0^{3|\theta|/2} \left\{1/x^5 + x^{5(\mathrm{a}_2-1)}\right\} x^{\mathrm{a}_1-1}\mathrm{e}^{-x^{\mathrm{a}_2}}\mathrm{d}x\,, \\ &\leq C\left(|\theta|^{\mathrm{a}_1}\int_0^{3/2} x^{\mathrm{a}_1-6}\mathrm{e}^{-(|\theta|x)^{\mathrm{a}_2}}\mathrm{d}x \right.\\ &\qquad \left. + |\theta|^{5\mathrm{a}_2+\mathrm{a}_1}\int_0^{3/2} x^{5(\mathrm{a}_2-1)+\mathrm{a}_1-1}\mathrm{e}^{-(|\theta|x)^{\mathrm{a}_2}}\mathrm{d}x\right)\,, \\ &\leq C(|\theta|^{\mathrm{a}_1} + |\theta|^{5\mathrm{a}_2+\mathrm{a}_1})\,. \end{aligned} \tag{S15}$$

On the other hand, as for all $x > -1$, $x/(x+1) \leq \log(1+x) \leq x$, for all $\theta < 0$, and $x \geq 3|\theta|/2$,

$$|\log(1+\theta/x) - \theta/x| \leq \frac{|\theta|^2}{x^2(1+\theta/x)} \leq 3|\theta|^2/x^2\,,$$

where the last inequality come from $|\theta|/x \leq 2/3$. Then, it yields

$$\begin{aligned}\int_{\mathbb{R}_+^*} |\mathcal{E}_2(x)|^5 \, \pi_\gamma(x)\mathrm{d}x &\leq C|\theta|^{10}\left(\int_{3|\theta|/2}^1 x^{\mathrm{a}_1-11}\mathrm{e}^{-x^{\mathrm{a}_2}}\mathrm{d}x + \int_1^{+\infty} x^{\mathrm{a}_1-11}\mathrm{e}^{-x^{\mathrm{a}_2}}\mathrm{d}x\right)\,, \\ &\leq C(|\theta|^{\mathrm{a}_1} + |\theta|^{10})\,. \end{aligned} \tag{S16}$$

For the last term, for all $\theta < 0$ and all $x \geq 3|\theta|/2$, using a Taylor expansion of $x \mapsto x^{\mathrm{a}_2}$, there exists $\zeta \in [x+\theta, x]$ such that

$$\left|(x+\theta)^{\mathrm{a}_2} - x^{\mathrm{a}_2} - \mathrm{a}_2\theta x^{\mathrm{a}_2-1}\right| \leq C|\theta|^2|\zeta|^{\mathrm{a}_2-2} \leq C|\theta|^2\,|x|^{\mathrm{a}_2-2}\,.$$

Then,

$$\int_{\mathbb{R}_+^*} |\mathcal{E}_3(x)|^5 \, \pi_\gamma(x)\mathrm{d}x \leq C|\theta|^{10}\int_{3|\theta|/2}^{+\infty} x^{5(\mathrm{a}_2-2)+\mathrm{a}_1-1}\mathrm{e}^{-x^{\mathrm{a}_2}}\mathrm{d}x \leq C(|\theta|^{5\mathrm{a}_2+\mathrm{a}_1}+|\theta|^{10})\,. \tag{S17}$$

Combining (S15), (S16),(S17) and using that $\mathrm{a}_1 > 6$ concludes the proof of **G** 1(i) for $p = 5$.

References

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