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SUPPLEMENTARY MATERIA: LOWER BOUND FOR THE EXPECTED SUPREMUM OF FRACTIONAL BROWNIAN MOTION USING COUPLING

KRZYSZTOF BISEWSKI,* *Université de Lausanne*

Appendix B. Calculations

Derivation of the formula for $\mathcal{E}_{1/2}(T,a)$ in Proposition 2. The derivation in cases $T=\infty$, $a>0$ and $T\in(0,\infty)$, $a=0$ is straightforward using Eq. (13) and Eq. (12) respectively. Till this end assume that $a\neq 0$ and $T\in(0,\infty)$. We have

$$\mathbb{P}(\tau(T,a)\in \mathrm{d}t)=\left(\frac{e^{-a^2t/2}}{\sqrt{\pi t}}-\frac{a}{\sqrt{2}}\operatorname{erfc}\left(\frac{a\sqrt{t}}{\sqrt{2}}\right)\right)\left(\frac{e^{-a^2(T-t)/2}}{\sqrt{\pi(T-t)}}+\frac{a}{\sqrt{2}}\operatorname{erfc}\left(-\frac{a\sqrt{T-t}}{\sqrt{2}}\right)\right),$$

see e.g. 2.1.12.4 in [2]. We now have

$$\begin{aligned}\mathcal{E}_{1/2}(T,a)&=\int_0^T t\mathbb{P}(\tau(T,a)\in\mathrm{d}t)\mathrm{d}t\\&=T\int_0^1 t\left(\frac{e^{-u^2t}}{\sqrt{\pi t}}-u\operatorname{erfc}(u\sqrt{t})\right)\left(\frac{e^{-u^2(1-t)}}{\sqrt{\pi(1-t)}}+u\operatorname{erfc}(-u\sqrt{1-t})\right)\mathrm{d}t,\end{aligned}$$

where we substituted $t=tT$ and put $u:=a\sqrt{T/2}$. Now, we have

$$\mathcal{E}_{1/2}(T,a)=T\cdot\left(\frac{e^{-u^2}}{\pi}\cdot J_1+\frac{u}{\sqrt{\pi}}\cdot(J_2(u)-J_3(u))-u^2J_4(u)\right), \tag{38}$$

where

$$J_1:=\int_0^1\sqrt{\frac{t}{1-t}}\mathrm{d}t=\frac{\pi}{2},\quad J_2(u):=\int_0^1\sqrt{t}e^{-u^2t}\operatorname{erfc}(-u\sqrt{1-t})\mathrm{d}t,$$
$$J_3(u):=\int_0^1\frac{t}{\sqrt{1-t}}e^{-u^2(1-t)}\operatorname{erfc}(u\sqrt{t})\mathrm{d}t,\quad J_4(u):=\int_0^1 t\operatorname{erfc}(u\sqrt{t})\operatorname{erfc}(-u\sqrt{1-t})\mathrm{d}t.$$

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* Postal address: Quartier UNIL-Chamberonne, Bâtiment Extranef, 1015 Lausanne, Switzerland

Using the fact that $\operatorname{erfc}(-z) = 2 - \operatorname{erfc}(z)$ and applying substitution $t = 1 - t$, we obtain

$$J_2(u) = 2\int_0^1 \sqrt{t}e^{-u^2 t}\mathrm{d}t - \int_0^1 \sqrt{1-t}e^{-u^2(1-t)}\operatorname{erfc}(u\sqrt{t})\mathrm{d}t.$$

In the following, let

$$J_5(u) := \int_0^1 \sqrt{t}e^{-u^2 t}\mathrm{d}t = \frac{\sqrt{\pi}\operatorname{erf}(u) - 2ue^{-u^2}}{2u^3}, \quad J_6(u) := \int_0^1 \frac{e^{-u^2(1-t)}}{\sqrt{1-t}}\operatorname{erfc}(u\sqrt{t})\mathrm{d}t,$$

where the first integral can be calculated using substitution $t = x^2$. Then

$$\begin{aligned} J_2(u) &= 2J_5(u) - \int_0^1 \frac{1-t}{\sqrt{1-t}}e^{-u^2(1-t)}\operatorname{erfc}(u\sqrt{t})\mathrm{d}t \\ &= 2J_5(u) + J_3(u) - J_6(u). \end{aligned}$$

Now, applying substitution $t = x^2/u^2$ and formula [3, 4.3.20] we obtain

$$J_6(u) = u^{-1}e^{-u^2}\int_0^u \frac{2xe^{x^2}}{\sqrt{u^2 - x^2}}\operatorname{erfc}(ux)\mathrm{d}t = \frac{\sqrt{\pi}}{u}\cdot\left(e^{-u^2} - \operatorname{erfc}(u)\right).$$

Further, using integration by parts we obtain

$$\begin{aligned} J_3(u) = & \frac{-2ue^{-u^2(1-t)}\sqrt{1-t} + \sqrt{\pi}(1-2u^2)\operatorname{erf}(u\sqrt{1-t})}{2u^3}\cdot\operatorname{erfc}(u\sqrt{t})\Big|_0^1 \\ & + \int_0^1 \frac{-2ue^{-u^2(1-t)}\sqrt{1-t} + \sqrt{\pi}(1-2u^2)\operatorname{erf}(u\sqrt{1-t})}{2u^3}\cdot\frac{ue^{-u^2 t}}{\sqrt{\pi t}}\mathrm{d}t. \end{aligned}$$

In the following let

$$J_7(u) := \int_0^1 \frac{e^{-u^2 t}}{\sqrt{t}}\mathrm{d}t = \frac{\sqrt{\pi}\operatorname{erf}(u)}{u},$$

which can be easily calculated using substitution $t = x^2$. Then

$$J_3(u) = \frac{2ue^{-u^2} - \sqrt{\pi}(1-2u^2)\operatorname{erf}(u)}{2u^3} - \frac{e^{-u^2}}{u\sqrt{\pi}}\cdot J_1 + \left(\tfrac{1}{2u^2} - 1\right)(J_7(u) - J_6(u))$$

and therefore

$$\begin{aligned} J_3(u) &= \frac{e^{-u^2}}{2u^3}\left(2u - \sqrt{\pi}(1-u^2) - e^{u^2}\sqrt{\pi}(2u^2-1)\operatorname{erfc}(u)\right) \\ J_2(u) &= \frac{e^{-u^2}}{2u^3}\left(-2u - \sqrt{\pi}(1+u^2) + e^{u^2}\sqrt{\pi}(1+\operatorname{erf}(u))\right). \end{aligned} \tag{39}$$

In the following, let

$$J_8(u) := \int_0^1 t\,\mathrm{erfc}(u\sqrt{t})\mathrm{d}t = \frac{1}{8}\left(4 - \frac{2e^{-u^2}(3+2u^2)}{\sqrt{\pi}u^3} + \left(\frac{3}{u^4} - 4\right)\mathrm{erf}(u)\right),$$

$$J_9(u) := \int_0^1 t\,\mathrm{erfc}(u\sqrt{t})\,\mathrm{erfc}(-u\sqrt{1-t})\mathrm{d}t,$$

where the first integral was calculated using integration by parts. Using the identity $\mathrm{erfc}(-z) = 2 - \mathrm{erfc}(z)$ we find that

$$J_4(u) = 2J_8(u) - J_9(u).$$

We will now find the value of the integral $J_9(u)$. See that After applying substitution $t = 1 - t$ we obtain

$$\begin{aligned} 2J_9(u) &= \int_0^1 t\,\mathrm{erfc}(u\sqrt{t})\,\mathrm{erfc}(u\sqrt{1-t})\mathrm{d}t + \int_0^1 (1-t)\,\mathrm{erfc}(u\sqrt{t})\,\mathrm{erfc}(u\sqrt{1-t})\mathrm{d}t \\ &= \int_0^1 \mathrm{erfc}(u\sqrt{t})\,\mathrm{erfc}(u\sqrt{1-t})\mathrm{d}t. \end{aligned}$$

After integration by parts we find that

$$\begin{aligned} 2J_9(u) = &\left(\frac{\mathrm{erf}(u\sqrt{t})}{2u^2} + t\,\mathrm{erfc}(u\sqrt{t}) - \frac{\sqrt{t}e^{-u^2 t}}{\sqrt{\pi}u}\right)\cdot \mathrm{erfc}(u\sqrt{1-t})\Bigg|_0^1 \\ &- \int_0^1 \left(\frac{\mathrm{erf}(u\sqrt{t})}{2u^2} + t\,\mathrm{erfc}(u\sqrt{t}) - \frac{\sqrt{t}e^{-u^2 t}}{\sqrt{\pi}u}\right)\cdot \frac{ue^{-u^2(1-t)}}{\sqrt{\pi(1-t)}}\mathrm{d}t, \end{aligned}$$

therefore

$$2J_9(u) = \left(\frac{\mathrm{erf}(u)}{2u^2} + \mathrm{erfc}(u) - \frac{e^{-u^2}}{\sqrt{\pi}u}\right) - \frac{1}{2\sqrt{\pi}u}\left(J_7(u) - J_6(u)\right) - \frac{u}{\sqrt{\pi}}\cdot J_3(u) + \frac{e^{-u^2}}{\pi}\cdot J_1.$$

After simple algebraic manipulations this yields

$$J_4(u) = \frac{3 - e^{-u^2}\left(2u^2 + \frac{6u}{\sqrt{\pi}}\right) + (2u^2-3)\,\mathrm{erfc}(u)}{4u^4}. \tag{40}$$

Finally, after plugging in the expressions for $J_2(u)$, $J_3(u)$ and $J_4(u)$ calculated in (39) and (40) into Eq. (38) we obtain

$$\mathcal{E}_{1/2}(T, a) = \frac{T}{2}\left(1 + \left(\frac{1}{2u^2} - 1\right)\mathrm{erf}(u) - \frac{e^{-u^2}}{\sqrt{\pi}u}\right),$$

which concludes the proof. □

Continuation of the proof of Lemma 1. First we show that

$$I_H(t,y) = \frac{t^{H+1/2}}{\sqrt{2\pi}y}\int_0^\infty\int_0^\infty \frac{q^{H-3/2}x^2}{(1+q)^{H+3/2}}\cdot\left[e^{-\frac{1}{2}\left(x-y\sqrt{\frac{q(s)}{t}}\right)^2} - e^{-\frac{1}{2}\left(x+y\sqrt{\frac{q(s)}{t}}\right)^2}\right]\mathrm{d}x\mathrm{d}q, \tag{41}$$

where $q(s) = s/(1-s)$, which generalizes [1, Proposition 2.1] in case $H = \frac{1}{2}$. From (41) we have

$$\begin{aligned} I_H(t,y) &= \frac{t^{3/2}}{y}\int_0^t\int_0^\infty \frac{x^2}{s^{3-H}\sqrt{2\pi(t-s)}}e^{-\frac{x^2}{2s}+\frac{y^2}{2t}}\cdot\left[e^{-\frac{(x-y)^2}{2(t-s)}} - e^{-\frac{(x+y)^2}{2(t-s)}}\right]\mathrm{d}x\mathrm{d}s \\ &= \frac{t^{3/2}}{\sqrt{2\pi}y}\int_0^t\int_0^\infty \frac{x^2}{s^{3-H}\sqrt{t-s}}\cdot\left[e^{-\frac{1}{2}\left(\frac{x}{\sigma_0}-y\mu_0\right)^2} - e^{-\frac{1}{2}\left(\frac{x}{\sigma_0}+y\mu_0\right)^2}\right]\mathrm{d}x\mathrm{d}s, \end{aligned}$$

where $\mu_0^2 := \frac{s}{(t-s)t}$, and $\sigma_0^2 := \frac{(t-s)s}{t}$. We now apply substitution $x := \sigma_0 x$, then $s := ts$, which gives us

$$I_H(t,y) = \frac{t^{H+1/2}}{\sqrt{2\pi}y}\int_0^1\int_0^\infty \frac{(1-s)x^2}{s^{3/2-H}}\cdot\left[e^{-\frac{1}{2}\left(x-y\sqrt{\frac{q(s)}{t}}\right)^2} - e^{-\frac{1}{2}\left(x+y\sqrt{\frac{q(s)}{t}}\right)^2}\right]\mathrm{d}x\mathrm{d}s,$$

which after substitution $q = s/(1-s)$ yields (41). Using [1, Eq. (D.2)] we find that for any $b \in \mathbb{R}$:

$$\int_0^\infty x^2\left(e^{-\frac{(x-b)^2}{2}} - e^{-\frac{(x+b)^2}{2}}\right)\mathrm{d}x = 2be^{-b^2/2} + (1+b^2)\sqrt{2\pi}\cdot\mathrm{erf}(b/\sqrt{2}).$$

Therefore,

$$I_H(t,y) = \frac{t^{H+1/2}}{\sqrt{\pi}y}(J_1(H,u) + J_2(H,u) + J_3(H,u)),$$

where we put $u := \frac{y}{\sqrt{2t}}$ and

$$\begin{aligned} J_1(H,u) &:= \int_0^\infty \frac{q^{H-3/2}}{(1+q)^{H+3/2}}\cdot 2u\sqrt{q}e^{-uq}\mathrm{d}q, \\ J_2(H,u) &:= \int_0^\infty \frac{q^{H-3/2}}{(1+q)^{H+3/2}}\cdot \sqrt{\pi}\,\mathrm{erf}(u\sqrt{q})\mathrm{d}q \\ J_3(H,u) &:= \int_0^\infty \frac{q^{H-3/2}}{(1+q)^{H+3/2}}\cdot 2u^2q^2\sqrt{\pi}\,\mathrm{erf}(u\sqrt{q})\mathrm{d}q \end{aligned}$$

Using (28) we find that

$$J_1(H,u) = 2u\Gamma(H)U(H,-\tfrac{1}{2},u^2).$$

Integration by parts yields

$$J_2(H,u)=\frac{\sqrt{\pi}(2+4H+4q)q^{H-\frac{1}{2}}}{(4H^2-1)(1+q)^{H+1/2}}\cdot \mathrm{erf}(u\sqrt{q})\bigg|_0^\infty-\int_0^\infty \frac{\sqrt{\pi}(2+4H+4q)q^{H-\frac{1}{2}}}{(4H^2-1)(1+q)^{H+1/2}}\cdot\frac{ue^{-u^2q}}{\sqrt{\pi q}}\mathrm{d}q$$

and after applying (28) we find that

$$J_2(H,u)=\frac{\sqrt{\pi}}{H^2-\frac{1}{4}}-\frac{u\Gamma(H)U(H,\frac{1}{2},u^2)}{H-\frac{1}{2}}-\frac{u\Gamma(H)U(1+H,\frac{3}{2},u^2)}{H^2-\frac{1}{4}}.$$

The integral $J_3(H,u)$ is computed analogously to $J_2(H,u)$ and it equals

$$J_3(u,H)=\frac{2\sqrt{\pi}u^2}{H+\frac{1}{2}}-\frac{2u^3\Gamma(1+H)U(1+H,\frac{3}{2},u^2)}{H+\frac{1}{2}}.$$

Now, after simple algebraic manipulations we obtain

$$J_1(H,u)+J_2(H,u)+J_3(H,u)=\frac{\sqrt{\pi}}{H^2-\frac{1}{4}}\left(1-\frac{\Gamma(H)}{\sqrt{\pi}}J_4(H,u)\right)+\frac{2\sqrt{\pi}u^2}{H+\frac{1}{2}},$$

where

$$\begin{aligned}J_4(H,u)=&-2u(H^2-\tfrac{1}{4})U(H,-\tfrac{1}{2},u^2)+u(H+\tfrac{1}{2})U(H,\tfrac{1}{2},u^2)\\&+uHU(H+1,\tfrac{3}{2},u^2)+2u^3(H-\tfrac{1}{2})HU(H+1,\tfrac{3}{2},u^2).\end{aligned}$$

Applying the relation (31) to the last term above we obtain

$$\begin{aligned}J_4(H,u)=&-2u(H^2-\tfrac{1}{4})\Big(U(H,-\tfrac{1}{2},u^2)+HU(H+1,\tfrac{1}{2},u^2\Big)+u(H+\tfrac{1}{2})U(H,\tfrac{1}{2},u^2)\\&+uHU(H+1,\tfrac{3}{2},u^2)+2u(H-\tfrac{1}{2})HU(H,\tfrac{1}{2},u^2).\end{aligned}$$

Applying the relation (32) to the first and third terms above we obtain

$$J_4(H,u)=uU(H,\tfrac{3}{2},u^2)=U(H-\tfrac{1}{2},\tfrac{1}{2},u^2),$$

where in the last equality we applied the Kummer's transformation (30). This concludes the proof. □

Lemma 4. *If $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$ and $u\in\mathbb{R}$, then*

$$\begin{aligned}&\int_0^\infty U(H-\tfrac{1}{2},\tfrac{1}{2},z^2)e^{-(z+u)^2}\mathrm{d}z\\&=\frac{\sqrt{\pi}e^{-u^2}}{2\Gamma(H+\frac{1}{2})}+\frac{\sqrt{\pi}|u|^{1-2H}}{2}\left(\frac{\gamma(H+\frac{1}{2},u^2)}{\Gamma(H+\frac{1}{2})}-\mathrm{sgn}(u)\frac{\gamma(H,u^2)}{\Gamma(H)}\right).\end{aligned}$$

Proof of Lemma 4. Using (29) we find that $U(H-\frac{1}{2},\frac{1}{2},z^2) = zU(H,\frac{3}{2},z^2)$ therefore, using (28) we have

$$\begin{aligned} A(u,H) &:= \int_0^\infty U(H-\tfrac{1}{2},\tfrac{1}{2},z^2)e^{-(z+u)^2}\mathrm{d}z \\ &= \frac{1}{\Gamma(H)}\int_0^\infty t^{H-1}(1+t)^{1/2-H}\int_0^\infty ze^{-(z+u)^2}e^{-z^2t}\mathrm{d}z\mathrm{d}t \\ &= \frac{e^{-u^2}}{\Gamma(H)}\int_0^\infty t^{H-1}(1+t)^{1/2-H}\int_0^\infty ze^{-z^2(1+t)-2zu}\mathrm{d}z\mathrm{d}t \\ &= \frac{e^{-u^2}}{\Gamma(H)}\int_0^\infty t^{H-1}(1+t)^{1/2-H}\cdot\frac{\sqrt{1+t}-\sqrt{\pi}ue^{u^2/(1+t)}\operatorname{erfc}(u/\sqrt{1+t})}{2(1+t)^{3/2}}\mathrm{d}t \end{aligned}$$

After substitution $s=\frac{1}{1+t}$ we obtain

$$A(u,H) = \frac{e^{-u^2}}{2\Gamma(H)}\left(J_1(u,H) - J_2(u,H) + J_3(u,H)\right),$$

where

$$\begin{aligned} J_1(u,H) &:= \int_0^1 s^{-1/2}(1-s)^{H-1}\mathrm{d}s, \\ J_2(u,H) &:= \sqrt{\pi}u\int_0^1 (1-s)^{H-1}e^{u^2s}\mathrm{d}s, \\ J_3(u,H) &:= \sqrt{\pi}u\int_0^1 (1-s)^{H-1}e^{u^2s}\operatorname{erf}(u\sqrt{z})\mathrm{d}s. \end{aligned}$$

Now, $J_1(u,H)$ can be easily found from the definition of the Beta function, while $J_2(u,H)$ can be found using (27), hence

$$J_1(u,H) = \frac{\sqrt{\pi}\Gamma(H)}{\Gamma(H+\frac{1}{2})}, \quad J_2(u,H) = \frac{\sqrt{\pi}u\Gamma(H)}{\Gamma(H+1)}{}_1F_1(1,H+1,u^2).$$

The integral $J_3(u,H)$ is now computed using the error function representation from (37) and the result in (36), i.e.

$$\begin{aligned} J_3(u,H) &= 2u^2\int_0^1 s^{1/2}(1-s)^{H-1}{}_1F_1(\tfrac{1}{2},\tfrac{3}{2},u^2s)\mathrm{d}s \\ &= 2u^2\cdot\frac{\Gamma(\frac{3}{2})\Gamma(H)}{\Gamma(H+\frac{3}{2})}{}_1F_1(1,H+\tfrac{3}{2},u^2). \end{aligned}$$

Finally, using (36) we obtain

$$A(u,H)=\frac{\sqrt{\pi}}{2}\left(\frac{e^{-u^2}}{\Gamma(H+\frac{1}{2})}-\frac{ue^{-u^2}}{\Gamma(H+1)}{}_1F_1(1,H+1,u^2)+\frac{u^2e^{-u^2}}{\Gamma(H+\frac{3}{2})}{}_1F_1(1,H+\tfrac{3}{2},u^2)\right)$$
$$=\frac{\sqrt{\pi}}{2}\left(\frac{e^{-u^2}}{\Gamma(H+\frac{1}{2})}-\frac{u|u|^{-2H}\gamma(H,u^2)}{\Gamma(H)}+\frac{|u|^{-2H+1}\gamma(H+\frac{1}{2},u^2)}{\Gamma(H+\frac{1}{2})}\right),$$

which ends the proof. □

Proof of Eq. (22). Recall that $H\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$, $a\neq 0$ and $T\in(0,\infty)$. We will first show that

$$\mathcal{J}_H^{(1)}(T,a):=\mathbb{E}\left(\tau^{H-\frac{1}{2}}\left(Y(\tau)+a\tau-\frac{\tau}{Y(\tau)}\right)\right)=0. \tag{42}$$

The joint density $p(t,y;a,T)$ of the pair $(\tau,Y(\tau))$ is well-known, see (12) in Section 2.2. We have $p(t,y;T,a)=p_1(t,y;T,a)p_2(t;T,a)$, where

$$p_1(t,y;T,a):=yt^{-3/2}\exp\left\{-\tfrac{(y+ta)^2}{2t}\right\}$$
$$p_2(t;T,a):=\frac{1}{\pi\sqrt{T-t}}\cdot\left(e^{-a^2(T-t)/2}+a\sqrt{\tfrac{\pi(T-t)}{2}}\ \mathrm{erfc}\left(-a\sqrt{\tfrac{T-t}{2}}\right)\right),$$

hence

$$\mathcal{J}_H^{(1)}(T,a)=\int_0^t\int_0^\infty t^{H-1/2}(y+at-\tfrac{t}{y})p(t,y;a,T)\mathrm{d}y\mathrm{d}t$$
$$=\int_0^t t^{H-2}p_2(t;T,a)\int_0^\infty(y^2+aty-t)\exp\left\{-\frac{(y+at)^2}{2t}\right\}\mathrm{d}y\mathrm{d}t$$
$$=\int_0^t t^{H-1/2}p_2(t;T,a)\int_{a\sqrt{t}}^\infty(z^2-a\sqrt{t}z-1)\exp\left\{-\frac{z^2}{2}\right\}\mathrm{d}z\mathrm{d}t,$$

where in the last line we substituted $z:=\frac{y+at}{\sqrt{t}}$. It can be seen that the definite integral with respect to z equals to 0 because for every $u\in\mathbb{R}$ we have

$$\int_u^\infty z^2e^{-\frac{z^2}{2}}\mathrm{d}z=ue^{-\frac{u^2}{2}}+\sqrt{\tfrac{\pi}{2}}\,\mathrm{erfc}\left(\tfrac{u}{\sqrt{2}}\right),\quad \int_u^\infty e^{-\frac{z^2}{2}}\mathrm{d}z=ue^{-\frac{u^2}{2}},$$
$$\int_u^\infty z^2e^{-\frac{z^2}{2}}\mathrm{d}z=\sqrt{\tfrac{\pi}{2}}\,\mathrm{erfc}\left(\tfrac{u}{\sqrt{2}}\right).$$

We have established that Eq. (42) holds and therefore it is left to calculate the integral

$$\mathcal{J}_H^{(2)}(T,a)=\frac{\Gamma(H)}{(H+\frac{1}{2})\sqrt{\pi}}\int_0^t t^{H-1}p_2(t;T,a)\int_0^\infty U\left(H-\tfrac{1}{2},\tfrac{1}{2},\tfrac{y^2}{2t}\right)\exp\left\{-\frac{(y+at)^2}{2t}\right\}\mathrm{d}y\mathrm{d}t.$$

Consider the innermost integral. After substituting $y = z\sqrt{2t}$ we find that

$$\int_0^\infty U\left(H - \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{y^2}{2t}\right) e^{-\frac{(y+at)^2}{2t}} \mathrm{d}y = \sqrt{2t} \int_0^\infty U\left(H - \tfrac{1}{2}, \tfrac{1}{2}, z^2\right) e^{-(z+a\sqrt{t/2})^2} \mathrm{d}z.$$

Using Lemma 4 we obtain

$$\mathcal{J}_H^{(2)}(T,a) = \frac{2^H |a|^{-2H}\Gamma(H)}{\sqrt{2\pi}(H+\frac{1}{2})} \Big(J_1(T,a) + J_2(T,a) - J_3(T,a) \Big), \tag{43}$$

where

$$\begin{aligned}
J_1(T,a) &:= \frac{\sqrt{\pi} 2^{-H}|a|^{2H}}{\Gamma(H+\frac{1}{2})} \int_0^T p_2(t;T,a) t^{H-1/2} e^{-\frac{a^2 t}{2}} \mathrm{d}t, \\
J_2(T,a) &:= \frac{\sqrt{\pi}|a|}{\sqrt{2}\Gamma(H+\frac{1}{2})} \int_0^T p_2(t;T,a) \gamma(H+\tfrac{1}{2}, \tfrac{a^2 t}{2}) \mathrm{d}t, \\
J_3(T,a) &:= \frac{\sqrt{\pi} a}{\sqrt{2}\Gamma(H)} \int_0^T p_2(t;T,a) \gamma(H, \tfrac{a^2 t}{2}) \mathrm{d}t.
\end{aligned}$$

It is left to calculate each of the integrals above. Using substitution $t = tT$ and putting $u := a\sqrt{T/2}$ we can find that

$$\begin{aligned}
J_1(T,a) &= \frac{|u|^{2H}}{u\Gamma(H+\frac{1}{2})} \int_0^1 \left(\frac{ue^{-u^2(1-t)}}{\sqrt{\pi(1-t)}} + u^2 \operatorname{erfc}(-u\sqrt{1-t}) \right) t^{H-1/2} e^{-u^2 t} \mathrm{d}t, \\
J_2(T,a) &= \frac{\operatorname{sgn}(u)}{\Gamma(H+\frac{1}{2})} \int_0^1 \left(\frac{ue^{-u^2(1-t)}}{\sqrt{\pi(1-t)}} + u^2 \operatorname{erfc}\Big(-u\sqrt{1-t} \Big) \right) \gamma(H+\tfrac{1}{2}, u^2 t) \mathrm{d}t, \\
J_3(T,a) &= \frac{1}{\Gamma(H)} \int_0^1 \left(\frac{ue^{-u^2(1-t)}}{\sqrt{\pi(1-t)}} + u^2 \operatorname{erfc}\Big(-u\sqrt{1-t} \Big) \right) \gamma(H, u^2 t) \mathrm{d}t.
\end{aligned}$$

Let us define:

$$f(t;u) := -\operatorname{erfc}(-u\sqrt{1-t}), \qquad g(t;u,H) := -|u|^{-2H-1}\Gamma(H+\tfrac{1}{2}, u^2 t).$$

Slightly abusing notation, for breviety we write $f(t) := f(t;u)$ and $g(t) := g(t;u,H)$. We then have

$$f'(t) = \frac{ue^{-u^2(1-t)}}{\sqrt{\pi(1-t)}}, \qquad g'(t) = t^{H-1/2} e^{-u^2 t},$$

and the quantities $J_i(T,a)$, $i \in \{1,2,3\}$ can be expressed as

$$J_1(T,a) = \frac{|u|^{2H}}{u\Gamma(H+\frac{1}{2})}\int_0^1 \Big(f'(t) - u^2 f(t)\Big) g'(t)\mathrm{d}t,$$
$$J_2(T,a) = \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^1 \Big(f'(t) - u^2 f(t)\Big)\gamma(H+\tfrac{1}{2}, u^2 t)\mathrm{d}t,$$
$$J_3(T,a) = \frac{1}{\Gamma(H)}\int_0^1 \Big(f'(t) - u^2 f(t)\Big)\gamma(H, u^2 t)\mathrm{d}t.$$

Before we calculate the values of $J_i(T,a)$ we introduce two useful functions, for $s > 0, u \in \mathbb{R}$:

$$h_1(u,s) := \int_0^1 f'(t)\gamma(s,u^2 t)\mathrm{d}t = \mathrm{sgn}(u)\cdot\frac{\Gamma(s)\gamma(s+\frac{1}{2},u^2)}{\Gamma(s+\frac{1}{2})},$$
$$h_2(u,s) := \int_0^1 (1-t)f'(t)\gamma(s,u^2 t)\mathrm{d}t = \mathrm{sgn}(u)\cdot\frac{\Gamma(s)\gamma(s+\frac{3}{2},u^2)}{2u^2\Gamma(s+\frac{3}{2})}.$$

The values of these functions were found by applying relation (36) and finding the value of the integral using (33). Now, integration by parts yields

$$\begin{aligned}
J_1(T,a) &= \frac{|u|^{2H}e^{-u^2}}{\Gamma(H+1)} + \frac{u|u|^{2H}}{\Gamma(H+\frac{1}{2})}\cdot\left(f(t)g(t)\Big|_0^1 - \int_0^1 f'(t)g(t)\mathrm{d}t\right) \\
&= \frac{|u|^{2H}e^{-u^2}}{\Gamma(H+1)} + \frac{\mathrm{sgn}(u)}{\Gamma(H+\frac{1}{2})}\Big(\gamma(H+\tfrac{1}{2},u^2) + h_1(u,H+\tfrac{1}{2})\Big) \\
&= \frac{\gamma(H,u^2)}{\Gamma(H)} - \frac{\gamma(H+1,u^2)}{\Gamma(H+1)} + \frac{\mathrm{sgn}(u)}{\Gamma(H+\frac{1}{2})}\Big(\gamma(H+\tfrac{1}{2},u^2) + h_1(u,H+\tfrac{1}{2})\Big),
\end{aligned}$$

where in the last line we used the recurrence relation for the incomplete Gamma function in Eq. (35). Now, notice that for any $s > 0, u \in \mathbb{R}$:

$$\int \gamma(s,u^2 t)\mathrm{d}t = t\gamma(s,u^2 t) + u^{-2}\Gamma(s+1,u^2 t) + C.$$

Therefore, integration by parts yields

$$\begin{aligned}
\int_0^1 f(t)\gamma(s,u^2 t)\mathrm{d}t &= -\gamma(s,u^2) - \int_0^1 tf'(t)\gamma(s,u^2 t) + u^{-2}\int_0^1 f'(t)\gamma(s+1,u^2 t))\mathrm{d}t \\
&= -\gamma(s,u^2) + h_2(u,s) - h_1(u,s) + u^{-2}h_1(u,s+1).
\end{aligned}$$

Finally, this gives us

$$J_2(T,a) = \frac{\mathrm{sgn}(u)}{\Gamma(H+\frac{1}{2})}\bigg((1+u^2)h_1(u,H+\tfrac{1}{2}) + u^2\gamma(H+\tfrac{1}{2},u^2) - u^2h_2(u,H+\tfrac{1}{2}) - \gamma(H+\tfrac{3}{2}) - h_1(u,H+\tfrac{3}{2})\bigg).$$

Using analogous methods, we find that

$$J_3(T,a) = \frac{1}{\Gamma(H)}\bigg((1+u^2)h_1(u,H)+u^2\gamma(H,u^2)-u^2h_2(u,H)-\gamma(H+1,u^2)-h_1(u,H+1)\bigg).$$

Through straightforward algebraic manipulations and applying simple recurrence relation (35), we finally obtain

$$J_1(T,a) + J_2(T,a) - J_3(T,a) = \frac{\gamma(H,u^2)}{\Gamma(H)},$$

which concludes the proof. □

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