Appendix C. Supplementary file for 'Monte Carlo Fusion' by Hongsheng Dai, Murray Pollock and Gareth Roberts – Proof of Lemma 2

Note that OU process transition density $p^{\text{ou}}_{T,c}\left(\boldsymbol{y}\,\middle|\,\boldsymbol{x}^{(c)}\right)$ in (19) of the main paper is given by

$$\begin{aligned} p^{\text{ou}}_{T,c}\left(\boldsymbol{y}\,\middle|\,\boldsymbol{x}^{(c)}\right) &= \mathbf{det}\,(2\pi\mathbf{V}_c)^{-1/2}\cdot\mathbf{etr}\left[-\frac{1}{2}\,(\boldsymbol{y}_T-\boldsymbol{m}_c)^{\otimes 2}\,\mathbf{V}_c^{-1}\right] \\ \boldsymbol{m}_c &:= \hat{\boldsymbol{\mu}}_c + e^{-\hat{\boldsymbol{\Lambda}}_c T}(\boldsymbol{x}^{(c)}-\hat{\boldsymbol{\mu}}_c) \\ \mathbf{V}_c &= \mathbf{Var}\left(\int_0^T e^{\hat{\boldsymbol{\Lambda}}_c(s-T)}\,\mathrm{d}\boldsymbol{W}^{(c)}_t\right) = \frac{\hat{\boldsymbol{\Lambda}}_c^{-1}}{2}\left(\mathbf{I}_d - e^{-2\hat{\boldsymbol{\Lambda}}_c T}\right). \end{aligned} \tag{20}$$

where $\mathbf{I}_d$ is the d-dimensional identity matrix.

From the definition of $A^{ou}_c(\boldsymbol{x})$ in (10) of the main paper and (20), we can then rewrite $\tilde{g}^{ou}$ in (19) as

$$\begin{aligned} &\tilde{g}^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)},\boldsymbol{y}) \\ \propto\; &\prod_{c=1}^{C}\left[f_c\left(\boldsymbol{x}^{(c)}\right)p^{\text{ou}}_{T,c}\left(\boldsymbol{y}\,\middle|\,\boldsymbol{x}^{(c)}\right)\cdot\mathbf{etr}\left(\frac{(\hat{\boldsymbol{\mu}}_c-\boldsymbol{y})^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c-(\hat{\boldsymbol{\mu}}_c-\boldsymbol{X}^{(c)}_0)^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c}{2}\right)\right] \\ \propto\; &\left[\prod_{c=1}^{C}f_c\left(\boldsymbol{x}^{(c)}\right)\right]\mathbf{etr}\left[-\frac{1}{2}\boldsymbol{y}^{\otimes 2}\sum_{c=1}^{C}\left(\mathbf{V}_c^{-1}-\hat{\boldsymbol{\Lambda}}_c\right)\right] \\ &\mathbf{etr}\left[\sum_{c=1}^{C}\left[\boldsymbol{y}\cdot\boldsymbol{m}_c^{tr}\cdot\mathbf{V}_c^{-1}-\boldsymbol{y}\cdot\hat{\boldsymbol{\mu}}_c^{tr}\cdot\hat{\boldsymbol{\Lambda}}_c\right]+\sum_{c=1}^{C}\left[\mathbf{V}_c^{-1}\cdot\boldsymbol{m}_c\cdot\boldsymbol{y}^{tr}-\hat{\boldsymbol{\Lambda}}_c\cdot\hat{\boldsymbol{\mu}}_c\cdot\boldsymbol{y}^{tr}\right]\right] \\ &\mathbf{etr}\left[-\frac{1}{2}\sum_{c=1}^{C}\boldsymbol{m}_c^{\otimes 2}\cdot\mathbf{V}_c^{-1}\right]\cdot\mathbf{etr}\left[-\frac{1}{2}\sum_{c=1}^{C}(\hat{\boldsymbol{\mu}}_c-\boldsymbol{x}^{(c)})^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c\right] \end{aligned} \tag{21}$$

It can be simplified as

$$\tilde{g}^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)},\boldsymbol{y}) \propto \left[\textstyle\prod_{c=1}^{C}f_c\left(\boldsymbol{x}^{(c)}\right)\right]\mathbf{etr}\left[-\tfrac{1}{2}\left[\boldsymbol{y}-\widetilde{\boldsymbol{x}}\right]^{\otimes 2}\mathbf{D}\right]\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)}), \tag{22}$$

$$\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)}) = \mathbf{etr}\left[\tfrac{1}{2}\widetilde{\boldsymbol{x}}^{\otimes 2}\mathbf{D}\right]\mathbf{etr}\left[-\tfrac{1}{2}\textstyle\sum_{c=1}^{C}\boldsymbol{m}_c^{\otimes 2}\cdot\mathbf{V}_c^{-1}\right]\cdot\mathbf{etr}\left[-\tfrac{1}{2}\textstyle\sum_{c=1}^{C}(\hat{\boldsymbol{\mu}}_c-\boldsymbol{x}^{(c)})^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c\right]$$

where

$$\begin{aligned} \widetilde{\boldsymbol{x}} &= \mathbf{D}^{-1}\left\{\sum_{c=1}^{C}\left(\mathbf{V}_c^{-1}\boldsymbol{m}_c-\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)\right\}, \\ \mathbf{D} &= \sum_{c=1}^{C}\mathbf{D}_c,\quad \mathbf{D}_c=\mathbf{V}_c^{-1}-\hat{\boldsymbol{\Lambda}}_c. \end{aligned}$$

Then Lemma 2 is proved by noticing that $\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)})$ in (22) can be rewritten as that in (12), which is given in the following lemma.

Lemma 1. *The result* $\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)})$ *in (22) can be rewritten as*

$$\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)}) \propto \mathbf{etr}\left\{-\frac{1}{2}\left[\mathbf{H}\mathbf{D}^{-1}+\sum_{c=1}^{C}\mathbf{M}_{1,c}\left(\boldsymbol{m}_c+\mathbf{M}_{1,c}^{-1}\mathbf{M}_{2,c}\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\right]\right\}.$$

where

$$\begin{aligned}
\mathbf{M}_{1,c} &= e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c-\mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1}\\
\mathbf{M}_{2,c} &= \mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1}-2\hat{\mathbf{\Lambda}}_c e^{2\hat{\mathbf{\Lambda}}_c T}\\
\mathbf{H} &= \left(\sum_{c=1}^{C}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\mathbf{V}_c^{-1}\right)\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)-\left\{\sum_{c=1}^{C}\mathbf{V}_c^{-1}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)\right\}^{\otimes 2}.
\end{aligned}$$

□

Proof. Using the following results

$$\begin{aligned}
\hat{\boldsymbol{\mu}}_c-\boldsymbol{x}^{(c)} &= -e^{\hat{\mathbf{\Lambda}}_c T}(\boldsymbol{m}_c-\hat{\boldsymbol{\mu}}_c)\\
e^{\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c e^{\hat{\mathbf{\Lambda}}_c T} &= e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c=\hat{\mathbf{\Lambda}}_c e^{2\hat{\mathbf{\Lambda}}_c T}
\end{aligned}$$

we can rewrite (22) as

$$\begin{aligned}
\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)}) &\propto \mathbf{etr}\left\{-\frac{1}{2}\left[\sum_{c=1}^{C}\left[\boldsymbol{m}_c^{\otimes 2}\mathbf{V}_c^{-1}+(\boldsymbol{m}_c-\hat{\boldsymbol{\mu}}_c)^{\otimes 2}e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_k\right]-\mathbf{D}^{-1}\mathbf{K}\right]\right\} \qquad (23)\\
\mathbf{K} &= (\mathbf{D}\widetilde{\boldsymbol{x}})^{\otimes 2}=\left\{\sum_{c=1}^{C}\left(\mathbf{V}_c^{-1}\boldsymbol{m}_c-\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)\right\}^{\otimes 2}=\left\{\sum_{c=1}^{C}\mathbf{V}_c^{-1}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)\right\}^{\otimes 2}
\end{aligned}$$

and then further as

$$\begin{aligned}
\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)}) \propto\ & \mathbf{etr}\left\{-\frac{1}{2}\left(\sum_{c=1}^{C}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\cdot\mathbf{V}_c^{-1}-\mathbf{K}\mathbf{D}^{-1}\right.\right. \qquad (24)\\
&\left.\left.+\sum_{c=1}^{C}\left[\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\hat{\mathbf{\Lambda}}_c+\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}\mathbf{V}_c^{-1}-\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c^{\otimes 2}\hat{\mathbf{\Lambda}}_c+(\boldsymbol{m}_c-\hat{\boldsymbol{\mu}}_c)^{\otimes 2}e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c\right]\right)\right\}
\end{aligned}$$

Noticing that

$$
\begin{aligned}
&\sum_{c=1}^{C}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\mathbf{V}_c^{-1}\cdot\mathbf{D}-\mathbf{K}\\
=\;&\underbrace{\left(\sum_{c=1}^{C}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\mathbf{V}_c^{-1}\right)\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)-\left\{\sum_{c=1}^{C}\mathbf{V}_c^{-1}\left(\boldsymbol{m}_c^{tr}-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)\right\}^{\otimes 2}}_{\mathbf{H}}\\
&-\left(\sum_{c=1}^{C}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\mathbf{V}_c^{-1}\right)\left(\sum_{c=1}^{C}\hat{\boldsymbol{\Lambda}}_c\right)
\end{aligned}
$$

we can write (24) as

$$
\begin{aligned}
&\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)})\\
\propto\;&\mathbf{etr}\Bigg\{-\frac{\mathbf{D}^{-1}}{2}\Bigg(\mathbf{H}-\left(\sum_{c=1}^{C}\left(\boldsymbol{m}_c-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\mathbf{V}_c^{-1}\right)\left(\sum_{c=1}^{C}\hat{\boldsymbol{\Lambda}}_c\right)\\
&+\sum_{c=1}^{C}\left[\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\hat{\boldsymbol{\Lambda}}_c+\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}\mathbf{V}_c^{-1}-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c+(\boldsymbol{m}_c-\hat{\boldsymbol{\mu}}_c)^{\otimes 2}e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\right]\cdot\mathbf{D}\Bigg)\Bigg\}\\
=\;&\mathbf{etr}\Bigg\{-\frac{\mathbf{D}^{-1}}{2}\Bigg(\mathbf{H}-\left(\sum_{c=1}^{C}\boldsymbol{m}_c^{\otimes 2}\mathbf{V}_c^{-1}+\left(\boldsymbol{m}_c^{\otimes 2}-\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\right)e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c+\mathbf{V}_ce^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\left(-\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}+\hat{\boldsymbol{\mu}}_c^{\otimes 2}\right)\mathbf{V}_c^{-1}\right)\left(\sum_{c=1}^{C}\hat{\boldsymbol{\Lambda}}_c\right)\\
&\qquad+\sum_{c=1}^{C}\Big[\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\hat{\boldsymbol{\Lambda}}_c+\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}\mathbf{V}_c^{-1}-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c\\
&\qquad+\left(\boldsymbol{m}_c^{\otimes 2}-\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\right)e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c+\mathbf{V}_ce^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\left(-\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}+\hat{\boldsymbol{\mu}}_c^{\otimes 2}\right)\mathbf{V}_c^{-1}\Big]\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)\Bigg)\Bigg\}\\
=\;&\mathbf{etr}\Bigg\{-\frac{\mathbf{D}^{-1}}{2}\mathbf{H}\\
&-\frac{\mathbf{D}^{-1}}{2}\sum_{c=1}^{C}\Big[\boldsymbol{m}_c^{\otimes 2}e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c+\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\left(\hat{\boldsymbol{\Lambda}}_c-e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\right)+\mathbf{V}_c\left(\mathbf{I}_d-e^{2\hat{\boldsymbol{\Lambda}}_cT}\right)\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}\mathbf{V}_c^{-1}\\
&\quad+\mathbf{V}_ce^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c^{\otimes 2}\mathbf{V}_c^{-1}-\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c^{\otimes 2}\hat{\boldsymbol{\Lambda}}_c\Big]\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)\\
&-\sum_{c=1}^{C}\left(\boldsymbol{m}_c^{\otimes 2}\left(\mathbf{V}_c^{-1}+e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\right)-\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}e^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c-\mathbf{V}_ce^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}\mathbf{V}_c^{-1}+\mathbf{V}_ce^{2\hat{\boldsymbol{\Lambda}}_cT}\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c^{\otimes 2}\mathbf{V}_c^{-1}\right)\left(\sum_{c=1}^{C}\hat{\boldsymbol{\Lambda}}_c\right)\Bigg\}\\
=\;&\mathbf{etr}\;\left(-\frac{1}{2}\left(\mathbf{H}\mathbf{D}^{-1}+\sum_{c=1}^{C}\left(\boldsymbol{m}_c^{\otimes 2}\mathbf{M}_{1,c}+\left(\boldsymbol{m}_c\hat{\boldsymbol{\mu}}_c^{tr}\hat{\boldsymbol{\Lambda}}_c\mathbf{V}_c+\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\boldsymbol{m}_c^{tr}\right)\mathbf{M}_{2,c}\right)+\mathbf{M}_0^{*}\right)\right)\\
=\;&\mathbf{etr}\;\left(-\frac{1}{2}\left(\mathbf{H}\mathbf{D}^{-1}+\sum_{c=1}^{C}\mathbf{M}_{1,c}\left(\boldsymbol{m}_c+\mathbf{M}_{1,c}^{-1}\mathbf{M}_{2,c}\mathbf{V}_c\hat{\boldsymbol{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}+\mathbf{M}_0\right)\right)
\end{aligned}
\tag{25}
$$

where $\mathbf{M}_0^*$ and $\mathbf{M}_0$ are some constant matrices (not depending on $\boldsymbol{x}^{(c)}$) and

$$\begin{aligned}\mathbf{M}_{1,c} &= e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)\mathbf{D}^{-1} - \left(\mathbf{V}_c^{-1} + e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c\right)\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1}\\ \mathbf{M}_{2,c} &= \mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)\mathbf{D}^{-1} - \mathbf{V}_c^{-1}e^{2\hat{\mathbf{\Lambda}}_c T}\end{aligned}$$

Therefore

$$\rho^{ou}(\boldsymbol{x}^{(1)},\cdots,\boldsymbol{x}^{(C)}) \propto \mathbf{etr}\left\{-\frac{1}{2}\left[\mathbf{H}\mathbf{D}^{-1} + \sum_{c=1}^{C}\mathbf{M}_{1,c}\left(\boldsymbol{m}_c + \mathbf{M}_{1,c}^{-1}\mathbf{M}_{2,c}\mathbf{V}_c\hat{\mathbf{\Lambda}}_c\hat{\boldsymbol{\mu}}_c\right)^{\otimes 2}\right]\right\}. \tag{26}$$

We can further simplify $\mathbf{M}_{1,c}$ and $\mathbf{M}_{2,c}$ as

$$\begin{aligned}\mathbf{M}_{1,c} &= e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)\mathbf{D}^{-1} - \left(\mathbf{V}_c^{-1} + e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c\right)\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1}\\ &= e^{2\hat{\mathbf{\Lambda}}_c T}\hat{\mathbf{\Lambda}}_c - \mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1}\end{aligned}$$

and

$$\begin{aligned}\mathbf{M}_{2,c} &= \mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\mathbf{V}_c^{-1}\right)\mathbf{D}^{-1} - \mathbf{V}_c^{-1}e^{2\hat{\mathbf{\Lambda}}_c T}\\ &= \mathbf{V}_c^{-1}\left(\mathbf{D} + \sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1} - \mathbf{V}_c^{-1}e^{2\hat{\mathbf{\Lambda}}_c T}\\ &= \mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1} + \mathbf{V}_c^{-1}\left(\mathbf{I}_d - e^{2\hat{\mathbf{\Lambda}}_c T}\right)\\ &= \mathbf{V}_c^{-1}\left(\sum_{c=1}^{C}\hat{\mathbf{\Lambda}}_c\right)\mathbf{D}^{-1} - 2\hat{\mathbf{\Lambda}}_c e^{2\hat{\mathbf{\Lambda}}_c T}\end{aligned}$$

□