5 Butler (2019a): Supplementary materials

5.1 Proofs for Section 2

5.1.1 Proof of Theorem 1

If $f(t)$ is locally of bounded variation at t, then the inversion formula in (4) holds for argument t; see Widder (1946, p. 241, Theorem 5a). To ensure that the second to last integral in (5) is negligible, we require that

$$\max_{b^- \leq x \leq b^+} |\mathcal{M}(x+iN)| \to 0 \qquad N \to \infty. \tag{48}$$

That portion of the uniform convergence in (48) with $x < b$ automatically holds because it is within the range of the convergence domain for $\mathcal{M}$. The range $x \in [b, b^+]$, however, is outside of this convergence domain so assumption $\mathcal{X}$ in Theorem 1 is needed to ensure that (48) holds.

Showing that the last integral in (5) is negligible uses an analytic continuation argument. Define $\mathcal{D}(s) = \overline{\mathcal{M}(s)} - \mathcal{M}(\bar{s})$ where $\bar{s}$ denotes complex conjugate. Function $\mathcal{D}(s) \equiv 0$ on $\{\text{Re}(s) < b\}$ so its analytic continuation is also zero on $\mathbb{C}$. Thus $|\mathcal{M}(x-iN)| = |\mathcal{M}(x+iN)|$ for $x \in (b, b^+)$ and assumption $\mathcal{X}$ in Theorem 1 also ensures that the last integral is negligible.

Denote the integral in assumption $\mathcal{UI}$ of Theorem 1 as $J(t)$. By the $\mathcal{UI}$ assumption, for any sufficiently small $\eta > 0$, there is a $\tau = \tau(\eta)$ such that

$$\left| J(t) - \int_{-\tau}^{+\tau} \mathcal{M}(b^+ + iy) e^{-iyt} dy \right| < \eta/2 \tag{49}$$

for all $t \geq T$. Now, since $\mathcal{M}$ is analytic from the boundary out to $\text{Re}(s) = b + \varepsilon_0$, the Riemann-Lebesgue lemma (Feller, 1971, p. 513) ensures the existence of a $T_1 = T_1(\tau, \eta)$ such that

$$\left| \int_{-\tau}^{+\tau} \mathcal{M}(b^+ + iy) e^{-iyt} dy \right| < \eta/2$$

for all $t \geq T_1$. Thus $|J(t)| < \eta$ for $t \geq \max\{T, T_1\}$, the integral in (7) is $o(1)$ as $t \to \infty$, and $R_1(t)$ in (7) is $o(e^{-b^+ t})$ as $t \to \infty$.

5.1.2 Example 1

To show that $\mathcal{X}$ holds, note that

$$\max_{\alpha \leq x \leq \alpha^+} |\mathcal{M}(x+iN)| \sim cN^{-\beta} \to 0 \qquad N \to \infty.$$

For $\beta \in (0,1]$, assumption $\mathcal{UI}$ follows from an integration-by-parts argument. Break the integral into $(-\infty, -Y) \cup [-Y, Y] \cup (Y, \infty)$ for $Y > 0$. Then,

$$\begin{aligned} \left| \int_Y^\infty \mathcal{M}(\alpha^+ + iy) e^{-iyt} dy \right| &= \left| \left\{ \mathcal{M}(\alpha^+ + iy) \frac{e^{-iyt}}{-it} \right\}_Y^\infty + i \int_Y^\infty \mathcal{M}'(\alpha^+ + iy) e^{-iyt} dy \right| \\ &\leq \frac{1}{t} |\mathcal{M}(\alpha^+ + iY)| + \int_Y^\infty |\mathcal{M}'(\alpha^+ + iy)| dy. \end{aligned} \tag{50}$$

The first term in (50) can be made uniformly small for $t > T$ by making Y sufficiently large. The second term is, upon differentiating $\mathcal{M}$,

$$\int_Y^\infty |\mathcal{M}(\alpha^+ + iy)|\,|\psi(\beta + z_y) - \psi(z_y)|dy, \tag{51}$$

where $z_y = -(\alpha^+ - \alpha) - iy$ and ψ is the digamma function. Using 5.11.2 of NIST DLMF in which $\psi(z) = \ln z - 1/(2z) + R(z)$ where $R(z) = O(z^{-2})$ as $|z| \to \infty$, then

$$|\psi(\beta + z_y) - \psi(z_y)| < |\ln(1 + \beta/z_y)| + \frac{\beta}{2|z_y|^2} + |R(\beta + z_y) - R(z_y)|.$$

Since $\beta/|z_y| \sim \beta/y$ as $y \to \infty$, we may use 4.5.6 and 4.5.2 of NIST DLMF so that

$$|\ln(1 + \beta/z_y)| \le -\ln(1 - \beta/|z_y|) < \frac{\beta/|z_y|}{1 - \beta/|z_y|}$$

for sufficiently large y. Thus,

$$|\psi(\beta + z_y) - \psi(z_y)| < c_2\beta/y + c_3/y^2$$

for some c_2 and c_3 with $y > Y_1$. Since $|\mathcal{M}(\alpha^+ + iy)| < c_1|y|^{-\beta}$ for $y > Y_2$, an upper bound on (51) for $Y > \max(Y_1, Y_2)$ is

$$c_1 \int_Y^\infty \frac{1}{y^\beta}\left(\frac{c_2\beta}{y} + \frac{c_3}{y^2}\right) dy = c_1\left(\frac{c_2}{Y^\beta} + \frac{c_3}{(\beta+1)Y^{1+\beta}}\right) \to 0 \qquad Y \to \infty. \tag{52}$$

The same argument applies to the integral over $(-\infty, -Y)$. For sufficiently large Y, the Riemann-Lebesgue theorem applies to the integral over $[-Y, Y]$. This proves the $\mathcal{UI}$ assumption.

5.1.3 Multiple poles on the convergence boundary $\{\mathrm{Re}(s) = b\}$

Corollary 6. *Suppose all conditions of Theorem 1 except replace $\mathcal{AC}$ with the following:*

($\mathcal{AC}_6$) *There exists $\varepsilon_0 > 0$ such that $\mathcal{M}$ can be analytically continued across the boundary to $\{b \le \mathrm{Re}(s) < b + \varepsilon_0\}$, save from a finite set of poles at b, $b \pm iy_1, \dots, b \pm iy_p$ with orders $\mathfrak{m}_0, \dots, \mathfrak{m}_p$ respectively.*

If $\sum_{k=1}^{\mathfrak{m}_0} \beta_{-k;b}(s-b)^{-k}$ denotes the principal part of the Laurent expansion of $\mathcal{M}$ at b, then

$$\begin{aligned} f(t) &= e^{-bt}\sum_{k=1}^{\mathfrak{m}_0} t^{k-1}\frac{(-1)^k\beta_{-k;b}}{(k-1)!} \\ &\quad + e^{-bt}\sum_{j=1}^{p}\sum_{k=1}^{\mathfrak{m}_j} t^{k-1}\frac{(-1)^k}{(k-1)!}\left\{e^{-iy_jt}\beta_{-k;b+iy_j} + e^{iy_jt}\beta_{-k;b-iy_j}\right\} + o(e^{-b^+t}) \\ &= e^{-bt}\sum_{k=1}^{\mathfrak{m}_0} t^{k-1}\frac{(-1)^k\beta_{-k;b}}{(k-1)!} + 2e^{-bt}\sum_{j=1}^{p}\sum_{k=1}^{\mathfrak{m}_j} t^{k-1}\frac{(-1)^k}{(k-1)!}\,\mathrm{Re}\left\{e^{-iy_jt}\beta_{-k;b+iy_j}\right\} + o(e^{-b^+t}) \end{aligned} \tag{53}$$

The proof uses Cauchy's theorem in the same manner as used in Theorem 1.

5.1.4 Proof of Theorem 2

The inversion formula for $S(t)$ is

$$S(t) = \frac{1}{2\pi i} \lim_{N\to\infty} \int_{b^- - iN}^{b^- + iN} \frac{\mathcal{M}(s)}{s} e^{-st} ds. \tag{54}$$

Let the inversion integral in (54) be part of the integral round the rectangular curve with the four corners $b^- \pm iN$ and $b^+ \pm iN$ as in (5) but with $\mathcal{M}(s)/s$ replacing $\mathcal{M}(s)$. Cauchy's residue theorem allows the integral in (54) to be written in terms of $-\operatorname{Res}\left\{s^{-1}\mathcal{M}(s)e^{-st}; b\right\}$ and integrals along the other 3 line segments of the rectangle as in (5). The two integrals along the top and bottom are negligible as $N \to \infty$ since

$$\max_{b\le x\le b^+} \frac{|\mathcal{M}(x+iN)|}{|x+iN|} \le \frac{1}{b} \max_{b\le x\le b^+} |\mathcal{M}(x+iN)| \to 0.$$

Condition $\mathcal{UI}^S$ allows the argument from Theorem 1, based on the Riemann-Lebesgue lemma, to be applied to the integral along $\operatorname{Re}(s) = b^+$ to show it is $o(1)$ as $t \to \infty$. The value for $-\operatorname{Res}\left\{s^{-1}\mathcal{M}(s)e^{-st}; b\right\}$ is the leading term in (11).

To evaluate this residue, we take the Laurent expansion of $\mathcal{M}$ about b and sum the residues of the resulting addends so that

$$-\operatorname{Res}\left\{\frac{1}{s}\mathcal{M}(s)e^{-st}; b\right\} = -\sum_{k=1}^{\mathfrak{m}} \beta_{-k} \operatorname{Res}\left\{\frac{1}{s}(s-b)^{-k}e^{-st}; b\right\}. \tag{55}$$

In (55), now expand functions $1/s$ and e^{-st} as Taylor series about b, take the product, and then extract the coefficient for the term $(s-b)^{k-1}$ to get the residue as

$$-\sum_{k=1}^{\mathfrak{m}} \beta_{-k} \frac{(-1)^{k-1}e^{-bt}}{b^k} \sum_{j=0}^{k-1} \frac{(bt)^j}{j!} = \sum_{k=1}^{\mathfrak{m}} \beta_{-k} \frac{(-1)^k}{b^k} S_{\mathrm{G}(k,b)}(t),$$

from the expression for $S_{\mathrm{G}(k,b)}(t)$ in (12).

5.1.5 Multiple poles on the convergence boundary $\{\operatorname{Re}(s) = b\}$

Corollary 7. *Suppose the conditions of Theorem 2 except replace $\mathcal{AC}$ with the condition $\mathcal{AC}_6$ in Corollary 6. Then*

$$S(t) = \sum_{k=1}^{\mathfrak{m}_0} S_{\mathrm{G}(k,b)}(t) \frac{(-1)^k \beta_{-k;b}}{b^k} + 2\sum_{j=1}^{p} \sum_{k=1}^{\mathfrak{m}_j} \operatorname{Re}\left\{ S_{\mathrm{G}(k,b+iy_j)}(t) \frac{(-1)^k \beta_{-k;b+iy_j}}{(b+iy_j)^k} \right\} + o(e^{-b^+ t}) \tag{56}$$

as $t \to \infty$, where $S_{\mathrm{G}(k,b+iy_j)}(t)$ is computed using (12).

The proof is the same as for Corollary 6 with $\mathcal{M}(s)/s$ replacing $\mathcal{M}(s)$ in the argument. The expansion in (56) is easily shown to be the tail area for the expansion given in (53).

5.1.6 Example 6, Noncentral $\chi^2(2m, \lambda)$

In the expression for $\mathcal{M}(s)$, expand $\exp\{\lambda/2(1-2s)^{-1}\}$ about 0. Also expand e^{-st} about $s = 1/2$. This gives

$$-\operatorname{Res}\{\mathcal{M}(s)e^{-st}; 1/2\} = -e^{-\lambda/2} \operatorname{Res}\left\{ \frac{1}{(1-2s)^m} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!(1-2s)^k} \sum_{j=0}^{\infty} \frac{t^j e^{-t/2}(1/2-s)^j}{j!}; 1/2 \right\}.$$

Collecting the terms with power -1 in $s-1/2$ gives the noncentral density in (20). The survival result in (21) has the same derivation.

To show $\mathcal{UI}$ holds for any $m \geq 1$, use the integration-by-parts argument of Example 1 in §5.1.2 to get the upper bound given in (50). The first upper bound term converges to 0 as $Y \to \infty$ since the exponential factor in $\mathcal{M}$ is bounded. For the second term, compute

$$\mathcal{M}'(s) = \mathcal{M}(s) \left\{ \frac{2m}{1-2s} + \frac{\lambda}{(1-2s)^2} \right\}. \tag{57}$$

Substituting $s = 1/2 + \varepsilon + iy$ into (57), then

$$|\mathcal{M}'(b^+ + iy)| \leq \frac{c_1}{(\varepsilon^2 + y^2)^{(m+1)/2}} + \frac{c_2}{(\varepsilon^2 + y^2)^{(m+2)/2}} \tag{58}$$

for some constants $c_1 > 0 < c_2$. The integrability of the upper bound in (58) ensures that $\mathcal{UI}$ holds.

5.1.7 Exact infinite residue expansions

Example 9 (Minus log-beta). The residue at $\alpha + j$ is

$$\beta_{-1;\alpha+j} = \operatorname{Res}\left\{ \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{\Gamma(\alpha-s)}{\Gamma(\alpha+\beta-s)}; \alpha+j \right\} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{(-1)^{j+1}}{j!\Gamma(\beta-j)}$$

so that from (15),

$$\begin{aligned} f_\infty(t) &= \sum_{j=0}^{\infty} e^{-(\alpha+j)t} \left\{ (-1) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} \frac{(-1)^{j+1}}{j!\Gamma(\beta-j)} \right\} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} e^{-\alpha t} \sum_{j=0}^{\infty} (-e^{-t})^j \binom{\beta-1}{j} \\ &= \frac{1}{B(\alpha,\beta)} e^{-\alpha t}(1-e^{-t})^{\beta-1} = f(t). \end{aligned} \tag{59}$$

From (17) and (59), the survival expansion is

$$\begin{aligned} S_\infty(t) &= \frac{e^{-\alpha t}}{B(\alpha,\beta)} \sum_{j=0}^{\infty} \frac{(-e^{-t})^j}{\alpha+j} \binom{\beta-1}{j} = \frac{e^{-\alpha t}}{\alpha B(\alpha,\beta)} \sum_{j=0}^{\infty} \frac{(\alpha)_j (1-\beta)_j}{j!(\alpha+1)_j} (e^{-t})^j \\ &= \frac{e^{-\alpha t}}{\alpha B(\alpha,\beta)} \, {}_2F_1(\alpha, 1-\beta; \alpha+1, e^{-t}), \end{aligned}$$

by the definition of the Gauss series in 15.2.1 of the NIST DLMF. This is the incomplete beta function $P\{\text{Beta}(\alpha,\beta) \leq e^{-t}\}$ expressed in terms of the Gauss hypergeometric function ${}_2F_1$ as in 8.17.7 of the NIST DLMF.

Example 10. (Extreme value (ExV), logistic (Log), and hyperbolic secant(θ)). The first two distributions have MGFs $\Gamma(1-s)$ and $\Gamma(1+s)\Gamma(1-s)$ respectively and both satisfy all parts of Corollary 2. The expressions for $f_\infty(t)$ and $S_\infty(t)$ are given in Table 6. Each distribution has simple poles at $j = 1, 2, \ldots$.

Dist.	$\mathrm{Res}(\mathcal{M}; j)$	$f(t) = f_\infty(t)$
ExV	$(-1)^j/(j-1)!$	$\exp(-t - e^{-t}) = \sum_{j=1}^{\infty} e^{-jt}(-1)^{j+1}/(j-1)!$
Log	$(-1)^j j$	$e^{-t}/(1+e^{-t})^2 = \sum_{j=1}^{\infty} e^{-jt}(-1)^{j+1} j$
		$S(t) = S_\infty(t)$
ExV		$1 - \exp(-e^{-t}) = \sum_{j=1}^{\infty} e^{-jt}(-1)^{j+1}/\{j(j-1)!\}$
Log		$1 - (1+e^{-t})^{-1} = \sum_{j=1}^{\infty} e^{-jt}(-1)^{j+1}$

Table 6. Residues and their expansions $f_\infty(t)$ and $S_\infty(t)$ for the extreme value (ExV) and logistic (Log) distributions.

The hyperbolic secant (θ) distribution (Butler, 2007, §7.1.1) also satisfies all parts of Corollary 2. Here, $\mathcal{M}(s) = \cos(\theta)/\cos(\theta + s)$ for $\theta \in (-\pi/2, \pi/2)$ and has simple poles at $\{(2j-1)\pi/2 - \theta : j = 1, 2, \ldots\}$ with residues $\{(-1)^j \cos\theta\}$. Conditions $\mathcal{AC}_m, \mathcal{X}_m, \mathcal{UI}_m$, and $\mathcal{UI}_m^S$ hold for all m as simple computations show that $|\mathcal{M}(x+iy)| \le \cos\theta/|\sinh y|$. The infinite residue expansion $f_\infty(t)$ for density $f(t)$ is

$$\cos(\theta)e^{\theta t}/\{2\cosh(\pi t/2)\} = \sum_{j=1}^{\infty} \exp\left[-t\{(2j-1)\pi/2 - \theta\}\right](-1)^{j+1}\cos\theta. \tag{60}$$

The infinite expansion $S_\infty(t)$ for $S(t)$ leads to

$$S(t) = \sum_{j=1}^{\infty} \exp\left[-t\{(2j-1)\pi/2 - \theta\}\right] \frac{(-1)^{j+1}\cos\theta}{(2j-1)\pi/2 - \theta}, \tag{61}$$

which also follows directly from integrating (60) and using Fubini's theorem to get (61).

Example 11. (Wilks' likelihood ratio statistic, $\mathcal{M}$ not rational). The factor $\Gamma(11/2 - s)/\Gamma(6-s)$ has an infinite sequence of simple poles at $\{11/2, 13/2, \ldots\}$ which contributes an infinite sum of residues. It is straightforward to show that $R_m(t) \to 0$ as $m \to \infty$ by taking $\varepsilon_m \equiv 1/4$ and using the fact that $|\mathcal{M}(b_m^+ + iy)|$ is uniformly integrable for large m. This follows by long and tedious computations showing that $|\mathcal{M}(b_m^+ + iy)| \le c|y|^{-15/2}$ for $|y| > Y_0$ and some constant $c > 0$, and also

$$\sup_{|y| \le Y_0} |\mathcal{M}(b_m^+ + iy)| \le c_2 \left(b_m^+\right)^{-15/2} \le c_2$$

for some $c_2 = c_2(Y_0) > 0$. Thus,

$$f(t) = f_\infty(t) = f_5(t) + \sum_{j=0}^{\infty} e^{-(11/2+j)t}\mathcal{M}_5(11/2+j)\frac{(-1)(-1)^{j+1}}{j!\Gamma(1/2-j)}$$

$$f(t) = f_5(t) + \frac{1}{\sqrt{\pi}} e^{-(11/2)t} \sum_{j=0}^{\infty} (e^{-t}/4)^j \mathcal{M}_5(11/2+j) \binom{2j}{j}$$
$$= f_5(t) - \frac{672}{\pi} e^{-(11/2)t} {}_4F_3(\tfrac{1}{2}, 1, 1, \tfrac{3}{2}; 3, \tfrac{7}{2}, 4; e^{-t})$$

The exact survival function has expansion

$$S(t) = S_\infty(t) = S_5(t) + \frac{1}{\sqrt{\pi}} e^{-(11/2)t} \sum_{j=0}^{\infty} (e^{-t}/4)^j \mathcal{M}_5(11/2+j) \binom{2j}{j} \frac{1}{11/2+j},$$

which can also be expressed in terms of ${}_4F_3$ hypergeometric functions.

5.1.8 Convolution examples

Example 12. To show that $|\mathcal{M}(b_m^+ + iy)|$ is integrable, write

$$\int_{-\infty}^{\infty} |\mathcal{M}(b_m^+ + iy)| dy = \int_{-\infty}^{\infty} \left| \exp\{(b_m^+ + iy)^2/2\} \Gamma(1 - b_m^+ - iy) \right| dy$$
$$= \exp\{(b_m^+)^2/2\} \int_{-\infty}^{\infty} e^{-y^2/2} |\Gamma(1 - b_m^+ - iy)| dy$$
$$\le \exp\{(b_m^+)^2/2\} \sqrt{2\pi} |\Gamma(1 - b_m^+)|.$$

by 5.6.6 of NIST DLMF.

Example 13. If f_{ExV} and S_{ExV} denote the density and survival function of the extreme value distribution, then the convolution density is

$$f(t) = \int_0^1 f_{\mathrm{ExV}}(t-u) du = S_{\mathrm{ExV}}(t) - S_{\mathrm{ExV}}(t-1) = \sum_{j=1}^{\infty} e^{-jt} \frac{(-1)^{j+1}}{(j-1)!} \frac{e^j - 1}{j} = f_\infty(t) \qquad (62)$$

for all t. Term-by-term integration of (62) provides the convergent expansion for $S(t)$.

Example 14. If $b_m^+ = m + 1/4$, then $|\Gamma(1 - b_m^+ - iy)|$ and $|\Gamma(1 - 2b_m^+ - 2iy)|$ are uniformly integrable for large m. This follows from

$$\left|\Gamma(1 - b_m^+)\right| \to 0 \leftarrow \left|\Gamma(1 - 2b_m^+)\right| \qquad m \to \infty$$

and the upper bound for $|\Gamma(1 - b_m^+ - iy)|$ as a function of y as given in NIST DLMF 5.8.3. Hence, the product is uniformly integrable for large m and the convolution density has a convergent infinite residue expansion.

The MGF $\Gamma(1-s)\Gamma(1-2s)$ has simple poles at $s = 1/2, 3/2, \ldots$ and double poles at $s = 1, 2, \ldots$. Laurent expansions for the two gamma functions at $s = j$ and $j + 1/2$ are derived below as

$$\Gamma(1-s) = \beta_{j;-1}(s-j)^{-1} + \beta_{j;-1}(\gamma - H_{j-1}) + O(s-j) \qquad (63)$$
$$\Gamma(1-2s) = \delta_{j;-1}(s-j)^{-1} + \delta_{j;-1} 2(\gamma - H_{2j-1}) + O(s-j)$$
$$\Gamma(1-2s) = \epsilon_{j+1/2;-1}(s-j-1/2)^{-1} + O(1)$$

where $\beta_{j;-1} = (-1)^j/(j-1)!$, $\delta_{j;-1} = 1/\{2(2j-1)!\}$, and $\epsilon_{j+1/2;-1} = -1/\{2(2j)!\}$ are residues, γ is Euler's constant, and $H_{j-1} = \sum_{k=1}^{j-1} 1/k$ is a Harmonic number. The Laurent coefficients for the MGF of $X+Y$ result from the product of the various expansions in (63). Summing over all simple and double poles leads to the correctly converging infinite expansion

$$\begin{aligned}
f_\infty(t) &= \sum_{j=1}^{\infty} e^{-jt} \{t\beta_{j;-1}\delta_{j;-1} - (\beta_{j;-1}\delta_{j;0} + \beta_{j;0}\delta_{j;-1})\} + \sum_{j=0}^{\infty} e^{-(j+1/2)t}(-1)\epsilon_{j+1/2;-1}\Gamma(1/2-j) \\
&= \sum_{j=1}^{\infty} e^{-jt}\beta_{j;-1}\delta_{j;-1}\{t - 2(\gamma - H_{2j-1}) - (\gamma - H_{j-1})\} - \sum_{j=0}^{\infty} e^{-(j+1/2)t}\epsilon_{j+1/2;-1}\Gamma(1/2-j),
\end{aligned}$$

where $H_0 = 0$.

To determine Laurent expansions for $\Gamma(1-2s)$,

$$\begin{aligned}
\delta_{j;-1} &= \mathrm{Res}\{\Gamma(1-2s); j\} = \lim_{s\to j}(s-j)\Gamma(1-2s) = \lim_{u\to 1-2j} \frac{u-(1-2j)}{-2}\Gamma(u) \\
&= \frac{(-1)^{2j-1}}{-2(2j-1)!} = \frac{1}{2(2j-1)!}.
\end{aligned}$$

The computations of $\beta_{j;-1}$ and $\epsilon_{j+1/2;-1}$ are similar. Thus we write $\Gamma(1-2s) = \delta_{j;-1}/(s-j) + \delta_{j;0} + O(s-j)$ where

$$\begin{aligned}
\delta_{j;0} &= \lim_{s\to j}\{(s-j)\Gamma(1-2s) - \delta_{j;-1}\}/(s-j) \\
&= \lim_{s\to j}\{(s-j)\Gamma(1-2s)\psi(1-2s)(-2) + \Gamma(1-2s)\}
\end{aligned}$$

using l'Hôpital's rule, where $\psi(\cdot)$ is the digamma function. Using the digamma recursion

$$\psi(1-2s) = \psi(2-2s) - \frac{1}{1-2s} = \cdots = \psi(2j-2s) - \sum_{k=1}^{2j-1} \frac{1}{k-2s},$$

this factors to

$$\begin{aligned}
\delta_{j;0} &= \lim_{s\to j}\{(s-j)\Gamma(1-2s)\} 2 \lim_{s\to j}\{-\psi(1-2s) + 1/(2s-2j)\} \\
&= \delta_{j;-1} 2 \lim_{s\to j}\left\{-\psi(2j-2s) + \sum_{k=1}^{2j-1}\frac{1}{k-2s} + \frac{1}{2s-2j}\right\} \\
&= \delta_{j;-1} 2 \lim_{s\to j}\left\{-\psi(2j-2s) + \frac{1}{2s-2j} - H_{2j-1}\right\} \\
&= \delta_{j;-1} 2(\gamma - H_{2j-1}),
\end{aligned} \tag{64}$$

where $H_{2j-1} = \sum_{k=1}^{2j-1} 1/k$ is a Harmonic number. The limit in (64) follows from

$$\lim_{u\to 0}\{\psi(u) + 1/u\} = \lim_{u\to 0}\psi(u+1) = \psi(1) = -\gamma.$$

The computation $\beta_{j;0} = \beta_{j;-1}(\gamma - H_{j-1})$ has a similar derivation.

5.2 Proofs for Section 3

5.2.1 Proof of Theorem 3

The conditions for Theorems 1 and 2 are shown to hold for absolutely continuous random variable R^+, defined as R given that $R \neq 0$. Its MGF $\mathcal{M}_{R^+}(s)$ is given in (29) and, in a neighbourhood of $s = 0$, as well as when $|\mathcal{M}_X(s)| < r$, it has the expansion

$$\mathcal{M}_{R^+}(s) = \frac{1}{1-p(0)} \sum_{k=1}^{\infty} p(k)\mathcal{M}_X(s)^k \qquad \{s \in \mathbb{C} : |\mathcal{M}_X(s)| < r\}. \tag{65}$$

To show that $f_{R^+}(t)$ is locally of bounded variation at $t > 0$, we first invert the right side of (65) term-by-term so that

$$f_{R^+}(t) = \frac{1}{1-p(0)} \sum_{k=1}^{\infty} p(k) f_X^{(*k)}(t) \qquad \text{a.e. } t, \tag{66}$$

where $f_X^{(*k)}$ is the k-fold convolution of f_X. A formal proof that the density of R^+ has this infinite mixture form follows from the argument used in Doetsch (1974, Theorem 30.1) for Laplace transforms but generalised to apply to our bilateral Laplace transforms with support in $(-\infty, \infty)$. It suffices to use the version of $f_{R^+}(t)$ on the right side of (66) to show that $f_{R^+}(t)$ is locally of bounded variation. To do this, we need the following lemma.

Lemma 3. *If density g has total variation $V(g)$ on $(-\infty, \infty)$, then for arbitrary density h, $V\{g * h\} \leq V(g)$.*

Proof. For any partition $P = \{t_0 < t_1 < \cdots < t_n\}$, its variation is

$$\begin{aligned} V_P\{g * h\} &:= \sum_{i=1}^{n} |(g*h)(t_i) - (g*h)(t_{i-1})| \leq \int_{-\infty}^{\infty} h(u) \sum_{i=1}^{n} |g(t_i - u) - g(t_{i-1} - u)| du \\ &= \int_{-\infty}^{\infty} h(u) V_{P-u}\{g\} du \leq \sup_u V_{P-u}(g), \end{aligned}$$

where $P - u$ is the partition $\{t_0 - u < t_1 - u < \cdots < t_n - u\}$. Thus, the total variation of $g * h$ is

$$V\{g * h\} = \sup_P V_P\{g * h\} \leq \sup_P \sup_u V_{P-u}(g) = V(g).$$

□

To show that $f_{R^+}(t)$ is locally of bounded variation, let $V_t(f_X^{(*k)})$ denote the total variation of $f_X^{(*k)}$ in $[t-\delta, t+\delta]$ where, according to condition $\mathcal{BV}^{CD}$, δ can be chosen small enough so that $V_t(f_X^{(*k)}) \leq V_0$ for $k = 1, \ldots q-1$ and some V_0. Using the expansion in (66) along with Lemma 3 with $g = f_X^{(*q)}$, then

$$\begin{aligned} V_t(f_{R^+}) &\leq \frac{1}{1-p(0)} \left[\sum_{k=1}^{q-1} p(k) V_t\{f_X^{(*k)}\} + \sum_{k=q}^{\infty} p(k) V_t\{f_X^{(*k)}\} \right] \\ &\leq \frac{1}{1-p(0)} \{V_0 + V(f_X^{(*q)})\} < \infty \end{aligned}$$

by $\mathcal{BV}^{CD}$. Therefore $f_{R^+}(t)$ is locally of bounded variation for all $t \in S \cap (0, \infty)$.

We now show that condition $\mathcal{AC}$ holds. In Butler (2017, Theorem 5), $\mathcal{P}\{\mathcal{M}_X(s)\}$ (with support $(0, \infty)$) was shown to be analytic on $\{\mathrm{Re}(s) \leq b\}$ apart from an $\mathfrak{m}$-pole at b, however, this is not sufficient to ensure that condition $\mathcal{AC}$ holds in this context. An $\varepsilon_0 > 0$ must exist such that $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic on $\{0 \leq \mathrm{Re}(s) < b + \varepsilon_0\}$ apart from the pole at b. To show this, choose $\eta_1 \in (\varepsilon_1, c - b)$ so that $\mathcal{M}_X(s)$ is analytic on $\{0 \leq \mathrm{Re}(s) \leq b + \eta_1\}$. Then, there exists $Y > 0$ such that

$$\max_{0 \leq x \leq b + \eta_1} |\mathcal{M}_X(x + iy)| < 1 \qquad y > Y.$$

This ensures that $|\mathcal{M}_X(s)| < 1 < r$ for $s \in \{0 \leq \mathrm{Re}(s) \leq b + \eta_1 \cap \mathrm{Im}(s) > Y\} =: A$ so that $\mathcal{M}_X(A)$ lies inside $\{|z| < r\}$ and $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic on A with expansion (65) holding for $s \in A$. Since b is an isolated pole for $\mathcal{P}\{\mathcal{M}_X(s)\}$, there exists $\eta_2 \in (0, \eta_1)$ such that $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic in $D(b, \eta_2)\backslash\{b\}$, an open disc of radius η_2 centered at b but without point b. Now, consider points on the line $\{b + iy : \eta_2/2 \leq y \leq Y + 1\}$. For every $y \in [\eta_2/2, Y + 1]$ there exits $D(b + iy, \eta_y)$ with $\eta_y > 0$ such that $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic on $D(b + iy, \eta_y)$. The argument for this is simply that

$$|\mathcal{M}_X(b + iy)| < \mathcal{M}_X(b) = r \qquad y \neq 0$$

so a sufficiently small $\eta_y > 0$ exists such that $\sup_{s \in D(b+iy,\eta_y)} |\mathcal{M}_X(s)| < r$. Thus, $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic on the open cover $\cup_{y \in [\eta_2/2, Y+1]} D(b + iy, \eta_y)$ for compact set $\{b + iy : \eta_2/2 \leq y \leq Y + 1\}$. Compactness guarantees a finite subcover $\cup_{j=1}^{n} D(b + iy_j, \eta_{y_j})$ with $y_1 < y_2 < \cdots < y_n$. The two circular neighbourhoods associated with contiguous points (b, iy_j) and (b, iy_{j+1}) create a rectangle $[b, b + \lambda_j] \times [iy_j, iy_{j+1}]$ with $\lambda_j > 0$ on which $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic. Thus, if ε_0 is taken to be $\varepsilon_0 = \min\{\eta_2, \lambda_1, \ldots \lambda_{n-1}\}$, then $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic on $\{0 \leq \mathrm{Re}(s) \leq b + \varepsilon_0\}\backslash\{b\}$ and condition $\mathcal{AC}$ holds.

For condition $\mathcal{X}$, take $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ and $b^+ = b + \varepsilon$ so the line $[iN, b^+ + iN] \subset A$, for $N > Y$, is in the convergence domain of $\mathcal{M}_X(s)$ and $\mathcal{M}_X(A)$ is in the convergence circle of $\mathcal{P}$. The Taylor expansion (65) is therefore valid on $[iN, b^+ + iN] \subset [iN, b + \eta_1 + iN]$ so that for sufficiently small $\eta > 0$,

$$\max_{0 \leq x \leq b + \eta_1} |\mathcal{M}_{R^+}(x + iN)| \leq \max_{0 \leq x \leq b + \eta_1} |\mathcal{M}_X(x + iN)| < \eta \qquad N > N_0 \tag{67}$$

and condition $\mathcal{X}$ holds.

We now show that condition $\mathcal{UI}$ holds for the MGF of R^+, i.e. for arbitrarily small $\eta_0 > 0$, there exists $T > 0$ such that

$$\int_{-\infty}^{\infty} \mathcal{M}_{R^+}(b^+ + iy) e^{-iyt} dy < \eta_0 \tag{68}$$

for $t > T$. First note that $\mathcal{M}_{R^+}$ is analytic on $A \cup \bar{A} \cup B$ where $\bar{A}$ is the complex conjugate set of points for A and $B := \{s \in \mathbb{C} : b < \mathrm{Re}(s) < b + \varepsilon_0\}$. Thus, by using Cauchy's theorem, the contour integral of $\mathcal{M}_{R^+}$ in (68) can be deformed into 5 line segments within $A \cup \bar{A} \cup B$. The deformed lines run from $b + \varepsilon_1 - i\infty$ to $b + \varepsilon_1 - iN$ to $b^+ - iN$ to $b^+ + iN$ to $b + \varepsilon_1 + iN$ to $b + \varepsilon_1 + i\infty$. We denote their respective integrals as $I_1 - I_5$. For any $N > Y$,

$$0 < I_3 = \int_{-N}^{N} \mathcal{M}_{R^+}(b^+ + iy) e^{-iyt} dy < \eta_0/6$$

uniformly for $t > T_1(N)$ by the Riemann-Lebesgue lemma. Also, note that the integral for $\mathcal{M}_{R^+}$ in (68) takes the contour integral form

$$\int_{-\infty}^{\infty} \mathcal{M}_{R^+}(b^+ + iy)e^{-iyt}dy = \frac{1}{i}e^{b^+t}\int_{b^+-i\infty}^{b^++i\infty} \mathcal{M}_{R^+}(s)e^{-st}ds. \tag{69}$$

Therefore, deforming the contour integral in (69) to path I_4 leads to

$$|I_4| = \left|\frac{1}{i}e^{b^+t}\int_{b^++iN}^{b+\varepsilon_1+iN} \mathcal{M}_{R^+}(s)e^{-st}ds\right| = e^{b^+t}\left|\int_{b^+}^{b+\varepsilon_1} \mathcal{M}_{R^+}(x+iN)e^{-(x+iN)t}dx\right|.$$

Using (67) with $\eta = \eta_0/6$, then

$$|I_4| \leq \frac{\eta_0}{6}e^{b^+t}\int_{b^+}^{b+\varepsilon_1} |e^{-(x+iN)t}|dx = \frac{\eta_0}{6t}\{1 - e^{-(\varepsilon_1-\varepsilon)t}\} < \frac{\eta_0}{6t}$$

for $N > N_1$. Likewise, $|I_2| = |I_4| < \eta_0/(6t)$. Finally,

$$\begin{aligned} I_1 + I_5 &= \frac{1}{i}e^{b^+t}\left(\int_{b+\varepsilon_1-i\infty}^{b+\varepsilon_1-iN} + \int_{b+\varepsilon_1+iN}^{b+\varepsilon_1+i\infty}\right)\mathcal{M}_{R^+}(s)e^{-st}ds \\ &= e^{-(\varepsilon_1-\varepsilon)t}\left(\int_{-\infty}^{-N} + \int_{N}^{\infty}\right)\mathcal{M}_{R^+}(b+\varepsilon_1+iy)e^{-ity}dy. \end{aligned} \tag{70}$$

The contours are in $A \cup \bar{A}$ so expansion (65) holds. We separate the first $p-1$ terms of this expansion from the remainder so

$$\mathcal{M}_{R^+}(s) = \mathcal{M}_p(s) + \mathcal{R}(s) := \frac{1}{1-p(0)}\sum_{k=1}^{p-1} p(k)\mathcal{M}_X(s)^k + \frac{1}{1-p(0)}\sum_{k=p}^{\infty} p(k)\mathcal{M}_X(s)^k. \tag{71}$$

Since $|\mathcal{R}(s)| \leq |\mathcal{M}_X(s)|^p$ in $A \cup \bar{A}$, the magnitude of the contribution of $\mathcal{R}(s)$ to $I_1 + I_5$ in (70) is at most

$$e^{-(\varepsilon_1-\varepsilon)t}2\int_N^{\infty} |\mathcal{M}_X(b+\varepsilon_1+iy)|^p dy < \eta_0/6$$

for $N > N_2$ by assumption $\mathcal{AI}_p^{CD}$. Integration of $\mathcal{M}_p(s)$, the first term in (71), contributes the following terms:

$$\begin{aligned} &e^{-(\varepsilon_1-\varepsilon)t}\frac{1}{1-p(0)}\left(\int_{-\infty}^{\infty} - \int_{-N}^{N}\right)\sum_{k=1}^{p-1} p(k)\mathcal{M}_X(b+\varepsilon_1+iy)^k e^{-ity}dy \\ &= e^{-(\varepsilon_1-\varepsilon)t}\frac{1}{1-p(0)}\left\{2\pi\sum_{k=1}^{p-1} p(k)e^{(b+\varepsilon_1)t}f_X^{(*k)}(t) - \int_{-N}^{N} \mathcal{M}_p(b+\varepsilon_1+iy)e^{-ity}dy\right\}. \end{aligned} \tag{72}$$

The last equality, which for each k inverts the characteristic function $\mathcal{M}_X(b+\varepsilon_1+iy)^k$ for $e^{(b+\varepsilon_1)t}f_X^{(*k)}(t)$, requires that each member of $\{f_X^{(*k)}(t) : k = 1, \ldots, p-1\}$ be locally of bounded variation. By assumption $\mathcal{BV}^{CD}$, this holds directly if $q \geq p-1$; if $q < p-1$, then this holds as a consequence of assumption $\mathcal{BV}^{CD}$ combined with Lemma 3. The integral term (times its leading factor) in (72) can be made $< \eta_0/6$ for $t > T_2(N)$ by the Riemann-Lebesgue lemma. For terms in the summation of (72), the uniform integrability assumption in $\mathcal{UI}^{CD}$ ensures that

$2\pi\exp\{(b+\varepsilon_1)t\}f_X^{(*k)}(t) < \eta_0/\{6(p-1)\}$ for $t > T_{3(k)}$ by the same argument used to prove Theorem 1 in §5.1.1. In this argument, uniform integrability for $\mathcal{M}_X^k$ ensures the inequality

$$2\pi e^{(b+\varepsilon_1)t}f_X^{(*k)}(t) = \int_{-\infty}^{\infty}\mathcal{M}_X(b+\varepsilon_1+iy)^k e^{-ity}dy < \eta_0/\{6(p-1)\} \qquad t > T_{3(k)}, \tag{73}$$

while the equality in (73) holds since $b+\varepsilon_1$ is in the convergence domain of $\mathcal{M}_X^k$. Now take $N > \max\{Y, N_1, N_2\}$ and $T > \max\{T_1(N), T_2(N), \max_k T_{3(k)}\}$ so all inequalities hold and $\int_{-\infty}^{\infty}\mathcal{M}_{R^+}(b+\varepsilon_1+iy)e^{-ity}dy < \eta_0$ for $t > T$ as required for condition $\mathcal{UI}$ to hold. All conditions of Theorem 1 are now satisfied.

To show that condition $\mathcal{UI}^S$ of Theorem 2 holds for the survival function expansion of R^+, we use the same argument and deform the inversion integral to the same five lines. The only term in these five that requires comment is $I_1 + I_5$. This integral is as given in (70) but with the additional integrand factor $(b+\varepsilon_1+iy)^{-1}$ so

$$I_1 + I_5 = e^{-(\varepsilon_1-\varepsilon)t}\left(\int_{-\infty}^{-N} + \int_N^{\infty}\right)\frac{\mathcal{M}_{R^+}(b+\varepsilon_1+iy)}{b+\varepsilon_1+iy}e^{-ity}dy.$$

Using Hölder's inequality,

$$\begin{aligned} I_1 + I_5 &\le e^{-(\varepsilon_1-\varepsilon)t}2\int_N^{\infty}\frac{|\mathcal{M}_{R^+}(b+\varepsilon_1+iy)|}{|b+\varepsilon_1+iy|}dy \\ &\le 2\left\{\int_N^{\infty}|\mathcal{M}_X(b+\varepsilon_1+iy)|^p\,dy\right\}^{1/p} \times \left\{\int_N^{\infty}\frac{1}{|b+\varepsilon_1+iy|^q}dy\right\}^{1/q}, \end{aligned} \tag{74}$$

where $1/p+1/q=1$. By condition $\mathcal{AI}_p^{CD}$, this can be made small for sufficiently large N. Thus, condition $\mathcal{UI}^S$ of Theorem 2 holds.

The computation of Laurent coefficients for $\mathcal{P}\{\mathcal{M}_X(s)\}$ begins by taking the Laurent expansion of $\mathcal{P}(z)$ at r and setting $z = \mathcal{M}_X(s)$ so that

$$\mathcal{P}\{\mathcal{M}_X(s)\} = \sum_{i=1}^{\mathfrak{m}}\frac{\rho_{-i}}{\{\mathcal{M}_X(s)-r\}^i} + O(1) = \sum_{i=1}^{\mathfrak{m}}\frac{\rho_{-i}}{\mathcal{N}(s)^i}(s-b)^{-i} + O(1), \tag{75}$$

where $\mathcal{N}(s) = \{\mathcal{M}_X(s)-r\}/(s-b)$ for $s \ne b$ and $\mathcal{N}(b) = \mathcal{M}_X'(b)$. Now substitute Taylor expansion

$$\frac{1}{\mathcal{N}(s)^i} = \sum_{j=0}^{i-1}\frac{1}{j!}\frac{d^j}{ds^j}\left.\frac{1}{\mathcal{N}(s)^i}\right|_{s=b}(s-b)^j + O(s-b)^i$$

into (75), rearrange terms and substitute $k = i-j$ so that

$$\mathcal{P}\{\mathcal{M}_X(s)\} = \sum_{k=1}^{\mathfrak{m}}\sum_{j=0}^{\mathfrak{m}-k}\frac{\rho_{-k-j}}{j!}\frac{d^j}{ds^j}\left.\frac{1}{\mathcal{N}(s)^{k+j}}\right|_{s=b}(s-b)^{-k}. \tag{76}$$

The kth coefficient of (76) is as given in (27). The derivatives in (76) are evaluated by expanding

$$\mathcal{N}(s) = \frac{\mathcal{M}_X(s)-\mathcal{M}_X(b)}{s-b} = \sum_{k=1}^{\infty}\frac{\mathcal{M}_X^{(k)}(b)}{k!}(s-b)^{k-1}$$

so that

$$\mathcal{N}^{(j)}(b) = \frac{\mathcal{M}_X^{(j+1)}(b)}{j+1}.$$

The expansions actually use the Laurent coefficients for the MGF of R^+ denoted as $\{\beta_{-k}^+\}$ rather than those for MGF of R denoted as $\{\beta_{-k}\}$. They are related by $\beta_{-k}^+ = \beta_{-k}/\{1-p(0)\}$. The densities are also related in the opposite way since for $t \neq 0$,

$$\begin{aligned} f_R(t) = \{1-p(0)\}f_{R^+}(t) &= \{1-p(0)\}e^{-bt}\sum_{k=1}^{\mathfrak{m}} t^{k-1}\frac{(-1)^k\beta_{-k}^+}{(k-1)!} + o(e^{-(b+\varepsilon)t}) \\ &= e^{-bt}\sum_{k=1}^{\mathfrak{m}} t^{k-1}\frac{(-1)^k\beta_{-k}}{(k-1)!} + o(e^{-(b+\varepsilon)t}), \end{aligned}$$

as given in (26) of Theorem 3.

5.2.2 Precise asymptotic orders for expansions errors in Theorem 3

With additional assumptions on the compound distribution MGF $\mathcal{P}\{\mathcal{M}_X(s)\}$, the expansions errors in Theorem 3 will hold for larger values of ε that those stated in the theorem. The proof of Theorem 3 requires that $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$, where $\varepsilon_0 > 0$ is a value constructed so that $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytically extendible from $\{\mathrm{Re}(s) < b\}$ to $\{\mathrm{Re}(s) \leq b+\varepsilon_0\}\backslash\{b\}$. By making additional assumptions about the extent to which $\mathcal{P}\{\mathcal{M}_X(s)\}$ can be analytically extended into $\{\mathrm{Re}(s) \geq b\}$ and correspondingly increasing the value of ε_1 in conditions $\mathcal{AI}_p^{CD}$ and $\mathcal{UI}^{CD}$, then expansions (26) and (28) will hold for larger values of ε and perhaps for $b+\varepsilon > c$ in the analytic continuation of $\mathcal{M}_X$. We summarise this in the next corollary based on the following revised conditions.

($\mathcal{AC}_8$) *Suppose* $\mathcal{M}_{R^+}(s)$ *can be analytically continued to* $\{\mathrm{Re}(s) < b+\varepsilon_0\}\backslash\{b\}$ *where perhaps* $b+\varepsilon_0 > c$.

($\mathcal{AI}_8^p$) *There exists an* $\varepsilon_1 \geq \varepsilon_0$ *and integer* $p \geq 1$ *such that* $|\mathcal{M}_X(b+\varepsilon_1+iy)|^p$ *is integrable in* y. *If* $b+\varepsilon_0 > c$, *then also assume* $\max_{c\leq x\leq b+\varepsilon_1}|\mathcal{M}_X(x+iN)| \to 0$ *as* $N\to\infty$.

Corollary 8. *Let* $\mathcal{P}(z)$ *and* $\mathcal{M}_X(s)$ *be as described before Theorem 3 for compound distribution* $\mathcal{P}\{\mathcal{M}_X(s)\}$.
(a) Under conditions $\mathcal{AC}_8$, $\mathcal{AI}_8^1$ *with* $p=1$, *and* $\mathcal{BV}^{CD}$ *(from Theorem 3), the density expansion (26) holds for any* $\varepsilon \in (0,\varepsilon_0)$.
(b) Under conditions $\mathcal{AC}_8$ and $\mathcal{AI}_8^p$, *survival expansion (28) holds for any* $\varepsilon \in (0,\varepsilon_0)$.

Proof. The results follow from the same arguments used in Theorem 3. We use the same deformed contour path and only comment on those aspects of the proof that differ from the last subsection. The latter parts of conditions $\mathcal{AI}_8^1$ and $\mathcal{AI}_8^p$ are needed to ensure that set $A = \{0 \leq \mathrm{Re}(s) \leq b+\varepsilon_1 \cap \mathrm{Im}(s) > Y\}$ is such that $\mathcal{M}_X(A) \subset \{\mathrm{Re}(z) < r\}$. When condition $\mathcal{AC}_8$ is paired with either $\mathcal{AI}_8^1$ or $\mathcal{AI}_8^p$, then Y may be chosen so that $|\mathcal{M}_X(s)| < 1 < r$ for $s \in A$ and this ensures that $\mathcal{P}\{\mathcal{M}_X(s)\}$ is analytic on $s \in A$ and also that $|\mathcal{M}_{R^+}(b+\varepsilon_1+iy)| \leq |\mathcal{M}_X(b+\varepsilon_1+iy)|$ for $y > Y$.

The arguments differ only for showing that I_1+I_5 in (70) is arbitrarily small. For $N > Y$ the contour for I_1+I_5 is in $A \cup \bar{A}$ so, by condition $\mathcal{AI}_8^1$ and the inequality $|\mathcal{M}_{R^+}(b+\varepsilon_1+iy)| \leq |\mathcal{M}_X(b+\varepsilon_1+iy)|$,

$$I_1 + I_5 < e^{-(\varepsilon_1-\varepsilon)t}\left(\int_{-\infty}^{-N} + \int_N^{\infty}\right)|\mathcal{M}_X(b+\varepsilon_1+iy)|dy < \eta_0/6 \qquad N > \max(Y, N_1).$$

Condition $\mathcal{AI}_8^p$ ensures that Hölder's inequality can be used for the same argument in (74) concerning inversion for the survival function.

5.2.3 Proof of Corollary 3

We show that the conditions of Corollary 1 hold for R^+ by broadly following the proof used in Theorem 3. Assumption $\mathcal{AC}_m^{CD}$ ensures that condition $\mathcal{AC}_m$ holds for R^+. Using the latter part of assumption $\mathcal{AI}_{mp}^{CD}$, we take Y to be such that

$$\max_{0\le x\le b_m^+} |\mathcal{M}_X(x+iN)| < 1 \qquad N > Y.$$

Then $A = \{s \in \mathbb{C} : 0 \le \mathrm{Re}(s) \le b_m^+ \cap \mathrm{Im}(s) > Y\}$ is in the convergence domain of $\mathcal{P}$ so that expansion (65) holds for $\mathcal{M}_{R^+}$ in $A \cup \bar{A}$. Therefore, if $N > Y$, then by expansion (65),

$$\max_{0\le x\le b_m^+} |\mathcal{M}_{R^+}(x+iN)| < \max_{0\le x\le b_m^+} |\mathcal{M}_X(x+iN)| \to 0 \qquad N \to \infty \tag{77}$$

so condition $\mathcal{X}_m$ holds.

To show $\mathcal{UI}_m$, partition $\int_{-\infty}^{\infty} \mathcal{M}_{R^+}(b_m^+ + iy)e^{-ity}dy$ into three disjoint parts over $\int_{-\infty}^{-N} + \int_{-N}^{N} + \int_N^{\infty} =: I_1 + I_2 + I_3$ for sufficiently large $N > Y$. For small $\eta > 0$, $I_2 < \eta/3$ for $t > T_1(N)$ by the Riemann-Lebesgue lemma. If $p = 1$ and $N > Y$, then by assumption $\mathcal{AI}_{mp}^{CD}$ with $p = 1$, there exists N_1 such that

$$I_1 + I_3 \le 2\int_N^{\infty} |\mathcal{M}_X(b_m^+ + iy)|dy < 2\eta/3$$

for $N > \max\{Y, N_1\}$. For $p \ge 2$, we use the expansion (71) with $N > Y$ so that

$$I_1 + I_3 = \left(\int_{-\infty}^{-N} + \int_N^{\infty}\right)\left\{\mathcal{M}_p(b_m^+ + iy) + \mathcal{R}(b_m^+ + iy)\right\} e^{-ity}dy =: J_1 + J_2,$$

where J_1 integrates $\mathcal{M}_p(b_m^+ + iy)$ and J_2 integrates $\mathcal{R}(b_m^+ + iy)$. Since $\mathcal{M}_p(b_m^+ + iy)$ is a linear combination of $\mathcal{M}_X(b_m^+ + iy)^k$ for $k = 1, \dots p-1$, then by assumption $\mathcal{UI}_m^{CD}$,

$$J_1 < \int_{-\infty}^{\infty} \mathcal{M}_p(b_m^+ + iy)e^{-ity}dy < \eta/3$$

for $t > T_2(N)$. Using the expansion for $\mathcal{R}(b_m^+ + iy)$ and since $N > Y$, then

$$J_2 \le 2\int_N^{\infty} |\mathcal{R}(b_m^+ + iy)|dy \le 2\int_N^{\infty} |\mathcal{M}_X(b_m^+ + iy)|^p dy < \eta/3$$

for $N > \max\{Y, N_2\}$ by assumption $\mathcal{AI}_m^{CD}$. Take $N > \max\{Y, N_1, N_2\}$ and $T > \max\{T_1(N), T_2(N)\}$ so $\int_{-\infty}^{\infty} \mathcal{M}_{R^+}(b_m^+ + iy)e^{-ity}dy < \eta$.

To show $\mathcal{UI}_m^S$, break the integral into the following three pieces

$$\int_{-\infty}^{\infty} \frac{\mathcal{M}_{R^+}(b_m^+ + iy)}{b_m^+ + iy}e^{-ity}dy = \left(\int_{-\infty}^{-N} + \int_{-N}^{N} + \int_N^{\infty}\right)\frac{\mathcal{M}_{R^+}(b_m^+ + iy)}{b_m^+ + iy}e^{-ity}dy =: I_1 + I_2 + I_3.$$

Then $I_2 < \eta/2$ for $t > T_1(N)$ by the Riemann-Lebesgue lemma. For $p = 1$ and with $N > Y$, then

$$I_1 + I_3 \leq 2 \int_N^\infty \frac{|\mathcal{M}_{R^+}(b_m^+ + iy)|}{|b_m^+ + iy|} dy \leq 2 \int_N^\infty \frac{|\mathcal{M}_X(b_m^+ + iy)|}{|b_m^+ + iy|} dy < \eta/2$$

for $N > \max\{Y, N_3\}$. For $p \geq 2$ and using the expansion (65) with $N > Y$, then

$$\begin{aligned} I_1 + I_3 &\leq 2 \int_N^\infty \frac{|\mathcal{M}_{R^+}(b_m^+ + iy)|}{|b_m^+ + iy|} dy \leq 2 \int_N^\infty \frac{|\mathcal{M}_X(b_m^+ + iy)|}{|b_m^+ + iy|} dy \\ &\leq 2 \left\{ \int_N^\infty \left|\mathcal{M}_X(b_m^+ + iy)\right|^p dy \right\}^{1/p} \left\{ \int_{-\infty}^\infty |b_m^+ + iy|^{-q} dy \right\}^{1/q}, \end{aligned}$$

for $1/p + 1/q = 1$ by Hölder's inequality. By assumption $\mathcal{AI}_{mp}^{CD}$, $I_1 + I_3 < \eta/2$ for $N > N_4$. For $N > \max\{Y, N_3, N_4\}$ and $t > T_1(N)$, then $I_1 + I_2 + I_3 < \eta$.

5.3 Expansions in the left tail

The expansions of Theorems 1 and 2 apply only to the right tail of the density or survival function for random variable X with support on $(-\infty, \infty)$. In the left tail, the asymptotics for these theorems are not justified. However, many of the infinite residue expansions, including those in Examples 9, 10, and 11 converge in both tails.

Expansions in the left tail are easily addressed in the abstract, since the left tail of X is the right tail of $-X$, and so all previous results apply. However, taking such an approach creates substantial notational problems when the final results are to be stated in terms of left-tail asymptotics for $f(t)$ and $F(t)$ as $t \to -\infty$. It is easier to deal directly with the left tail as considered in Theorems 7 and 8 below. These theorems provide single residue density and distribution function expansions $f_{L1}(t)$ and $F_1(t)$ which by Cauchy's theorem relate to their true counterparts as $f(t) = f_{L1}(t) + L_1(t)$ and $F(t) = F_1(t) - L_1^F(t)$, where L_1 and $-L_1^F$ are integral expressions for the errors.

Expansions in the left tail depend on poles at the boundary $a < 0$ and in the analytic continuation $\{\mathrm{Re}(s) < a\}$. Thus, if $\mathcal{M}$ admits a sequence of decreasing real or complex conjugate pairs of poles at $a = a_1 > a_2 > \cdots$ or $-a = |a_1| < |a_{2\pm}| > \cdots$, then higher-order and infinite expansions as in Corollaries 1 and 2 can be formulated to generalise Theorems 7 and 8 in an obvious manner.

Theorem 7. (Left tail density expansions). *Suppose absolutely continuous X has density $f(t)$ on $S \subseteq (-\infty, \infty)$, which is locally of bounded variation for all $t < 0$. Let $\mathcal{M}$ have convergence boundary $\{s \in \mathbb{C} : \mathrm{Re}(s) = a\}$ on the left with $-\infty < a < 0$. Subject to conditions $\mathcal{AC}^L$, $\mathcal{X}^L$, and $\mathcal{UI}^L$ below, then*

$$f(t) = f_{L1}(t) + L_1(t) := -e^{-at} \sum_{k=1}^{\mathfrak{m}} t^{k-1} \frac{(-1)^k \alpha_{-k}}{(k-1)!} + L_1(t),$$

where $\sum_{k=1}^{\mathfrak{m}} \alpha_{-k}(s-a)^{-k}$ is the principal part of the Laurent expansion for $\mathcal{M}$ at a, and

$$L_1(t) = e^{-a^- t} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{M}(a^- + iy) e^{-iyt} dy = o(e^{-a^- t}) \qquad t \to -\infty, \tag{78}$$

with $a^- = a - \varepsilon$.

($\mathcal{AC}^L$) *There exists* $\varepsilon_0 > 0$ *such that* $\mathcal{M}$ *can be analytically extended across the boundary* $\{\text{Re}(s) = a\}$ *to* $\{a - \varepsilon_0 < \text{Re}(s) \leq a\}$, *save from an* $\mathfrak{m}$*-pole at* a.

($\mathcal{X}^L$) *For some* $a^- = a - \varepsilon$ with $\varepsilon \in (0, \varepsilon_0)$, $\max_{a^- \leq x \leq a} |\mathcal{M}(x + iN)| \to 0$ as $N \to \infty$.

($\mathcal{UI}^L$) *The principal-value integral* $\int_{-\infty}^{+\infty} \mathcal{M}(a^- + iy)e^{-iyt}dy$ *converges uniformly in* t *for* $t \leq -T < 0$.

Proof. The proof uses Cauchy's theorem as in Theorem 1 but rather integrates anti-clockwise round the rectangle with corners $a^- \pm Ni$ and $a^+ \pm Ni$, with $a^+ = a + \varepsilon$. This results in the identity $f(t) = \text{Res}\{\mathcal{M}(s)e^{-st}; b\} + L_1(t)$ without the negative sign in front of the residue. This accounts for the extra minus sign in the expression for $f_{L1}(t)$. □

Theorem 8. (Left tail CDF expansions). *Suppose absolutely continuous* X *with support in* $(-\infty, \infty)$ *has distribution function* $F(t)$ *and* $\mathcal{M}$ *has convergence boundary* $\{s \in \mathbb{C} : \text{Re}(s) = a\}$ *with* $-\infty < a < 0$. *Subject to conditions* $\mathcal{AC}^L$ *and* $\mathcal{X}^L$ *of Theorem 7 as well as* $\mathcal{UI}^{LF}$ *below,*

$$F(t) = F_1(t) - L_1^F(t) := \sum_{k=1}^{\mathfrak{m}} S_{\text{G}(k,a)}(t)\frac{(-1)^k \alpha_{-k}}{a^k} - L_1^F(t), \tag{79}$$

$$L_1^F(t) = e^{-a^- t}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\mathcal{M}(a^- + iy)}{a^- + iy}e^{-iyt}dy = o(e^{-a^- t}) \qquad t \to -\infty, \tag{80}$$

where $S_{\text{G}(k,a)}(t) = S_{\text{G}(k,-a)}(-t)$ *is evaluated using the survival function expression in (12).*

($\mathcal{UI}^{LF}$) *For some* $T > 0$, *the principal-value integral* $\int_{-\infty}^{+\infty} \mathcal{M}(a^-+iy)/(a^-+iy)e^{-iyt}dy$ *converges uniformly in* t *for* $t \leq -T < 0$.

Proof. The proof is the same as used in Theorem 2 but applied using the anti-clockwise integration of Theorem 7 in the negative half plane $\{\text{Re}(s) < 0\}$. Such integration round the rectangle with corners $a^- \pm Ni$ and $a^+ \pm Ni$ gives

$$\frac{1}{2\pi i}\int_{a^+ - i\infty}^{a^+ + i\infty} \frac{1}{s}\mathcal{M}(s)e^{-st}ds = \text{Res}\left\{\frac{1}{s}\mathcal{M}(s)e^{-st}; a\right\} + \frac{1}{2\pi i}\int_{a^- - i\infty}^{a^- + i\infty} \frac{1}{s}\mathcal{M}(s)e^{-st}ds. \tag{81}$$

Since $\{\text{Re}(s) = a^+\}$ is in the convergence domain of $\mathcal{M}$ but $a^+ < 0$, the integral on the left is $-F(t)$; see Widder (1946, Theorem 5b, p. 242). The integral on the right is $L_1^F(t)$ as in (80). Thus, (81) reduces to $F(t) = -\text{Res}\left\{s^{-1}\mathcal{M}(s)e^{-st}; a\right\} - L_1^F(t)$. The expression for the residue follows the same derivation as used in Theorem 2. □

5.3.1 Numerical example

Example 22. (Extreme value claims, as in Example 17, left tail). Consider the case $\mathfrak{m} = 1$. The left tail of $\mathcal{M}_{R^+}(s)$ has a simple pole at $a_1 = -2$, and a pair of complex conjugate simple poles at $a_{2\pm} = -3.539 \pm 4.100i$. Using this pole structure, we approximate the density and distribution function at $t = -0.65$ and -1 as shown in Table 7. In all instances, the expansions using all three poles are most accurate. At $t = -0.65$, the SP\ saddlepoint methods which remove the point mass at 0 are more accurate than the first-order expansions. This accuracy gets reversed at $t = -1$ and further into the left tail.

To approximate the expansion errors in each computation of f_1 and F_1, saddlepoints are located in between a_1 and the complex conjugate poles at $a_{2\pm}$. For example, consider the computation of $f_1(-0.65)$. Saddlepoints for the computation of $\hat{L}_1(-0.65)$ are $\hat{s}_{2\pm} = -3.235 \pm$

$2.693i$. This computation requires adding the contributions from each of these saddlepoints which themselves compute as complex conjugates and add to -0.03583 as seen in the table. With a complex saddlepoint, the steepest descent direction for the upper saddlepoint $\hat{s}_{2+}$ is computed as $\pi/2 - \text{Arg}\{\mathcal{K}''_{R^+}(\hat{s}_{2+})\}/2 = 0.3689 = 21.1°$; see Murray (1984, p. 47). Perpendicular to this direction, or at 111.1° is the direction of steepest ascent which points roughly in the direction of the pole a_{2+} with a bearing of 102.2° from $\hat{s}_{2+}$. The geometry of the lower saddlepoint $\hat{s}_{2-}$ and its relationship to pole a_{2-} are the mirror image of this.

	$t=-0.65$	$t=-1$		$t=-0.65$	$t=-1$
f	0.1050172	0.0719123	F	0.0693469	0.0382398
f_{L1}	0.**1**477	0.0**7**333	F_1	0.0**7**383	0.0**3**666
$f_{L1}+\hat{L}_1$	0.1**1**19	0.0**7**123	$F_1-\hat{L}_1^F$	0.0**7**028	0.03**8**13
f_{L2}	0.0**9**557	0.0**7**051	F_2	0.0**67**170	0.03**8**40
f_{SP}	**0**.1679	0.**0**8275	F_{SP}	0.**0**8888	0.03**7**52
$f_{\text{SP}\backslash}$	0.**0**830	0.**0**5439	$F_{\text{SP}\backslash}$	0.0**6**770	0.0**4**069
$\hat{L}_1{}^a$	-0.03583	$-0.0^2 1034$	$-\hat{L}_1^{F\,a}$	$-0.0^2 3557$	$0.0^2 1465$
L_1	-0.04265	$-0.0^2 1417$	$-L_1^F$	$-0.0^2 4487$	$0.0^2 1575$

Table 7. Various density approximations (f_{L1} and f_{L2}) and distribution approximations (F_1 and F_2) using residue expansion terms about the poles a_1 and $\{a_1, a_{2\pm}\}$ respectively. See Table 3 in §3.2 for a description of other entries. [a] Saddlepoint approximations for the true errors L_1 and $-L_1^F$ given in (78) and (80).

When $\hat{L}_1$ is used to improve the approximation, then $f_{L1} + \hat{L}_1 = 0.1119$ is better than all other approximations. Likewise, $F_1 - \hat{L}_1^F = 0.07028$ is most accurate. For $t = -1$, $f_{L1} + \hat{L}_1 = 0.07123$ is best as is also $F_1 - \hat{L}_1^F = 0.03813$. Saddlepoint error corrections for second-order expansions f_{L2} and F_2 were not computed since saddlepoints could not be found to the left of $a_{2\pm}$.

5.4 Proofs for Section 4

5.4.1 Proof of Theorem 4

The conditions of Theorems 1 and 2 need to be satisfied. Conditions $\mathcal{AC}$ and $\mathcal{X}$ hold because $\mathcal{M}_{R^+}$ is analytic on $\{\text{Re}(s) < b + \varepsilon_0\}$ for some $\varepsilon_0 > 0$ as was shown in the proof of Theorem 3.

To show density f_{R^+} is locally of bounded variation, it suffices to show that density f_{L^+} satisfies the bounded variation assumptions of condition $\mathcal{BV}^{CD}$. This leads to local bounded variation for density f_{R^+} as described in the proof of Theorem 3. The total variation of f_{L^+} is

$$\int_{0^-}^{\infty} |df_{L^+}(t)| \leq \int_{0^-}^{\infty} e^{(b+\varepsilon_1)t} |df_{L^+}(t)| < \infty$$

since the integral on the right is absolutely integrable by condition $\mathcal{AC}^{SA}$. Thus, density f_{L^+} satisfies the bounded variation condition $\mathcal{BV}^{CD}$ with $q = 1$.

To show condition $\mathcal{UI}$, we let $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ and $b^+ = b + \varepsilon$ and proceed to show that

$$\left(\int_{-\infty}^{-N} + \int_{-N}^{N} + \int_{N}^{\infty} \right) \mathcal{M}_{R^+}(b^+ + iy) e^{-iyt} dy =: I_1 + I_2 + I_3 \tag{82}$$

is uniformly small for sufficiently large $t > T$. For arbitrarily small $\eta_0 > 0$, $I_2 < \eta_0/4$ for $t > T_1(N)$ by the Riemann-Lebesgue lemma. For $I_1 + I_3$, we take $N > Y$ where Y is sufficiently large so that

$$\max_{0 \le x \le b+\varepsilon_1} |\mathcal{M}_{L^+}(x+iy)| < 1 \qquad y > Y.$$

Now use the expansion for $\mathcal{M}_{R^+}(b^+ + iy)$ given in (71) with $p = 2$ so that

$$I_1 + I_3 = \left(\int_{-\infty}^{-N} + \int_N^{\infty}\right)\{\mathcal{M}_2(b^+ + iy) + \mathcal{R}(b^+ + iy)\}e^{-iyt}dy, \tag{83}$$

where $\mathcal{M}_2(s) = p(1)\mathcal{M}_{L^+}(s)/\{1 - p(0)\}$. For the first component, b^+ is in the convergence domain for $\mathcal{M}_{L^+}$ so that the integral from $-\infty$ to ∞ of $\mathcal{M}_2(b^+ + iy)$ is the inversion integral for $e^{b^+t} f_{L^+}(t)$; thus its contribution to $I_1 + I_3$ is

$$\frac{p(1)}{1 - p(0)}\left\{2\pi e^{b^+t} f_{L^+}(t) - \int_{-N}^{N} \mathcal{M}_{L^+}(b^+ + iy)\}e^{-iyt}dy\right\}. \tag{84}$$

For any $N > 0$, the last integral in (84) times its leading factor has magnitude $< \eta_0/4$ for $t > T_2(N)$ by the Riemann-Lebesgue lemma.

To show that the tilted density in (84) must converge to 0 as $t \to \infty$, we use condition $\mathcal{AC}^{SA}$ which ensures that the Laplace-Stieltjes integral

$$\mathcal{M}_D(s) = \int_{0-}^{\infty} e^{st} df_{L^+}(t)$$

is absolutely convergent at $s = b + \varepsilon_1 > b^+$. Using Widder (1946, Theorem 2.2b, p. 40), $f_{L^+}(\infty)$ exists and $f_{L^+}(t) - f_{L^+}(\infty) = o(e^{-(b+\varepsilon_1)t})$. Since f_{L^+} is integrable, then $f_{L^+}(\infty) = 0$ and $f_{L^+}(t) = o(e^{-(b+\varepsilon_1)t})$ so that $e^{b^+t} f_{L^+}(t) \to 0$. Thus the tilted density factor in (84) is $< \eta_0/4$ for $t > T_3$.

For the second integral in (83) involving $\mathcal{R}$, define

$$\mathcal{M}_E(s) = \int_{0-}^{\infty} e^{st} d\{e^{b^+t} f_{L^+}(t)\} = b^+\mathcal{M}_{L^+}(b^+ + s) + \mathcal{M}_D(b^+ + s), \tag{85}$$

which, by condition $\mathcal{AC}^{SA}$, is absolutely convergent for $\mathrm{Re}(s) \le \varepsilon_1 - \varepsilon > 0$. Then, using integration by parts,

$$\begin{aligned}\mathcal{M}_E(s) &= \lim_{N\to\infty}\left\{ e^{(b^+ + s)t} f_{L^+}(t)\Big|_0^N - s\int_0^N e^{(b^+ + s)t} f_{L^+}(t)dt\right\}\\ &= -f_{L^+}(0) - s\mathcal{M}_{L^+}(b^+ + s) \qquad \mathrm{Re}(s) \le \varepsilon_1 - \varepsilon,\end{aligned} \tag{86}$$

since $f_{L^+}(N) = o(e^{-(b+\varepsilon_1)N})$. From (86),

$$\mathcal{M}_{L^+}(b^+ + s) = \frac{-1}{s}\{\mathcal{M}_E(s) + f_{L^+}(0)\} \qquad \mathrm{Re}(s) \le \varepsilon_1 - \varepsilon. \tag{87}$$

Using (85),

$$\begin{aligned}|\mathcal{M}_E(iy)| &\le b^+|\mathcal{M}_{L^+}(b^+ + iy)| + |\mathcal{M}_D(b^+ + iy)|\\ &\le b^+|\mathcal{M}_{L^+}(b^+)| + \int_{0-}^{\infty} e^{b^+t}|df_{L^+}(t)|.\end{aligned}$$

Therefore, from (87), $|\mathcal{M}_{L^+}(b^+ + iy)| < d/|y|$ with

$$d = b^+ \mathcal{M}_{L^+}(b^+) + \int_{0^-}^{\infty} e^{b^+ t} |df_{L^+}(t)| + f_{L^+}(0) < \infty.$$

This leads to the upper bound

$$|\mathcal{R}(b^+ + iy)| \le \frac{1}{1 - p(0)} \sum_{k=2}^{\infty} p(k) \frac{d^k}{|y|^k} < \frac{d^2}{y^2} \qquad |y| > \max(Y, d),$$

so that

$$\left(\int_{-\infty}^{-N} + \int_{N}^{\infty} \right) \mathcal{R}(b^+ + iy) e^{-iyt} dy \le 2 \int_{N}^{\infty} \frac{d^2}{y^2} dy = 2d^2/N < \eta_0/4 \qquad N > N_1,$$

where $N_1 = \max(Y, d, 8d^2/\eta_0)$. Take $N > N_1$ and $T > \max\{T_1(N), T_2(N), T_3\}$ so $I_1 + I_2 + I_3 < \eta_0$ for $t > T$. This proves uniform integrability for $\mathcal{M}_{R^+}(b^+ + iy)e^{-ity}$ in y for $t > T$ so that condition $\mathcal{UI}$ holds.

We now show condition $\mathcal{UI}^S$ of Theorem 2 holds so that $\int_{-\infty}^{\infty} \mathcal{M}_{R^+}(b^+ + iy)(b^+ + iy)^{-1} e^{-ity} dy$ is uniformly integrable for $t > T$ when $b^+ = b + \varepsilon$ and $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$. The proof is the same as that used in Theorem 3 to show the same condition. The contour $\{\mathrm{Re}(s) = b^+\}$ is deformed into the same five lines used in Theorem 3 which are $b + \varepsilon_1 - i\infty$ to $b + \varepsilon_1 - iN$ to $b^+ - iN$ to $b^+ + iN$ to $b + \varepsilon_1 + iN$ to $b + \varepsilon_1 + i\infty$. The moduli for the integrals along the middle three legs are small by the same arguments used in Theorem 3. The moduli of the integrals for the first and last legs are the same so only the last leg requires comment. On the last leg, set $b_1^+ = b + \varepsilon_1$ so that $s = b_1^+ + iy$ for $y \ge N$ on this leg. Taking $N > Y$ as used in expansion (65), then $|\mathcal{M}_{L^+}(s)| < 1$ so that

$$|\mathcal{M}_{R^+}(s)|^p = \left| \frac{e^{-B}}{1 - (1 - e^{-B})\mathcal{M}_{L^+}(s)} \mathcal{M}_{L^+}(s) \right|^p \le |\mathcal{M}_{L^+}(s)|^p. \tag{88}$$

Thus, by (88) and Hölder's inequality,

$$\left| \int_N^{\infty} \frac{\mathcal{M}_{R^+}(b_1^+ + iy)}{b_1^+ + iy} e^{-iyt} dy \right| \le \left\{ \int_N^{\infty} |\mathcal{M}_{L^+}(b_1^+ + iy)|^p dy \right\}^{1/p} \left(\int_N^{\infty} \frac{1}{|b_1^+ + iy|^q} dy \right)^{1/q} \tag{89}$$

where $1/p + 1/q = 1$. Assumption $\mathcal{AI}_p^{SA}$ ensures that (89) can be make uniformly small in t for large N.

Assumption $\mathcal{AC}^{SA}$ also suffices for showing condition $\mathcal{UI}^S$ of Theorem 2. Again, only the last of the five legs in the deformation of the contour integral needs comment. Using expansion (71) with $p = 2$ so that $\mathcal{M}_2(s) = p(1)\mathcal{M}_{L^+}(s)/\{1 - p(0)\}$, and also $|\mathcal{M}_{L^+}(b^+ + iy)| < d/|y|$, then for $N > \min\{Y, d\}$

$$\begin{aligned} \left| \int_N^{\infty} \frac{\mathcal{M}_{R^+}(b^+ + iy)}{b^+ + iy} e^{-iyt} dy \right| &\le \int_N^{\infty} \left| \frac{\mathcal{M}_2(b^+ + iy)}{b^+ + iy} \right| dy + \int_N^{\infty} \left| \frac{\mathcal{R}(b^+ + iy)}{b^+ + iy} \right| dy \\ &< \frac{1}{1 - p(0)} \left(p(1) \int_N^{\infty} \frac{d}{y^2} dy + \int_N^{\infty} \frac{d^2}{y^3} dy \right) \to 0 \qquad N \to \infty. \end{aligned}$$

5.4.2 Proof of Corollary 4

Here, $f_{L^+}(t) = S_X(t)/\mu$ and so $e^{(b+\varepsilon_1)t} f'_{L^+}(t) = -e^{(b+\varepsilon_1)t} f_X(t)/\mu$ is absolutely integrable for all $0 < \varepsilon_1 < c - b$. Thus, condition $\mathcal{AC}^{SA}$ of Theorem 4 holds.

The excess life MGF is $\mathcal{M}_{L^+}(s) = \{1 - \mathcal{M}_X(s)\}/(-\mu s)$. Taking $0 < \varepsilon_1 < c - b$, then $|\mathcal{M}_X(b + \varepsilon_1 + iy)| < 1$ for $y > Y$ so that $|\mathcal{M}_{L^+}(b + \varepsilon_1 + iy)|^p \le 2^p/|\mu y|^p$. For any $p > 1$, this is integrable so condition $\mathcal{AI}_p{}^{SA}$ holds. We can also easily show that condition $\mathcal{UI}^S$ of Theorem 2 holds. With $0 < \varepsilon_1 < c - b$ and for $N > Y$,

$$\left| \int_N^\infty \frac{1 - \mathcal{M}_X(b + \varepsilon_1 + iy)}{-\mu(b + \varepsilon_1 + iy)^2} e^{-iyt} dy \right| \le \int_N^\infty \frac{2}{\mu y^2} dy = \frac{2}{\mu N} < \eta_0$$

for $N > 2/(\mu\eta_0)$ for all $t > 0$. Thus uniform integrability holds.

5.4.3 Orders for expansions errors in Theorem 4 and Corollary 4

The following conditions are used to extend the asymptotic orders for errors in these expansions.

($\mathcal{AC}_9$) *For some $\varepsilon_0 \in (0, c - b)$, suppose $\mathcal{M}_{R^+}(s)$ can be analytically continued to $\{\mathrm{Re}(s) < b + \varepsilon_0\} \backslash \{b\}$. Also, for some $\varepsilon_1 \in [\varepsilon_0, c - b)$, suppose $\int_{0-}^\infty \exp\{(b + \varepsilon_1)t\} df_{L^+}(t)$ is absolutely convergent.*

($\mathcal{AI}_{p9}$) *For some $\varepsilon_0 > 0$ (not necessarily bounded by $c - b$), suppose $\mathcal{M}_{R^+}(s)$ can be analytically continued to $\{\mathrm{Re}(s) < b + \varepsilon_0\} \backslash \{b\}$. For some $\varepsilon_1 \ge \varepsilon_0$, suppose $|\mathcal{M}_{L^+}(b + \varepsilon_1 + iy)|^p$ is integrable in y for some integer $p > 1$ and, if $\varepsilon_1 \ge c - b$, then $\max_{c \le x \le b + \varepsilon_1} |\mathcal{M}_{L^+}(x + iN)| \to 0$ as $N \to \infty$.*

Corollary 9. (Sparre Andersen model). *Reconsider Theorem 4 with $\mathcal{AC}_9$ and $\mathcal{AI}_{p9}$ used in place of $\mathcal{AC}^{SA}$ and $\mathcal{AI}_p^{SA}$ respectively.*

Assuming $\mathcal{AC}_9$, the density expansion (35) holds to order $o(e^{-(b+\varepsilon)t})$ for any $\varepsilon < \varepsilon_0$.

Assuming $\mathcal{AI}_{p9}$, the survival expansion (36) holds to order $o(e^{-(b+\varepsilon)t})$ for any $\varepsilon < \varepsilon_0$.

Proof. The proof is the same as used in Theorem 4. For the density expansions, these proofs restrict $\varepsilon_1 < c - b$ but no such restrictions apply to the survival expansions. □

In the Cramér-Lundberg model, if $\mathcal{M}_{R^+}(s)$ can be analytically extended to $\{\mathrm{Re}(s) < b + \varepsilon_0\} \backslash \{b\}$ for $\varepsilon_0 < c - b$, then the expansions for both the density in (37) and the survival function in (38) hold to order $o(e^{-(b+\varepsilon)t})$ for any $\varepsilon < \varepsilon_0$. If the analytic extension extends to $\varepsilon_0 \ge c - b$, then the density expansion (37) still holds to lower order $o(e^{-(b+\varepsilon)t})$ for any $\varepsilon < c - b$ while the survival function expansion is valid to the higher order $o(e^{-(b+\varepsilon)t})$ for $\varepsilon < \varepsilon_0$ under the condition that $\mathcal{AI}_{p9}$ holds.

5.4.4 Proof of Corollary 5

The proof follows by showing that the conditions of Corollary 3 hold. Condition $\mathcal{AC}_m^{CD}$ holds by assumption. Condition $\mathcal{X}_m^{CL}$ ensures that $\mathcal{M}_{L^+}(s) = \{1 - \mathcal{M}_X(s)\}/(-\mu s)$ satisfies condition $\mathcal{AI}_{mp}^{CD}$ in Corollary 3. Since $f_{L^+}(t) = S_X(t)/\mu$ has bounded variation, so does $f_{L^+}^{(*k)}(t)$ for $k \ge 2$ by Lemma 3 §5.2.1; hence $\mathcal{BV}^{CD}$ holds.

In the Cramér-Lundberg context, condition $\mathcal{AI}_{mp}^{CD}$ implies $\mathcal{UI}_m^{CD}$ as we now show. To show uniform integrability for $k = 1$ in $\mathcal{UI}_m^{CD}$, use $\mathcal{M}_{L^+}(s) = \{1 - \mathcal{M}_X(s)\}/(-\mu s)$ and break the integral into

$$\int_{-\infty}^{\infty} \mathcal{M}_{L^+}(b_m^+ + iy)e^{-iyt}dy = \int_{-\infty}^{\infty} \left\{ \frac{1}{-\mu(b_m^+ + iy)} + \frac{\mathcal{M}_X(b_m^+ + iy)}{\mu(b_m^+ + iy)} \right\} e^{-iyt}dy =: I_1 + I_2. \quad (90)$$

Integral I_1 is uniformly integrable for large t and I_2 is absolutely integrable after Hölder's inequality is used in conjunction with assumption $\mathcal{AI}_{mp}^{CD}$. Thus the integral in (90) is uniformly integrable. To show uniform integrability for $k \geq 2$, note that all terms in $\int_{-\infty}^{\infty} \mathcal{M}_{L^+}(b_m^+ + iy)^k e^{-iyt}dy$ are absolutely integrable. For example, when $k = 2$, then $\mathcal{M}_{L^+}(b_m^+ + iy)^2 = O(y^{-2})$ so the integral is absolutely integrable.

The expressions in (40) are derived in the same manner as the first-order approximations in (37) and (38). From Corollary 3, the density approximation is

$$f_R(t) = \sum_{j=1}^{m} e^{-b_j t} \frac{-\rho_{-1}}{\mathcal{M}'_{L^+}(b_j)}, \quad (91)$$

where $-\rho_{-1} = (1 - \rho)/\rho$ is the negative residue from the geometric PGF. Straightforward differentiation of $\mathcal{M}_{L^+}(s)$ at b_j gives

$$\mathcal{M}'_{L^+}(b_j) = \frac{1}{\mu b_j^2} \left\{ b_j \mathcal{M}'_X(b_j) + 1 - \mathcal{M}_X(b_j) \right\}. \quad (92)$$

However, since b_j is a zero of the denominator of $\mathcal{M}_R(s)$, then $\mathcal{M}_{L^+}(b_j) = 1/\rho$ so that $1 - \mathcal{M}_X(b_j)$ can be replaced by $-\mu b_j/\rho$ and

$$\mathcal{M}'_{L^+}(b_j) = \frac{1}{\mu b_j} \left\{ \mathcal{M}'_X(b_j) - \sigma/\lambda \right\}.$$

Substitution into (91) and additional simplification leads to (40).

5.4.5 Proof of Theorem 5

Numerical inversions of the ruin density and survival functions using the vertical contour $\{\mathrm{Re}(s) = x\}$ in (34) are plagued by heavy tails as a result of the form for the excess life distribution, with MGF $\mathcal{M}_{L^+}(s) = \{1 - \mathcal{M}_X(s)\}/(-\mu s)$. Taking $s = x + iy$, then as $y \to \infty$, the term $\mathcal{M}_X(x + iy) = o(1)$ but also $1/(-\mu s) = -\mu^{-1}(x + iy)^{-1}$ so that when inverting to compute $f_R(t)$,

$$\begin{aligned} \mathrm{Re}\{\mathcal{M}_{R^+}(x+iy)e^{-t(x+iy)}\} &= \mathrm{Re}\left\{ \frac{(1-\rho)\mathcal{M}_{L^+}(x+iy)}{1-\rho\mathcal{M}_{L^+}(x+iy)} e^{-t(x+iy)} \right\} \\ &\sim (1-\rho)e^{-xt}\,\mathrm{Re}\left\{ \frac{-\mu^{-1}(x+iy)^{-1}}{1+\rho\mu^{-1}(x+iy)^{-1}} e^{-ity} \right\} \\ &\sim -\frac{1-\rho}{\mu} e^{-xt}\,\mathrm{Re}\left(\frac{e^{-ity}}{x+iy} \right) \\ &\sim \frac{1-\rho}{\mu} e^{-xt} \left\{ \frac{y\sin(ty) - x\cos(ty)}{x^2+y^2} \right\} \\ &\sim \frac{(1-\rho)}{\mu} e^{-xt}\, \frac{\sin(ty)}{y} \qquad y \to \infty. \end{aligned} \quad (93)$$

Using the comparison theorem for integrals, then the left-hand side of (93) is not absolutely integrable. The argument is that if the left-hand side is absolutely integrable, then so is the right-hand side, which is a contradiction.

The corresponding integrand for determining $S_R(t)$ has the additional factor $(x+iy)^{-1}$ so it has order

$$-\frac{(1-\rho)e^{-xt}}{\mu}\operatorname{Re}\left\{\frac{e^{-ity}}{(x+iy)^2}\right\} \sim \frac{(1-\rho)e^{-xt}}{\mu}\frac{\cos(ty)}{y^2} \qquad y\to\infty. \tag{94}$$

By the comparison test, the left-hand side of (94) is absolutely integrable.

Under the conditions of Corollary 5, the same asymptotic orders apply when x is replaced by $x\in[b_m, b_m+\varepsilon)$ to compute mth order error terms $R_m(t)$ and $R_m^S(t)$ for the higher-order expansions $f_m(t)$ and $S_m(t)$.

The order of the integrand for density inversion differs slightly from the inversion integral used by Abate and Whitt (1992). Instead of (34), they used the inversion integral

$$f_{R^+}(t) \approx 2e^{-xt}\frac{1}{\pi}\int_0^N \operatorname{Re}\{\mathcal{M}_{R^+}(x+iy)\}\cos(ty)dy \tag{95}$$

with $x=0$, which applies only to densities on $(0,\infty)$ so it is less general than (34). Using $\operatorname{Re}\{\mathcal{M}_{R^+}(x+iy)\} \sim -(1-\rho)\operatorname{Re}\{\mu^{-1}(x+iy)^{-1}\} = O(y^{-2})$, then the integrand in (95) has order $O\{y^{-2}\cos(ty)\}$ as $y\to\infty$ so the integral converges absolutely but still slowly.

5.4.6 Proof of Theorem 6

To alleviate the inversion difficulties for both density and survival, we determine the direction for ultimate steepest descent as the value $\theta\in[0,\pi/2]$ that minimises $|\mathcal{M}_{R^+}(re^{i\theta})e^{-it(re^{i\theta})}|$ using various distance values such as $r=30$ and 60. Now, instead of integrating from $\hat{s}_0 - i\infty$ to $\hat{s}_0 + i\infty$ in (34), we integrate along the 3 lines from $\hat{s}_0 - i\pi + e^{-i\theta}\infty$ to $\hat{s}_0 - i\pi$ to $\hat{s}_0 + i\pi$ to $\hat{s}_0 + i\pi + e^{i\theta}\infty$ so

$$f_{R^+}(t) = \frac{1}{2\pi i}\left(\int_{\hat{s}_0 - i\pi + e^{-i\theta}\infty}^{\hat{s}_0 - i\pi} + \int_{\hat{s}_0 - i\pi}^{\hat{s}_0 + i\pi} + \int_{\hat{s}_0 + i\pi}^{\hat{s}_0 + i\pi + e^{i\theta}\infty}\right)\mathcal{M}_{R^+}(s)e^{-ts}ds = I_1 + I_2 + I_3.$$

Using $\mathcal{M}_{R^+}(\bar{s}) = \overline{\mathcal{M}_{R^+}(s)}$, the middle integral reduces to integration from $\hat{s}_0$ to $\hat{s}_0 + i\pi$ of the form

$$I_2 = \frac{1}{\pi}\int_0^\pi \operatorname{Re}\{\mathcal{M}_{R^+}(\hat{s}_0+iy)e^{-t(\hat{s}_0+iy)}\}dy,$$

which is (46). Likewise the first and third integrals are along complex conjugate curves and can be combined to integrate from $s_\pi = \hat{s}_0 + i\pi$ to $s_\pi + e^{i\theta}\infty$. Denote $\mathcal{N}(s) = \mathcal{M}_{R^+}(s)\exp(-ts)$ and write their sum as

$$\begin{aligned} I_1 + I_3 &= \frac{1}{2\pi i}\left\{\int_0^\infty \mathcal{N}(s_\pi + re^{i\theta})e^{i\theta}dr + \int_\infty^0 \mathcal{N}(\bar{s}_\pi + re^{-i\theta})e^{-i\theta}dr\right\} \\ &= \frac{1}{2\pi i}\int_0^\infty \left\{\mathcal{N}(s_\pi + re^{i\theta})e^{i\theta} - \overline{\mathcal{N}(s_\pi + re^{i\theta})e^{i\theta}}\right\}dr \\ &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left\{\mathcal{N}(s_\pi + re^{i\theta})e^{i\theta}\right\}dr. \end{aligned}$$

This gives expression (47).

It remains to show that integration from s_π to $s_\pi + i\infty$ can be deformed to the ray from s_π to $s_\pi + e^{i\theta}\infty$. This follows from Cauchy's deformation theorem by showing that the integral over the following arc is arbitrarily small for large N :

$$A = \int_{s_\pi + Ne^{i\theta}}^{s_\pi + Ne^{i\pi/2}} \mathcal{N}(s)ds = \int_\theta^{\pi/2} \mathcal{M}_{R^+}(s_\pi + Ne^{i\phi}) \exp(-s_\pi t - tNe^{i\phi}) Nie^{i\phi} d\phi.$$

Using the assumptions of Theorem 6, for arbitrarily small $\eta_0 > 0$,

$$\max_{\theta \le \phi \le \pi/2} |\mathcal{M}_{R^+}(s_\pi + Ne^{i\phi})| \le \eta_0 t/\pi \qquad N \ge N(\eta_0). \tag{96}$$

Then,

$$|A| \le \frac{\eta_0 t}{\pi} N e^{-\hat{s}_0 t} \int_\theta^{\pi/2} \exp(-Nt\cos\phi) d\phi.$$

Over $\phi \in [0, \pi/2]$, $\cos\phi \ge 1 - 2\phi/\pi$ so that

$$\begin{aligned} |A| &\le \frac{\eta_0 t}{\pi} N e^{-\hat{s}_0 t} \int_0^{\pi/2} \exp\{-Nt(1 - 2\phi/\pi)\} d\phi \\ &= \frac{\eta_0 t}{\pi} N e^{-\hat{s}_0 t} \frac{\pi}{2Nt}(1 - e^{-Nt}) = \frac{\eta_0}{2} e^{-\hat{s}_0 t}(1 - e^{-Nt}) < \eta_0. \end{aligned}$$

Sufficient conditions for (96) to hold are:

$$\begin{array}{lll} \max_{0 \le \phi \le \pi/2} |\mathcal{M}_{R^+}(s_\pi + Ne^{i\phi})| \to 0 & \text{if} & \theta = 0 \\ \max_{\theta \le \phi \le \pi/2} |\mathcal{M}_{R^+}(Ne^{i\phi})| \to 0 & \text{if} & \theta \in (0, \mathrm{Arg}(s_\pi)] \\ \max_{\theta^- \le \phi \le \pi/2} |\mathcal{M}_{R^+}(Ne^{i\phi})| \to 0 & \text{if} & \theta \in (\mathrm{Arg}(s_\pi), \pi/2) \end{array} \qquad N \to \infty, \tag{97}$$

for some $\theta^- \in (0, \theta)$.

5.4.7 Raleigh distribution

A Raleigh distribution with mean 1 has MGF

$$\mathcal{M}_X(s) = 1 + se^{s^2/\pi}\{\mathrm{erf}(s/\sqrt{\pi}) + 1\} = 1 + se^{s^2/\pi}\{2 - \mathrm{erfc}(s/\sqrt{\pi})\},$$

where $\mathrm{erfc}(t) = 1 - \mathrm{erf}(t)$. Using expansion 7.12.1 of NIST DLMF for $\mathrm{erfc}(t)$, we determine that

$$\mathcal{M}_X(s) = 2se^{s^2/\pi} + \pi/(2s^2) + O(s^{-4}) \qquad |\,\mathrm{Arg}(s)| < 3\pi/4$$

so that

$$\mathcal{M}_{L^+}(s) = \frac{1 - \mathcal{M}_X(s)}{-s} = 2e^{s^2/\pi} - 1/s + \pi/(2s^3) + O(s^{-5}). \tag{98}$$

Taking $s = \hat{s}_0 + iy$, then

$$\mathcal{M}_{R^+}(\hat{s}_0 + iy) = \frac{(1-\rho)\mathcal{M}_{L^+}(s)}{1 - \rho\mathcal{M}_{L^+}(s)} = -\frac{1-\rho}{\hat{s}_0 + iy} + O(y^{-2}) \qquad y \to \infty. \tag{99}$$

Therefore, the integrand for density inversion is

$$\frac{e^{-\hat{s}_0 t}}{\pi} \mathrm{Re}\{\mathcal{M}_{R^+}(\hat{s}_0 + iy)e^{-ity}\} \sim \frac{e^{-\hat{s}_0 t}}{\pi} \mathrm{Re}\left\{-\frac{1-\rho}{\hat{s}_0 + iy}\right\} \sim \frac{(1-\rho)e^{-\hat{s}_0 t}}{\pi}\frac{\sin(ty)}{y} \qquad y \to \infty,$$

as given in (44).

To determine the direction for the path of ultimate steepest descent, we first note from (99) that $\mathcal{M}_{R^+}(\hat{s}_0 + re^{i\theta}) \approx (1-\rho)\mathcal{M}_{L^+}(\hat{s}_0 + re^{i\theta})$ for large r. From (98),

$$\mathcal{M}_{L^+}(\hat{s}_0 + re^{i\theta}) \approx 2\exp\{(\hat{s}_0 + re^{i\theta})^2/\pi\} - 1/(\hat{s}_0 + re^{i\theta}), \tag{100}$$

so determining this direction involves trading off the sum of the two components in (100). The steepest descent direction for the modulus of the first term is $\theta = \pi/2$ so that $\exp\{(re^{i\theta})^2\} = \exp(-r^2)$ while the second term has the same steepest descent direction as the MGF of an exponential density which is $\theta = 0$. The first term is slightly more dominating as reflected in the overall steepest descent direction $\theta = 9\pi/32 \approx 50.6°$ as determined directly from $\mathcal{M}_{R^+}(\hat{s}_0 + re^{i\theta})$. In this setting $\theta < \text{Arg}(s_\pi) = 81.4°$. For $\theta > 45°$, (97) and (96) hold so the conditions for Theorem 6 hold.

5.4.8 Truncated extreme value distribution

The MGF is

$$\begin{aligned}\mathcal{M}_X(s) &= \frac{1}{1-e^{-1}} \int_0^\infty \exp(st - t - e^{-t})dt \\ &= \frac{1}{1-e^{-1}} \int_0^1 e^{-u}u^{-s}du \end{aligned} \tag{101}$$

upon substituting $u = e^{-t}$. From 13.4.1 of NIST DLMF, this is

$$\mathcal{M}_X(s) = \frac{1}{1-e^{-1}} \frac{\Gamma(1-s)}{\Gamma(2-s)}\, {}_1F_1(1-s; 2-s; -1)$$

which reduces to (43).

To show that $\max_{0\leq\phi\leq\pi/2} |\mathcal{M}_{R^+}(\pi i + Ne^{i\phi})| \to 0$ as $N \to \infty$ as required in Theorem 6, first use the Kummer transformation from 13.2.39 of NIST DLMF in (102) below, followed by the expansion for ${}_1F_1$ given in 13.8.2 of NIST DLMF in (103). With $s = \pi i + Ne^{i\phi}$, these give

$$\begin{aligned}{}_1F_1(1-s; 2-s; -1) &= e^{-1}\, {}_1F_1(1; 2-s; 1) & (102)\\ &= \frac{\Gamma(2-s)}{\Gamma(1-s)} \left\{(2-s)^{-1} + O(2-s)^{-2}\right\} \to 1 & (103)\end{aligned}$$

as $N \to \infty$ for $0 \leq \phi \leq \pi/2$. Therefore $\max_{0\leq\phi\leq\pi/2} |\mathcal{M}_X(\pi i + Ne^{i\phi})| \leq c_1/N$ and $\max_{0\leq\phi\leq\pi/2} |\mathcal{M}_{L^+}(\pi i + Ne^{i\phi})| \leq c_2/N$ so that

$$|\mathcal{M}_{R^+}(\pi i + Ne^{i\phi})| = \frac{(1-\rho)|\mathcal{M}_{L^+}(\pi i + Ne^{i\phi})|}{|1 - \rho\mathcal{M}_{L^+}(\pi i + Ne^{i\phi})|} \leq \frac{(1-\rho)|\mathcal{M}_{L^+}(\pi i + Ne^{i\phi})|}{|1 - \rho|\mathcal{M}_{L^+}(\pi i + Ne^{i\phi})||}.$$

Thus,

$$\max_{0\leq\phi\leq\pi/2} |\mathcal{M}_{R^+}(\pi i + Ne^{i\phi})| \leq \frac{(1-\rho)c_2/N}{1-\rho c_2/N} \to 0 \qquad N \to \infty.$$