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SUPPLEMENTARY MATERIAL: TAIL ASYMPTOTICS OF AN INFINITELY DIVISIBLE SPACE-TIME MODEL WITH CONVOLUTION EQUIVALENT LÉVY MEASURE

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SM1. Proofs of Section 3

Proof of Lemma 3.1. For sufficiently large x we find that

$$\begin{aligned}\mathbb{P}(Z\phi(U,S) > x) &= \frac{1}{\nu(A)} F\Big(\Big\{(u,s,z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+ \,:\, z\phi(u,s) > x\Big\}\Big)\\ &= \frac{1}{\nu(A)} \int_{B'\times T'} L\Big(\frac{x}{c}\Big) \exp\Big(-\beta \frac{x}{c}\Big) m(\mathrm{d}u, \mathrm{d}s)\\ &\quad + \frac{1}{\nu(A)} \int_{(B'\times T')^c} L\Big(\frac{x}{\phi(u,s)}\Big) \exp\Big(-\beta \frac{x}{\phi(u,s)}\Big) m(\mathrm{d}u, \mathrm{d}s),\end{aligned}$$

where the first term equals $L(x/c)\exp(-\beta x/c)$ times the desired limit. The result follows when the latter integral is shown to be of order $o(L(x/c)\exp(-\beta x/c))$, as $x \to \infty$. Let $h(u,s;x)$ denote the integrand. For all $(u,s) \notin B'\times T'$ we have $\phi(u,s) < c$. Combined with (2.4), this implies the existence of $\gamma > 0$ and $C > 0$ such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \leq C\exp(-\gamma x)$$

for sufficiently large x. Thus, the integrand $h(u,s;x)$ is $o(L(x/c)\exp(-\beta x/c))$ at infinity. By dominated convergence, the integral is of order $o(L(x/c)\exp(-\beta x/c))$ if we can find an integrable function $g : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \leq g(u,s)$$

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for all $(u,s) \in \mathbb{R}^d \times \mathbb{R}$. Returning to (2.5) we see that for all $0 < \gamma < \beta/c$ there is $C > 0$ and x_0 such that

$$\frac{h(u,s;x)}{L(x/c)\exp(-\beta x/c)} \leq C \exp\Big(-x_0(\beta - \gamma c)\Big(\frac{1}{\phi(u,s)} - \frac{1}{c}\Big)\Big) \tag{SM1.1}$$

for all $x \geq x_0$. Independent of (u,s) we can find a finite constant $\tilde{C}$ such that the right hand side of (SM1.1) is bounded by $\tilde{C}\phi(u,s)$, which is integrable by assumption. This shows the desired order of convergence.

From [4, Lemma 2.4(i)] the distribution of $Z\phi(U,S)$ is convolution equivalent with index β/c. The integrability result follows from [4, Corollary 2.1(ii)]. □

Corollary SM1.1. *If $V^1, V^2, \dots$ are i.i.d. fields with distribution ν_1, then*

$$\mathbb{E}\Big[\exp\Big(\beta \sup_{u\in B} \sup_{s\in[0,T]} \lambda_{u,s}\left((V^1_{v,t} + \cdots + V^n_{v,t})_{(v,t)}\right)\Big)\Big] < \infty$$

for all $n \in \mathbb{N}$.

Proof. Because each V^i can be represented by $(Z^i f(|v - U^i|, t - S^i)_{(v,t)\in B'\times T'}$, the result follows from (3.8) and (3.10). □

Proof of Theorem 3.2. We will show the claim by induction over n: We note that the case $n = 1$ follows easily from Theorem 3.1. Now assume that the result holds true for some $n \in \mathbb{N}$ and let for convenience $V^{*n} = V^1 + \cdots + V^n$. Also, let $y^* = \sup_{(v,t)\in B'\times T'} y_{v,t}$. Using (3.7) and the representation $V^i = Z^i f(|v - U^i|, t - S^i)$, we find

$$\begin{aligned}
&\mathbb{P}(\Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x) \\
&\quad\leq \mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) > \frac{x-y^*}{2}\,, Z^{n+1}\phi(U^{n+1},S^{n+1}) > \frac{x-y^*}{2}\,, \\
&\qquad\quad \Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x\Big) \\
&\qquad + \mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) \leq \frac{x-y^*}{2}\,, \Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x\Big) \\
&\qquad + \mathbb{P}\Big(Z^{n+1}\phi(U^{n+1},S^{n+1}) \leq \frac{x-y^*}{2}\,, \Psi(V^{*n}_{v,t} + V^{n+1}_{v,t} + y_{v,t}) > x\Big)
\end{aligned}\;. \tag{SM1.2}$$

The first term in (SM1.2) is bounded from above by

$$\mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) > \frac{x-y^*}{2}\Big)\mathbb{P}\Big(Z^{n+1}\phi(U^{n+1},S^{n+1}) > \frac{x-y^*}{2}\Big).$$

In Lemma 3.1 we showed that the distribution of $Z^i\phi(U^i,S^i)$ is convolution equivalent with index β/c, and hence, from [3, Corollary 2.11] and (3.9), both factors are asymptotically equivalent to $\rho_1((x/(2c),\infty))$ as $x\to\infty$. Following the proof of [2, Lemma 2] we see that the product is $o((\rho_1*\rho_1)((x/c,\infty)))$, and as such the first term in (SM1.2) is $o(\rho_1((x/c,\infty)))$ due to the convolution equivalence of ρ_1. By Theorem 3.1 it is of order $o(\mathbb{P}(\Psi(V^1_{v,t})>x))$ as $x\to\infty$.

By independence, the two remaining terms in (SM1.2) divided by $\mathbb{P}(\Psi(V^1_{v,t})>x)$ are

$$\begin{aligned}&\int_{C_x}\frac{\mathbb{P}(\Psi(\sum_{i=1}^n z^i f(|v-u^i|,t-s^i)+V^{n+1}_{v,t}+y_{v,t})>x)}{\mathbb{P}(\Psi(V^1_{v,t})>x)}\\&\qquad\times F_1^{\otimes n}(\mathrm{d}(u^1,s^1,z^1;\dots;u^n,s^n,z^n))\\&+\int_{\tilde{C}_x}\frac{\mathbb{P}(\Psi(V^{*n}_{v,t}+z^1 f(|v-u^1|,t-s^1)+y_{v,t})>x)}{\mathbb{P}(\Psi(V^1_{v,t})>x)}F_1(\mathrm{d}(u^1,s^1,z^1)),\end{aligned}\tag{SM1.3}$$

where $F_1^{\otimes n}$ is the n-fold product measure of F_1 and

$$\begin{aligned}C_x&=\Big\{(u^1,s^1,z^1;\dots;u^n,s^n,z^n)\,:\,\sum_{i=1}^n z^i\phi(u^i,s^i)\le\frac{x-y^*}{2}\Big\},\\\tilde{C}_x&=\Big\{(u^1,s^1,z^1)\,:\,z^1\phi(u^1,s^1)\le\frac{x-y^*}{2}\Big\}.\end{aligned}$$

Above we used the representation $V^i=Z^if(|v-U^i|,t-S^i)$ again. By Theorem 3.1 and the induction assumption, the integrands of (SM1.3) have the following limits as $x\to\infty$,

$$\begin{aligned}&f_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n)\\&=\frac{\int_B\int_0^T\exp\left(\beta\lambda_{u,s}\left(\sum_{i=1}^n z^if(|v-u^i|,t-s^i)+y_{v,t}\right)\right)\mathrm{d}s\mathrm{d}u}{m(B\times[0,T])},\\&f_2(u^1,s^1,z^1)\\&=\frac{n\int_B\int_0^T\mathbb{E}\left[\exp\left(\beta\lambda_{u,s}\left(V^1_{v,t}+\cdots+V^{n-1}_{v,t}+z^1f(|v-u^1|,t-s^1)+y_{v,t}\right)\right)\right]\mathrm{d}s\mathrm{d}u}{m(B\times[0,T])},\end{aligned}$$

respectively. When integrated with respect to the relevant measures we find

$$\begin{aligned}&\int f_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n)F_1^{\otimes n}(\mathrm{d}(u^1,s^1,z^1;\dots;u^n,s^n,z^n))\\&\qquad+\int f_2(u^1,s^1,z^1)F_1(\mathrm{d}(u^1,s^1,z^1))\\&=\frac{n+1}{m(B\times[0,T])}\int_B\int_0^T\mathbb{E}\left[\exp\left(\beta\lambda_{u,s}\left(V^1_{v,t}+\cdots+V^n_{v,t}+y_{v,t}\right)\right)\right]\mathrm{d}s\mathrm{d}u,\end{aligned}$$

which is the desired expression. To show convergence of the integrals in (SM1.3), using Fatou's lemma, it suffices to find integrable functions $g_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n;x)$ and $g_2(u^1,s^1,z^1;x)$ that are upper bounds of the integrands such that their limits exist when $x\to\infty$ and such that

$$\begin{aligned}&\int g_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n;x)F_1^{\otimes n}(\mathrm{d}(u^1,s^1,z^1;\dots;u^n,s^n,z^n))\\&\qquad+\int g_2(u^1,s^1,z^1;x)F_1(\mathrm{d}(u^1,s^1,z^1))\\&\to\int \lim_{x\to\infty} g_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n;x)F_1^{\otimes n}(\mathrm{d}(u^1,s^1,z^1;\dots;u^n,s^n,z^n))\\&\qquad+\int \lim_{x\to\infty} g_2(u^1,s^1,z^1;x)F_1(\mathrm{d}(u^1,s^1,z^1))\end{aligned}$$

as $x\to\infty$. Using (3.7) and properties of Ψ, we can choose the functions

$$g_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n;x)=\mathbf{1}_{C_x}\frac{\mathbb{P}(Z^1\phi(U^1,Z^1)>x-y^*-\sum_{i=1}^n z^i\phi(u^i,s^i))}{\mathbb{P}(\Psi(V_{v,t}^1)>x)}$$

and

$$g_2(u^1,s^1,z^1;x)=\mathbf{1}_{\tilde{C}_x}\frac{\mathbb{P}(\sum_{i=1}^n Z^i\phi(U^i,Z^i)>x-y^*-z^1\phi(u^1,s^1))}{\mathbb{P}(\Psi(V_{v,t}^1)>x)}.$$

From Theorem 3.1 and (3.9) we find that

$$\mathbb{P}(Z^1\phi(U^1,S^1)>x)\sim\frac{m(B'\times T')}{m(B\times[0,T])}\mathbb{P}(\Psi(V_{v,t}^1)>x) \tag{SM1.4}$$

as $x\to\infty$. The fact that the distribution of $Z^1\phi(U^1,S^1)$ is convolution equivalent and in particular has an exponential tail implies

$$g_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n;x)\to\frac{m(B'\times T')}{m(B\times[0,T])}\exp\Big(\frac{\beta}{c}\Big(y^*+\sum_{i=1}^n z^i\phi(u^i,s^i)\Big)\Big)$$

as $x\to\infty$. Similarly, (SM1.4) and an application of [3, Corollary 2.11] gives

$$\begin{aligned}&g_2(u^1,s^1,z^1;x)\\&\quad\to\frac{m(B'\times T')}{m(B\times[0,T])}n\exp\Big(\frac{\beta}{c}(y^*+z^1\phi(u^1,s^1))\Big)\Big(\mathbb{E}\exp\Big(\frac{\beta}{c}Z^1\phi(U^1,S^1)\Big)\Big)^{n-1}\end{aligned}$$

as $x\to\infty$, and we conclude that

$$\begin{aligned}&\int \lim_{x\to\infty} g_1(u^1,s^1,z^1;\dots;u^n,s^n,z^n;x)F_1^{\otimes n}(\mathrm{d}(u^1,s^1,z^1;\dots;u^n,s^n,z^n))\\&\qquad+\int \lim_{x\to\infty} g_2(u^1,s^1,z^1;x)F_1(\mathrm{d}(u^1,s^1,z^1))\\&=\frac{m(B'\times T')}{m(B\times[0,T])}(n+1)\exp(\beta y^*/c)\Big(\mathbb{E}\exp\Big(\frac{\beta}{c}Z^1\phi(U^1,S^1)\Big)\Big)^n.\end{aligned}\tag{SM1.5}$$

For notational convenience, we let μ denote the distribution of $Z^i\phi(U^i, S^i)$. Then, again by [3, Corollary 2.11] and (SM1.4), (SM1.5) equals

$$\lim_{x\to\infty} \frac{m(B'\times T')}{m(B\times[0,T])} \frac{\mu^{*(n+1)}((x-y^*,\infty))}{\mu((x,\infty))} = \lim_{x\to\infty} \frac{\mu^{*(n+1)}((x-y^*,\infty))}{\mathbb{P}(\Psi(V^1_{v,t})>x)}. \tag{SM1.6}$$

Furthermore, we see

$$\begin{aligned}&\mathbb{P}(\Psi(V^1_{v,t})>x)\Big(\int g_1(z^1;\dots;z^n;x)\mu^{\otimes n}(\mathrm{d}(z^1;\dots;z^n)) + \int g_2(z;x)\mu(\mathrm{d}z)\Big)\\ &\quad = \int_0^{(x-y^*)/2} \mu((x-y^*-z,\infty))\mu^{*n}(\mathrm{d}z) + \int_0^{(x-y^*)/2} \mu^{*n}((x-y^*-z,\infty))\mu(\mathrm{d}z).\end{aligned}$$

Since, in particular, the tails of μ and μ^{*n} are exponential with index β/c, we see from [2, Lemma 2] that the sum of integrals is asymptotically equivalent to $\mu^{*(n+1)}((x-y^*,\infty))$. Returning to (SM1.6) concludes the proof. □

Before proving the theorem on the extremal behaviour of X^1, we need the following lemma for a dominated convergence argument.

Lemma SM1.1. *Let $V^1, V^2, \dots$ be i.i.d. fields with distribution ν_1, and let (U, S, Z) be distributed according to F_1. There exist a constant K such that*

$$\mathbb{P}(\Psi(V^1_{v,t} + \cdots + V^n_{v,t}) > x) \le K^n \mathbb{P}(Z\phi(U,S) > x)$$

for all $n \in \mathbb{N}$ and all $x \ge 0$.

Proof. By Lemma 3.1 the distribution of $Z\phi(U,S)$ is convolution equivalent, and it follows from [3, Lemma 2.8] that there is a constant K such that

$$\mathbb{P}\Big(\sum_{i=1}^n Z^i\phi(U^i,S^i) > x\Big) \le K^n \mathbb{P}(Z\phi(U,S)>x),$$

for i.i.d. variables $(U^1, S^1, Z^1), (U^2, S^2, Z^2), \dots$ with distribution F_1. The result follows directly from (3.7). □

Proof of Theorem 3.3. From (3.8) and the representation $V^i = (Z^i f(|v-U^i|, t-S^i))_{(v,t)}$, we see that

$$\mathbb{E}\left[\exp\!\big(\beta\lambda_{u,s}\left(X^1_{v,t}\right)\big)\right] \le \exp\Big(\nu(A)\big(\mathbb{E}[\exp(\tfrac{\beta}{c}Z\phi(U,S)] - 1\big)\Big).$$

The first claim now follows from (3.10).

For the limit result, we find by independence and Lemma SM1.1,

$$\begin{aligned}
&\mathbb{P}(\Psi(X^1_{v,t}+y_{v,t})>x)\\
&\quad=\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}\mathbb{P}(\Psi(V^1_{v,t}+\cdots+V^n_{v,t}+y_{v,t})>x)\\
&\quad\le \mathbb{P}(Z\phi(U,S)>x-y^*)\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^nK^n}{n!},
\end{aligned}$$

where $y^*=\sup_{(v,t)}y_{v,t}$ and $\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^nK^n}{n!}<\infty$. With the convention that $V^1_{v,t}+\cdots+V^{n-1}_{v,t}=0$ for $n=1$, by dominated convergence, Theorems 3.1 and 3.2 and Lemma 3.1 yield

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\mathbb{P}(\Psi(X^1_{v,t}+y_{v,t})>x)}{L(x/c)\exp(-\beta x/c)}\\
&=\frac{n}{\nu(A)}\mathrm{e}^{-\nu(A)}\sum_{n=1}^{\infty}\frac{\nu(A)^n}{n!}\int_B\int_0^T\mathbb{E}\left[\mathrm{e}^{\beta\lambda_{u,s}\left(V^1_{v,t}+\cdots+V^{n-1}_{v,t}+y_{v,t}\right)}\right]\mathrm{d}s\mathrm{d}u\\
&=\mathrm{e}^{-\nu(A)}\sum_{n=0}^{\infty}\frac{\nu(A)^n}{n!}\int_B\int_0^T\mathbb{E}\left[\mathrm{e}^{\beta\lambda_{u,s}\left(V^1_{v,t}+\cdots+V^{n}_{v,t}+y_{v,t}\right)}\right]\mathrm{d}s\mathrm{d}u\\
&=\int_B\int_0^T\mathbb{E}\left[\mathrm{e}^{\beta\lambda_{u,s}\left(X^1_{v,t}+y_{v,t}\right)}\right]\mathrm{d}s\mathrm{d}u.
\end{aligned}$$

This concludes the proof. □

Proof of Lemma 3.2. First we show that

$$\mathbb{E}\exp(\gamma\sup_{(v,t)\in B'\times T'}X^2_{v,t})<\infty \tag{SM1.7}$$

for all $\gamma>0$. Applying arguments as in Section 2, we write X^2 as the independent sum $X^2_{v,t}=Y^1_{v,t}+Y^2_{v,t}$. Here Y^1 is a compound Poisson sum

$$Y^1_{v,t}=\sum_{k=1}^{M}J^k_{v,t}$$

with finite intensity $\nu(A^c\cap D)<\infty$ and jump distribution $\nu_2=\nu_{A^c\cap D}/\nu(A^c\cap D)$, where $D=\{z\in\mathbb{R}^{\mathbb{K}}:\inf_{(v,t)\in\mathbb{K}}z_{v,t}<-1\}$. Furthermore, Y^2 is infinitely divisible with Lévy measure $\nu_{A^c\cap D^c}$, the restriction of ν to the set $A^c\cap D^c=\{z\in\mathbb{R}^{\mathbb{K}}:\sup_{(v,t)\in\mathbb{K}}|z_{v,t}|\le 1\}$. By arguments as before, both fields have t-càdlàg extensions to $B'\times T'$. For each k, $J^k_{v,t}\le 0$ for all $(v,t)\in B'\times T'$ almost surely, and in particular $\mathbb{E}\exp(\gamma\sup_{(v,t)\in B'\times T'}Y^1_{v,t})<\infty$ for all $\gamma>0$. As $(Y^2_{v,t})_{(v,t)\in B'\times T'}$ is t-càdlàg on the compact set $B' \times T'$, we find that $\mathbb{P}(\sup_{(v,t)\in B'\times T'} |Y^2_{v,t}| < \infty) = 1$. Since also $\nu_{A^c\cap D^c}(\{z \in \mathbb{R}^{\mathbb{K}} : \sup_{(v,t)\in\mathbb{K}} |z_{v,t}| > 1\}) = 0$, we obtain from [1, Lemma 2.1] that $\mathbb{E}\exp(\gamma \sup_{(v,t)\in B'\times T'} |Y^2_{v,t}|) < \infty$ for all $\gamma > 0$, which yields the claim (SM1.7).

Appealing to properties of $\lambda_{u,s}$ we find that

$$\lambda_{u,s}\left(X_{v,t}\right) \leq \lambda_{u,s}\big(X^1_{v,t} + \sup_{(v,t)\in B'\times T'} X^2_{v,t}\big) = \lambda_{u,s}\left(X^1_{v,t}\right) + \frac{\sup_{(v,t)\in B'\times T'} X^2_{v,t}}{c}.$$

The assertion now follows from (SM1.7) and the first claim of Theorem 3.3. □

Proof of Theorem 3.4. Let π be the distribution of $(X^2_{v,t})_{(v,t)\in B'\times T'}$. Conditioning on $(X^2_{v,t})_{(v,t)} = (y_{v,t})_{(v,t)}$ we find by independence that

$$\frac{\mathbb{P}(\Psi(X_{v,t}) > x)}{\mathbb{P}(\Psi(X^1_{v,t}) > x)} = \int \frac{\mathbb{P}(\Psi(X^1_{v,t} + y_{v,t}) > x)}{\mathbb{P}(\Psi(X^1_{v,t}) > x)}\pi(\mathrm{d}y) = \int f(y;x)\pi(\mathrm{d}y)$$

with $f(y;x) = \mathbb{P}(\Psi(X^1_{v,t}+y_{v,t}) > x)/\mathbb{P}(\Psi(X^1_{v,t}) > x)$, which, according to Theorem 3.3, satisfies

$$f(y;x) \to f(y) = \frac{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}\left(X^1_{v,t} + y_{v,t}\right)\right)\right] \mathrm{d}s\mathrm{d}u}{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}\left(X^1_{v,t}\right)\right)\right] \mathrm{d}s\mathrm{d}u}$$

as $x \to \infty$. By another application of Theorem 3.3 and since

$$\int f(y)\pi(\mathrm{d}y) = \frac{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}\left(X_{v,t}\right)\right)\right] \mathrm{d}s\mathrm{d}u}{\int_B \int_0^T \mathbb{E}\left[\exp\left(\beta\lambda_{u,s}\left(X^1_{v,t}\right)\right)\right] \mathrm{d}s\mathrm{d}u},$$

the proof is completed if we can find non-negative and integrable functions $g(y;x)$ and $g(y) = \lim_{x\to\infty} g(y;x)$ such that $f(y;x) \leq g(y;x)$ and such that

$$\int g(y;x)\pi(\mathrm{d}y) \to \int g(y)\pi(\mathrm{d}y)$$

as $x \to \infty$. With $y^* = \sup_{(v,t)\in B'\times T'} y_{v,t}$ we use the function

$$g(y;x) = \mathbb{P}(\Psi(X^1_{v,t}) + y^* > x)/\mathbb{P}(\Psi(X^1_{v,t}) > x)$$

which, according to properties of $\lambda_{u,s}$ and Theorem 3.3, satisfies

$$g(y;x) \to g(y) = \exp(\beta y^*/c)$$

as $x \to \infty$. From [4, Lemma 2.4(i)] and Theorem 3.3 the distribution of $\Psi(X^1_{v,t})$ is convolution equivalent with index β/c. Now let G and H denote the distributions of

$\Psi(X^1_{v,t})$ and $\sup_{(v,t)\in B'\times T'} X^2_{v,t}$, respectively. If $\overline{H}(x) = o(\overline{G}(x))$, $x\to\infty$, it follows from the integrability statement (SM1.7) and [4, Lemma 2.1] that

$$\int g(y;x)\pi(\mathrm{d}y) = \frac{\mathbb{P}(\Psi(X^1_{v,t}) + \sup_{(v,t)\in B'\times T'} X^2_{v,t} > x)}{\mathbb{P}(\Psi(X^1_{v,t}) > x)}$$
$$\to \mathbb{E}\exp\Big(\frac{\beta}{c}\sup_{(v,t)\in B'\times T'} X^2_{v,t}\Big) = \int g(y)\pi(\mathrm{d}y)$$

as $x\to\infty$. From (SM1.7) we find that $\lim_{x\to\infty} \mathrm{e}^{\gamma x}\mathbb{P}(\sup_{(v,t)\in B'\times T'} X^2_{v,t} > x) = 0$ for all $\gamma > 0$. Combined with the convolution equivalence of the distribution of $\Psi(X^1_{v,t})$, this yields $\overline{H}(x) = o(\overline{G}(x))$ and the claim follows. □

SM2. Proofs of Section 5

Proof of Lemma 5.2. Let $\omega\in\Omega_1'$ and $(s_n)\subset\tilde{S}$ such that $s_n\downarrow t\in[0,S]$. For all $k\in\mathbb{N}$ there exists $N\in\mathbb{N}$ such that

$$\|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{s_N}(\omega)\|_\infty \le \tfrac{1}{k} \quad \text{for all} \quad n\ge N. \tag{SM2.1}$$

This is seen by contradiction as follows: Assume that for any $N\in\mathbb{N}$ there exists $n\ge N$ such that

$$\|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{s_N}(\omega)\|_\infty > \tfrac{1}{k}.$$

Now fix $p\in\mathbb{N}$. By this there exist $n_0<n_1<n_2<\cdots<n_p$ such that

$$\|\boldsymbol{Z}_{s_{n_j}}(\omega) - \boldsymbol{Z}_{s_{n_{j-1}}}(\omega)\|_\infty > \tfrac{1}{k} \quad \text{for} \quad j=1,\dots,p$$

and we conclude that $\boldsymbol{Z}(\omega)$ has $\frac{1}{k}$-oscillation p times in $\tilde{S}$ for any p. Hence $\omega\in A_k^c$, which is a contradiction. From (SM2.1) and the fact that the metric space $(\mathcal{C}(K,\mathbb{R}), \|\cdot\|_\infty)$ is complete, we know that $\lim_{n\to\infty}\boldsymbol{Z}_{s_n}(\omega)$ exists with respect to $\|\cdot\|_\infty$ as a continuous function on K. To show uniqueness of the limit, let $(t_n)\subset\tilde{S}$ be another sequence such that $t_n\downarrow t$. Then $\lim_{n\to\infty}\boldsymbol{Z}_{s_n}(\omega) = \lim_{n\to\infty}\boldsymbol{Z}_{t_n}(\omega)$: Let (r_n) be the union of (s_n) and (t_n) ordered such that $r_n\downarrow t$. Then similarly for any $\epsilon>0$ there is N' such that

$$\|\boldsymbol{Z}_{r_n}(\omega) - \boldsymbol{Z}_{r_{N'}}(\omega)\|_\infty < \tfrac{\epsilon}{2} \quad \text{for} \quad n\ge N'.$$

Also there is $N\in\mathbb{N}$ such that $(s_n)_{n\ge N}, (t_n)_{n\ge N}\subseteq (r_n)_{n\ge N'}$, and hence

$$\|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{t_n}(\omega)\|_\infty \le \|\boldsymbol{Z}_{s_n}(\omega) - \boldsymbol{Z}_{r_{N'}}(\omega)\|_\infty + \|\boldsymbol{Z}_{t_n}(\omega) - \boldsymbol{Z}_{r_{N'}}(\omega)\|_\infty < \epsilon$$

for all $n \geq N$. Thus, the limit $\lim_{s\in\mathbb{Q},s\downarrow t} \boldsymbol{Z}_s(\omega)$ exists uniquely with respect to $\|\cdot\|_\infty$. Similarly for $\lim_{s\in\mathbb{Q},s\uparrow t} \boldsymbol{Z}_s(\omega)$. $\square$

We let

$$B(p,\epsilon,D) = \{\omega \in \Omega \mid \boldsymbol{Z}(\omega) \text{ has } \epsilon\text{-oscillation } p \text{ times in } D\},$$

with $D \subseteq \mathbb{Q} \cap [0,\infty)$, and

$$\alpha_\epsilon(r) = \sup\{\mathbb{P}(\|\boldsymbol{Z}_t\|_\infty \geq \epsilon) \mid t \in [0,r]\cap\mathbb{Q}\}.$$

Note that a direct consequence of the stochastic continuity from Lemma 5.1 is that $\alpha_\epsilon(r) \to 0$ as $r \to 0$ for all $\epsilon > 0$.

Lemma SM2.1. *Let p be a positive integer, $D = \{t_1,\dots,t_n\} \subseteq \mathbb{Q}\cap[0,\infty)$ and $u, r \in \mathbb{Q}$ such that $0 \leq u \leq t_1 < \cdots < t_n \leq r$. Then $\mathbb{P}(B(p,4\epsilon,D)) \leq (2\alpha_\epsilon(r-u))^p$.*

Proof. We will show the statement by induction in p. For this, define

$$\begin{aligned} C_k &= \{\|\boldsymbol{Z}_{t_j} - \boldsymbol{Z}_u\|_\infty \leq 2\epsilon,\ j = 1,\dots,k-1,\ \|\boldsymbol{Z}_{t_k} - \boldsymbol{Z}_u\|_\infty > 2\epsilon\}\ , \\ D_k &= \{\|\boldsymbol{Z}_{t_k} - \boldsymbol{Z}_r\|_\infty > \epsilon\} \end{aligned}$$

and note that $C_1,\dots,C_n$ are disjoint and

$$\begin{aligned} B(1,4\epsilon,D) &\subseteq \bigcup_{k=1}^n \{\|\boldsymbol{Z}_{t_k} - \boldsymbol{Z}_u\|_\infty > 2\epsilon\} = \bigcup_{k=1}^n C_k \\ &= \bigcup_{k=1}^n (C_k \cap D_k^c) \cup (C_k \cap D_k) \subseteq \{\|\boldsymbol{Z}_r - \boldsymbol{Z}_u\|_\infty \geq \epsilon\} \cup \bigcup_{k=1}^n (C_k \cap D_k). \end{aligned}$$

By the Lévy properties in Lemma 5.1 we have $\mathbb{P}(\|\boldsymbol{Z}_r - \boldsymbol{Z}_u\|_\infty \geq \epsilon) \leq \alpha_\epsilon(r-u)$ and furthermore that $\mathbb{P}(C_k \cap D_k) = \mathbb{P}(C_k)\mathbb{P}(D_k) \leq \mathbb{P}(C_k)\alpha_\epsilon(r-u)$. The fact that $C_1,\dots,C_n$ are disjoint then implies

$$\mathbb{P}(B(1,4\epsilon,D)) \leq \mathbb{P}(\|\boldsymbol{Z}_r - \boldsymbol{Z}_u\|_\infty \geq \epsilon) + \sum_{k=1}^n \mathbb{P}(C_k \cap D_k) \leq 2\alpha_\epsilon(r-u),$$

which is the desired expression for $p = 1$. We now assume the result to be true for arbitrary $p \in \mathbb{N}$. We define the sets

$$\begin{aligned} F_k = \{\omega : \boldsymbol{Z}(\omega) &\text{ has } 4\epsilon\text{-oscillation } p \text{ times in } \{t_1,\dots,t_k\}, \\ &\text{but does not have } 4\epsilon\text{-oscillation } p \text{ times in } \{t_1,\dots,t_{k-1}\}\}\ , \end{aligned}$$

$$G_k = \{\omega : \boldsymbol{Z}(\omega) \text{ has } 4\epsilon\text{-oscillation one time in } \{t_k,\dots,t_n\}\}.$$

Then $F_1, \dots, F_n$ are disjoint, and $\mathbb{P}(F_k \cap G_k) = \mathbb{P}(F_k)\mathbb{P}(G_k)$ for all $k = 1, \dots, n$ due to the Lévy properties. Also $B(p, 4\epsilon, D) = \cup_{k=1}^n F_k$, and furthermore

$$B(p+1, 4\epsilon, D) = \bigcup_{k=1}^{n} (F_k \cap G_k)$$

with the inclusion $\subseteq$ seen as follows: If $\omega \in B(p+1, 4\epsilon, D)$ then $\boldsymbol{Z}(\omega)$ has 4ϵ-oscillation $p+1$ times in some $\{t_{n_0}, \dots, t_{n_{p+1}}\} \subseteq D$ with $n_0 < n_1 < \cdots < n_{p+1}$. Hence there is $k \leq n_p$ such that $\omega \in F_k$. Also $\|\boldsymbol{Z}_{t_{n_{p+1}}}(\omega) - \boldsymbol{Z}_{t_{n_p}}(\omega)\|_\infty > 4\epsilon$ and as such also $\omega \in G_k$. From the induction assumption, the case $p = 1$ and the fact that $F_1, \dots, F_n$ are disjoint we find that

$$\begin{aligned}\mathbb{P}(B(p+1, 4\epsilon, D)) &= \sum_{k=1}^{n} \mathbb{P}(G_k)\mathbb{P}(F_k) \leq 2\alpha_\epsilon(r-u)\mathbb{P}\Big(\bigcup_{k=1}^{n} F_k\Big) \\ &= 2\alpha_\epsilon(r-u)\mathbb{P}(B(p, 4\epsilon, M)) \leq (2\alpha_\epsilon(r-u))^{p+1}.\end{aligned}$$

□

Proof of Lemma 5.3. To show that $\mathbb{P}(\Omega_1') = 1$ it suffices to prove $\mathbb{P}(A_k^c) = 0$ for any fixed $k \in \mathbb{N}$. Since $\alpha_\epsilon(r) \to 0$ as $r \downarrow 0$ for any $\epsilon > 0$, we can choose $\ell \in \mathbb{N}$ such that $2\alpha_{1/(4k)}(S/\ell) < 1$. Then by continuity of $\mathbb{P}$ we get

$$\begin{aligned}\mathbb{P}(A_k^c) &\leq \mathbb{P}(\boldsymbol{Z} \text{ has } \tfrac{1}{k}\text{-oscillation infinitely often in } \tilde{S}) \\ &\leq \sum_{j=1}^{\ell} \mathbb{P}(\boldsymbol{Z} \text{ has } \tfrac{1}{k}\text{-oscillation infinitely often in } [\tfrac{j-1}{\ell}S, \tfrac{j}{\ell}S] \cap \mathbb{Q}) \\ &= \sum_{j=1}^{\ell} \lim_{p \to \infty} \mathbb{P}(B(p, \tfrac{1}{k}, [\tfrac{j-1}{\ell}S, \tfrac{j}{\ell}S] \cap \mathbb{Q})).\end{aligned}$$

Now fix $j = 1, \dots, \ell$, and enumerate the elements of $[\frac{j-1}{\ell}S, \frac{j}{\ell}S] \cap \mathbb{Q}$ by $(t_m)_{m \in \mathbb{N}}$. From Lemma SM2.1 we know that

$$\mathbb{P}(B(p, \tfrac{1}{k}, \{t_1, \dots, t_n\})) \leq (2\alpha_{1/(4k)}(\tfrac{S}{\ell}))^p$$

for any $n \in \mathbb{N}$. By continuity of $\mathbb{P}$ we see that

$$\mathbb{P}(B(p, \tfrac{1}{k}, [\tfrac{j-1}{\ell}S, \tfrac{j}{\ell}S] \cap \mathbb{Q})) = \lim_{n \to \infty} \mathbb{P}(B(p, \tfrac{1}{k}, \{t_1, \dots, t_n\})) \leq (2\alpha_{1/(4k)}(\tfrac{S}{\ell}))^p$$

which tends to 0 as $p \to \infty$ since ℓ is chosen such that $2\alpha_{1/(4k)}(S/\ell) < 1$. As this holds for all $j = 1, \dots, \ell$ we conclude that $\mathbb{P}(A_k^c) = 0$. □

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