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SUPPLEMENTARY MATERIAL: BOUNDS FOR THE CHI-SQUARE APPROXIMATION OF THE POWER DIVERGENCE FAMILY OF STATISTICS

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Proof of Lemma 3.6. (i) Without loss of generality, we let $a = 1$; the general $a > 0$ case follows by rescaling. We therefore need to prove that, for $x \geq 0$,

$$|f(x)| \leq |x-1|^3, \tag{1}$$

where

$$f(x) := 2x\log(x) - 2(x-1) - (x-1)^2.$$

It is readily checked that inequality (1) holds for $x = 0$ and $x = 2$. For $0 < x < 2$ (that is $|x-1| < 1$), we can use a Taylor expansion to obtain the bound

$$|f(x)| = 2|x-1|^3\left|\sum_{k=0}^{\infty}\frac{(-1)^k(x-1)^k}{(k+2)(k+3)}\right| \leq 2|x-1|^3\sum_{k=0}^{\infty}\frac{1}{(k+2)(k+3)} = |x-1|^3,$$

so inequality (1) is satisfied for $0 < x < 2$. Now, suppose $x > 2$. We have that $f'(x) = 2\log(x) - 2(x-1)$ and $\frac{\mathrm{d}}{\mathrm{d}x}\big((x-1)^3\big) = 3(x-1)^2$. By the inequality $\log(u) \leq u - 1$, for $u \geq 1$, we get that

$$|f'(x)| = |2\log(x) - 2(x-1)| = 2(x-1) - 2\log(x) \leq (x-1)^2 \leq 3(x-1)^2,$$

where the final inequality holds because $x > 2$. Therefore, for $x > 2$, $(x-1)^3$ grows faster than $|f(x)|$. Since $|f(2)| = (2-1)^3 = 1$, it follows that inequality (1) holds for all $x > 2$. We have now shown that inequality (1) is satisfied for all $x \geq 0$, as required.

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(ii) Again, without loss of generality, we may set $a = 1$. We therefore need to prove that, for $x \geq 0$,

$$|g_\lambda(x)| \leq \frac{(\lambda-1)\lambda(\lambda+1)}{6}\big(1+x^{\lambda-2}\big)|x-1|^3, \tag{2}$$

where

$$g_\lambda(x) := x^{\lambda+1} - 1 - (\lambda+1)(x-1) - \frac{\lambda(\lambda+1)}{2}(x-1)^2. \tag{3}$$

By a Taylor expansion of $x^{\lambda+1}$ about $x = 1$ we have that

$$g_\lambda(x) = \frac{(\lambda-1)\lambda(\lambda+1)}{6}\xi^{\lambda-2}(x-1)^3, \tag{4}$$

where $\xi > 0$ is between 1 and x. Now, as ξ is between 1 and x and because $\lambda \geq 2$, we have that

$$\xi^{\lambda-2} \leq (\max\{1,x\})^{\lambda-2} \leq 1 + x^{\lambda-2},$$

and applying this inequality to (4) gives us (2), as required.

(iii) Suppose now that $\lambda \in (-1,2) \setminus \{0\}$. Without loss of generality, we set $a = 1$, and it therefore suffices to prove that, for $x \geq 0$,

$$|g_\lambda(x)| \leq \frac{|(\lambda-1)\lambda|}{2}|x-1|^3. \tag{5}$$

We shall verify inequality (5) by treating the cases $0 < x \leq 2$ and $x \geq 2$ separately (it is readily checked that the inequality holds at $x = 0$). For $0 < x < 2$ (that is $|x-1| < 1$) we can use a Taylor expansion to write

$$g_\lambda(x) = (x-1)^3 G_\lambda(x),$$

where

$$G_\lambda(x) = \sum_{k=0}^{\infty} \binom{\lambda+1}{k+3}(x-1)^k,$$

and the generalised binomial coefficient is given by $\binom{a}{j} = [a(a-1)(a-2)\cdots(a-j+1)]/j!$, for $a > 0$ and $j \in \mathbb{N}$. We now observe that, since $\lambda < 2$, the generalised binomial coefficients $\binom{\lambda+1}{k+3}$ are either positive for all even $k \geq 0$ and negative for all odd $k \geq 1$, or are negative for all even $k \geq 0$ and positive for all odd

$k \geq 1$ (or, exceptionally always equal to zero if $\lambda = 1$, which is a trivial case in which $g_\lambda(x) = 0$ for all $x \geq 0$). Hence, for $0 < x < 2$, $G_\lambda(x)$ is bounded above by $|G_\lambda(0)|$, and a short calculation using the expression (3) (note that $G_\lambda(x) = g_\lambda(x)/(x-1)^3$) shows that $G_\lambda(0) = |(\lambda-1)\lambda|/2$. Thus, for $0 \leq x < 2$, we have the bound

$$|g_\lambda(x)| \leq \frac{|(\lambda-1)\lambda|}{2}|x-1|^3.$$

Suppose now that $x \geq 2$. Recall from (4) that

$$g_\lambda(x) = \frac{(\lambda-1)\lambda(\lambda+1)}{6}\xi^{\lambda-2}(x-1)^3,$$

where $\xi > 0$ is between 1 and x. In fact, because we are considering the case $x \geq 2$, we know that $\xi > 1$. Therefore, since $\lambda < 2$, we have that $\xi^{\lambda-2} < 1$. Therefore, for $x \geq 2$,

$$|g_\lambda(x)| = \frac{|(\lambda-1)\lambda(\lambda+1)|}{6}|x-1|^3 \leq \frac{|(\lambda-1)\lambda|}{2}|x-1|^3,$$

where the second inequality follows because $\lambda \in (-1,2) \setminus \{0\}$. We have thus proved inequality (5), which completes the proof of the lemma. $\square$