Derivatives and Market (Il)liquidity

Internet Appendix

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This internet appendix contains the following parts:

- Section S1 studies a path-dependent general quadratic derivative; and
- Section S2 provides additional useful lemmas.

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S1 Introducing a path-dependent derivative

This internet appendix studies a possibly path-dependent quadratic derivative, $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$. In particular, we allow $f(D)$ to depend on the underlying asset price P_1 (even though $f(D)$ is realized at $t = 2$), hence "path-dependent."

Proposition S1. *With $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$, there exists a unique equilibrium at $t = 1$. The demand schedules for the underlying are*

$$X_{1d}(p, q; s, z) = X_{1d}^{nd}(p; s, z) + [(b-2)p + c]Y_{1d}(p, q; s, z); \text{ and}$$
$$X_{1s}(p, q) = X_{1s}^{nd}(p) + [(b-2)p + c]Y_{1s}(p, q).$$

The demand schedules for the general variance swap are

$$Y_{1d}(p, q; s, z) = \frac{1}{2\alpha}\left(\left(q + \left((a+b-1)p^2 + (c+e)p + f\right)\right)^{-1} - G_{1d}\right); \text{ and}$$
$$Y_{1s}(p, q) = \frac{1}{2\alpha}\left(\left(q + \left((a+b-1)p^2 + (c+e)p + f\right)\right)^{-1} - G_{1s}\right).$$

The underlying's market clears at $P_1 = P_1^{nd}$, the same as in the benchmark (Equation (6)). The derivative's market clears at $Q_1 = G_1^{-1} - (a+b-1)P_1^2 - (c+e)P_1 - f$. The conditional precision $\{G_{1d}, G_{1s}, G_1\}$ are the same as those defined in Proposition 1.

Proof. Consider a type-j investor. Her terminal wealth W_{2j} is given by

$$W_{2j} = W_0 + (P_1 - P_0)X_0 + (D - P_1)X_{1j} + (f(D) - Q_1)Y_{1j} + (D - \bar{D})z_j. \tag{S1}$$

Lemma 1 ensures that she holds the same posterior distribution for D with or without the derivative. In particular, D remains conditionally normal. Let $z_s = 0$, $z_d = z$, and $W_1 = W_0 + (p - P_0)X_0$.

Evaluating the expected utility (e.g., Lemma A.1 of Marín and Rahi (1999)) yields,

$$
\begin{aligned}
&\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right] \\
&= -\frac{1}{\sqrt{1+2\alpha \mathrm{var}_{1j}[D]Y_{1j}}} \exp\left[\alpha\left(-W_1 + z_j(\bar{D}-p) - Y_{1j}\left((1-a-b)p^2 - (c+e)p - f - q\right)\right)\right] \\
&\quad \cdot \exp\left[-\alpha\left(X_{1j}+z_j+((2-b)p-c)Y_{1j}\right)(\mathbb{E}_{1j}[D]-p) - \alpha Y_{1j}(\mathbb{E}_{1j}[D]-p)^2\right] \\
&\quad \cdot \exp\left[\frac{\alpha^2 \mathrm{var}_{1j}[D]\left(X_{1j}+z_j+Y_{1j}(2\mathbb{E}_{1j}[D]-bp-c)\right)^2}{2\left(1+2\alpha \mathrm{var}_{1j}[D]Y_{1j}\right)}\right].
\end{aligned}
$$

The first-order condition with respect to X_{1j} yields

$$X_{1j} = \frac{\mathbb{E}_{1j}[D]-p}{\alpha \mathrm{var}_{1j}[D]} - z_j - ((2-b)p-c)Y_{1j}.$$

Plug this back to $\mathbb{E}_{1j}[-e^{-\alpha W_{2j}}]$ and evaluate the first-order condition with respect to Y_{1j} to get:

$$Y_{1j} = \frac{1}{2\alpha}\left(\frac{1}{q+(a+b-1)p^2+(c+e)p+f} - \frac{1}{\mathrm{var}_{1j}[D]}\right).$$

Finally, clearing the market yields the equilibrium prices $p = P_1$ and $q = Q_1$ as stated in the proposition. (The utility maximization problem is a strictly concave one. Hence, the above solution implied by the first-order conditions is unique.) □

With the variance swap $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$, the two liquidity measures can be found, following Proposition 2, as

$$
\begin{aligned}
\lambda &= \frac{\alpha}{1-\pi}\frac{G_1-G_0}{G_1-G_{1s}}\frac{1}{G_1-0.5b(G_1-G_0)} = \frac{G_0}{G_1-\frac{b}{2}(G_1-G_0)}\lambda^{nd} \quad \text{and} \\
\gamma &= \left(1-\frac{G_0}{G_1}\right)\left(1-\frac{G_{1s}}{G_1}\right)\frac{1}{G_{1s}-G_0} = \gamma^{nd}.
\end{aligned}
$$

As can be seen, the possibly path-dependent derivative might change the result of $\lambda < \lambda^{nd}$ from Corollary 1. This happens if and only if the coefficient $b \geq 2$, i.e., the loading on DP_1. This is because with such path-dependent derivatives, the built-in dependence of $f(D)$ on P_1 creates some "mechanical" delta-hedging needs for the investors. In the quadratic example above, the total delta-hedging ratio is $\hat{\mathbb{E}}_{1j}\left[\frac{\partial f}{\partial D}\right] = 2P_1 - bP_1 - c$, and we can see that the term $-bP_1$ contributes to it, simply because of the built-in interaction between the actual terminal payoff D and the intermediate price P_1. In particular, when $b \geq 2$, the sign of the delta-hedging ratio above changes, mechanically affecting the information-to-noise ratio in the underlying and, hence, also the price impact λ.

On the other hand, the price reversal γ is unaffected, because for both types of investors, $j \in \{d, s\}$, the above delta hedging ratio remains the same. Hence, the net delta-hedging trading remains zero, as in the case of a path-independent variance swap in the paper, and there is no additional price pressure, ensuring $P_1 = P^{nd}$. As such, γ remains unaffected.

Proposition S2. *With $f(D) = D^2 - aP_1^2 - bDP_1 - cD - eP_1 - f$, the underlying's $t = 0$ equilibrium price remains the same as stated in Proposition 6.*

Proof. The proof of Proposition S1 gives an investor's expected utility at $t = 1$, $\mathbb{E}_{1j}\left[-e^{-\alpha W_{2j}}\right]$, taking $p = P_1$ and $q = Q_1$ as given. Consider a demander ($j = d$) first. Expanding with $W_1 = W_0 + (P_1 - P_0)X_0$ and $\mathbb{E}_{1d}[D]$ with s and z gives

$$\mathbb{E}_{1d}\left[-e^{-\alpha W_{2d}}\right] = -\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}} \cdot e^{-\alpha \cdot (W_0 + (P_1 - P_0)X_0 + (P_1 - \bar{D})z)} \cdot e^{-\frac{G_{1d}}{2}\left(\bar{D} + \frac{G_{1d} - G_0}{G_{1d}}(s - \bar{D}) - P_1\right)^2}$$

where $P_1 = P_1^{nd}$ can be further written as a linear combination of s and z. Taking the expectation of the above over $\{s, z\}$ yields the "interim" utility U_{0d} of a demander; that is, the expected utility after the type realizes but before the signal and the endowment shock are observed:

$$U_{0d} = -\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1} - \alpha W_{0d}} \left(1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2 \left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right) - \frac{\alpha^2}{G_0 \tau_z}\right)^{-\frac{1}{2}},$$

where

$$\begin{aligned} W_{0d} :=& W_0 + (\bar{D} - P_0)X_0 - \frac{\alpha}{G_0}X_0\bar{X} + \frac{\alpha}{2G_0}\bar{X}^2 - \frac{\alpha}{2}\left[1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2\left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right) - \frac{\alpha^2}{G_0 \tau_z}\right]^{-1} \\ &\cdot \left\{\left(\frac{G_1 - G_0}{G_1}\right)^2\left(\frac{1}{G_0} + \frac{1}{\tau_\varepsilon}\right)\left(1 + \frac{\alpha^2}{\tau_\varepsilon \tau_z}\right)(X_0 - \bar{X})^2 \right. \\ &\left. + \left(\frac{1}{G_0} + \frac{1}{\tau_\varepsilon}\right)^2 \frac{\alpha^2}{\tau_z}\left[2\left(1 - \frac{G_0}{G_1}\right)\left(1 - \frac{G_0}{G_{1d}}\right)X_0\bar{X} + \left[\frac{G_0}{G_{1d}}\left(1 - \frac{G_{1d}}{G_1}\right)^2 - \left(1 - \frac{G_0}{G_1}\right)^2\right]\bar{X}^2\right]\right\}. \end{aligned}$$

Note that condition $\alpha^2 G_0^{-1}\tau_z^{-1} < 1$ ensures U_{0d} is well-defined; in particular, the term inside the brackets is always positive. Similarly, the interim utility U_{0s} of liquidity suppliers can be derived as

$$U_{0s} = -\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1} - \alpha W_{0s}} \left(1 + \frac{G_0}{G_{1s} - G_0}\left(1 - \frac{G_{1s}}{G_1}\right)^2\right)^{-\frac{1}{2}},$$

where

$$W_{0s} = W_0 + (\bar{D} - P_0)X_0 - \frac{\alpha}{2G_0}X_0^2 + \frac{\alpha G_{1s}}{2G_1^2}\left[1 + \frac{G_0}{G_{1s} - G_0}\left(1 - \frac{G_{1s}}{G_1}\right)^2\right]^{-1} \cdot \left(X_0 - \bar{X}\right)^2.$$

At $t = 0$, investors choose X_0 to maximize

$$\pi U_{0d} + (1 - \pi)U_{0s}.$$

The first-order condition, together with the market clearing condition $X_0 = \bar{X}$, leads to

$$\pi \cdot \left(\bar{D} - p - \alpha G_0^{-1}\bar{X} - \alpha\Sigma\bar{X}\right)M + (1 - \pi)\left(\bar{D} - p - \alpha G_0^{-1}\bar{X}\right) = 0,$$

where

$$M = e^{\frac{G_{1s} - G_{1d}}{2G_1}} \sqrt{\frac{G_{1d}}{G_{1s}}} \exp\left(\frac{\alpha}{2}\Delta_2\bar{X}^2\right)\sqrt{\frac{1 + \pi^2\Delta_0}{1 + \Delta_0(1 - \pi)^2 - \alpha^2/(\tau_z G_0)}},$$

and Δ_0 and Δ_2 are the same coefficients as given in the proof of Propositions 5 and 1. (Note that the second-order conditions are satisfied as well as both U_{0d} and U_{0s} are monotone transformations of quadratic terms in X_0.) It can be seen that the above first-order condition is linear in the market clearing price p, which then uniquely solves the equilibrium P_0 stated in the proposition.

Conditional on the realization of P_0, in the no-derivative benchmark, following Vayanos and Wang (2012), the liquidity demanders' interim utility is

$$U_{0d}^{nd} = -e^{-\alpha W_{0d}^{nd}}\left(1 + \frac{G_0}{G_{1d} - G_0}\left(1 - \frac{G_{1d}}{G_1}\right)^2\left(1 + \frac{\alpha^2}{\tau_\varepsilon\tau_z}\right) - \frac{\alpha^2}{G_0\tau_z}\right)^{-\frac{1}{2}},$$

where $W_{0d}^{nd} = W_{0d}$. As $0 < \frac{G_1}{G_{1d}} < 1$, $\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}}$ is a decreasing function of $\frac{G_{1d}}{G_1}$. Therefore, $\sqrt{\frac{G_{1d}}{G_1}} \cdot e^{\frac{G_1 - G_{1d}}{2G_1}} < 1$ and $U_{0d} > U_{0d}^{nd}$. Likewise, for the liquidity suppliers, $\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1}}$ is an increasing function of $\frac{G_{1s}}{G_1}$ because $\frac{G_1}{G_{1s}} > 1$. Then $\sqrt{\frac{G_{1s}}{G_1}} \cdot e^{\frac{G_1 - G_{1s}}{2G_1}} < 1$ and $U_{0s} > U_{0s}^{nd}$. □

As we have seen above, while the path-dependence of the derivative payoff affects an individual investor's delta hedging at $t = 1$, in aggregate, the net delta-hedging trade remains zero. As such, intuitively, the path-dependent derivative does not create additional trading gains nor does it affect the split of the "pie." One step back to $t = 0$, therefore, the evaluation of the underlying asset is unaffected.

S2 Additional lemmas

Lemma S1

> **Lemma S1 (Decomposition of a call).** *Suppose the underlying price at $t = 1$ is P_1. The $t = 2$ payoff of an out-of-the-money call option with strike $K \geq P_1$ can be decomposed into*
>
> $$\max\{0, D - K\} = \frac{1}{2}|D - P_1| + \frac{1}{2}\left(1 - 2\mathbb{1}_{\{P_1 \leq D \leq K\}}\right)(D - P_1) + \mathbb{1}_{\{D>K\}}(P_1 - K);$$
>
> *and that of an in-the-money call with $K \leq P_1$ can be decomposed into*
>
> $$\max\{0, D - K\} = \frac{1}{2}|D - P_1| + \frac{1}{2}\left(1 + 2\mathbb{1}_{\{K \leq D \leq P_1\}}\right)(D - P_1) + \mathbb{1}_{\{D>K\}}(P_1 - K).$$

Proof. Consider the out-of-the-money call with $K \geq P_1$.

$$\max\{0, D - K\} = \frac{1}{2}|D - K| + \frac{1}{2}(D - K) = \frac{1}{2}|V - (K - P_1)| + \frac{1}{2}(V - (K - P_1))$$

where $V := D - P_1$ as a shorthand notation. Compare $|V - (K - P_1)|$ to $|V|$:

$$|V - (K - P_1)| - |V| = \begin{cases} K - P_1, & \text{if } V < 0 \\ -2V + (K - P_1), & \text{if } 0 \leq V \leq K - P_1 \\ -(K - P_1), & \text{if } V > K - P_1 \end{cases}.$$

Therefore, $|V - (K - P_1)| = |V| + \mathbb{1}_{\{V<0\}}(K - P_1) + \mathbb{1}_{\{0 \leq V \leq K - P_1\}}(-2V + K - P_1) - \mathbb{1}_{\{V > K - P_1\}}(K - P_1)$. Substituting into the call's payoff expression and simplifying gives the expression stated in the lemma. The proof for the decomposition of the in-the-money call repeats the above steps and is omitted. □

Lemma S2

Lemma S2 (Risk-neutral pricing). *The equilibrium underlying price* P_1 *and the derivative price* Q_1 *must satisfy*

$$P_1 = \hat{\mathbb{E}}_{1j}[D] = \int_{\mathbb{R}} D\hat{\phi}_{1j}(D)\mathrm{d}D \ \ and \ \ Q_1 = \hat{\mathbb{E}}_{1j}[f(D)] = \int_{\mathbb{R}} f(D)\hat{\phi}_{1j}(D)\mathrm{d}D \tag{S2}$$

where $\hat{\phi}_{1j}(D)$ *is a type-*j *investor's risk-neutral density, defined as*

$$\hat{\phi}_{1j}(D) := \frac{h_{1j}(D)}{\int_{\mathbb{R}} h_{1j}(D)\mathrm{d}D}, \ with \ h_{1j}(D) := e^{-\alpha\cdot\left(DX_{1j}+f(D)Y_{1j}\right)-\frac{G_{1j}}{2}D^2+(G_0\bar{D}+(G_{1j}-G_0)\eta)D}. \tag{S3}$$

Proof. The risk-neutral pricing formulas follow the first-order conditions

$$\frac{\partial U_{1j}}{\partial X_{1j}} = \mathbb{E}_{1j}\left[\alpha\cdot(D-P_1)e^{-\alpha W_{2j}}\right] = 0 \text{ and } \frac{\partial U_{1j}}{\partial Y_{1j}} = \mathbb{E}_{1j}\left[\alpha\cdot(f(D)-Q_1)e^{-\alpha W_{2j}}\right] = 0$$

which imply

$$P_1 = \frac{\mathbb{E}_{1j}[De^{-\alpha W_{2j}}]}{\mathbb{E}[e^{-\alpha W_{2j}}]} = \frac{\int_{\mathbb{R}} De^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}{\int_{\mathbb{R}} e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D} \text{ and } Q_1 = \frac{\mathbb{E}_{1j}[f(D)e^{-\alpha W_{2j}}]}{\mathbb{E}[e^{-\alpha W_{2j}}]} = \frac{\int_{\mathbb{R}} f(D)e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}{\int_{\mathbb{R}} e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}$$

where $\phi_{1j}(D)$ is the type-j investor's posterior density (conditional on the prices) of D. Letting $\hat{\phi}_{1j}(D) := \frac{e^{-\alpha W_{2j}}\phi_{1j}(D)}{\int_{\mathbb{R}} e^{-\alpha W_{2j}}\phi_{1j}(D)\mathrm{d}D}$, one obtains the risk-neutral pricing formula given in the lemma. It remains to simplify the expression of $\hat{\phi}_{1j}(D)$. To do so, recall $W_{2j} = W_1 + (D-P_1)X_{1j} + (f(D)-Q_1)Y_{1j} + (D-\bar{D})z_j$, where $W_1 = W_0 + (P_1 - P_0)X_0$ and z_j is a type-j investor's endowment shock ($z_d = z$ and $z_s = 0$). In addition, by Lemma 1, $\phi_{1d}(D)$ is the normal density with mean $\frac{G_0}{G_{1d}}\bar{D}+\frac{G_{1d}-G_0}{G_{1d}}s$ and variance G_{1d}^{-1}; and $\phi_{1s}(D)$ is the normal density with mean $\frac{G_0}{G_{1s}}\bar{D} + \frac{G_{1s}-G_0}{G_{1s}}\eta$ (with $\eta := s - \frac{\alpha}{\tau_\varepsilon}z$) and variance G_{1s}^{-1}. The simplified expression of $\hat{\phi}_{1j}(D)$ with $h_{1j}(D)$ follows by plugging these expressions into $\hat{\phi}_{1j}(D)$ and offsetting common terms in the numerator and the denominator. □