Appendix D: Detail of calculations involved in the determination of the inertial lift force

In the case of a fluid at rest at infnity, the decomposition of the right-hand side of Eq. (42) into a set of elementary contributions yields

$$
\begin{aligned}
&4\pi R_\mu\left(\sum_{n=0}^{2}(-1)^n J_D^n\right)V_{B3}^{(G)} = \\
&\quad\left(V_{B1}^{(0)}\right)^2\Big\{R_\mu^2 K_0 E_0^1 - \kappa R_\mu^3 K_1 E_1^1 + \kappa^2 R_\mu D_\mu E_2^1 \\
&\quad + 2R_S I_S\left[R_\mu\left(R_\mu I_2 - I_1\right) + D_\mu\left(D_\mu I_6 - I_5\right) + R_\mu D_\mu I_7\right] \\
&\quad + R_S J_S\left[R_\mu\left(R_\mu I_4 - I_3\right) + D_\mu\left(D_\mu I_8 - I_9\right) + R_\mu D_\mu I_{10}\right] \qquad \text{(D1)} \\
&\quad - J_S\Big[R_\mu\left(J_2 - R_\mu J_1\right) + R_\mu D_\mu\left(I_{21} - 6\left(I_{12} - R_\mu I_{11}\right)\right) - 6D_\mu^2\left(I_{14} - D_\mu I_{13}\right) \\
&\quad + 6R_\mu D_\mu^2 I_{15} + 3\bar{\rho} M_\mu^3 H_1\Big] - I_S\Big[R_\mu\left(J_4 - R_\mu J_3\right) + R_\mu D_\mu\left(I_{20} - 4R_\mu I_{16}\right) \\
&\quad + 4D_\mu^2\left(I_{17} - D_\mu I_{18}\right) - 4R_\mu D_\mu^2 I_{19} + \bar{\rho} M_\mu^3 H_2\Big]\Big\}
\end{aligned}
$$

where the E_i^j are three integrals to be evaluated in the entire domain $\overline{V_\mathrm{O}} \cup \overline{V_\mathrm{I}} \cup \overline{V_\mathrm{B}}$, the I_i are twenty-one nonzero integrals to be evaluated within $V_\mathrm{I} \cup V_\mathrm{O}$, the J_i and H_i are six integrals to be evaluated within V_B, and the K_i are defined by

$$K_0 = 1 - I_D - J_D + I_D^2 + J_D^2 + I_D J_D\ , \ K_1 = 1 - 2I_D - J_D.$$

To express the "outer" integrals we first define the following velocity fields

$$\overline{\mathbf{U}}_1 = -\frac{1}{2}\left(\frac{\mathbf{e}_1}{\bar{\tau}} + \frac{\overline{x_1}\overline{\mathbf{x}}}{\bar{\tau}^3}\right) + \frac{1+\overline{x_3}}{\bar{\tau}^5}\left(3\overline{x_1}\left(\overline{\mathbf{x}} + 2\mathbf{e}_3\right) - \bar{\tau}^2\mathbf{e}_3\right)\ ,\ \ \overline{\mathbf{U}}_{\mathrm{Sto}} = \overline{\mathbf{U}}_1 + \frac{1}{2}\left(\frac{\mathbf{e}_1}{\bar{r}} + \frac{\overline{x_1}\overline{\mathbf{x}}}{\bar{r}^3}\right)$$

$$\overline{\mathbf{U}}_3 = -\frac{\mathbf{e}_1}{\bar{\tau}^3} + 3\frac{\overline{x_1}\overline{\mathbf{x}}}{\bar{\tau}^5} + 6\frac{\left(1+\overline{x_3}\right)\left(2+\overline{x_3}\right)}{\bar{\tau}^7}\left[\left(\bar{\tau}^2\mathbf{e}_1 - 5\overline{x_1}\left(\overline{\mathbf{x}} + 2\mathbf{e}_3\right)\right)\right],\ \ \overline{\mathbf{U}}_{\mathrm{Dip}} = \overline{\mathbf{U}}_3 + \frac{\mathbf{e}_1}{\bar{r}^3} - 3\frac{\overline{x_1}\overline{\mathbf{x}}}{\bar{r}^5}$$

$$\overline{\mathbf{u}}_1 = -\frac{1}{2}\left(\frac{\mathbf{e}_3}{\bar{\tau}} + \frac{\overline{x_3}\overline{\mathbf{x}}}{\bar{\tau}^3}\right) - \frac{1+\overline{x_3}}{\bar{\tau}^5}\left(3\left(\overline{x_3} + 2\right)\left(\overline{\mathbf{x}} + 2\mathbf{e}_3\right) + \bar{\tau}^2\mathbf{e}_3\right),\ \overline{\mathbf{u}}_{\mathrm{Sto}} = \overline{\mathbf{u}}_1 + \frac{1}{2}\left(\frac{\mathbf{e}_3}{\bar{r}} + \frac{\overline{x_3}\overline{\mathbf{x}}}{\bar{r}^3}\right)$$

$$\bar{\mathbf{u}}_3 = -\frac{\mathbf{e}_3}{\bar{\tau}^3} - 3\frac{(\bar{x}_3+2)\bar{\mathbf{x}}}{\bar{\tau}^5} - 6\frac{1+\bar{x}_3}{\bar{\tau}^7}\left(\left(\bar{\tau}^2 - 5(\bar{x}_3+2)^2\right)(\bar{\mathbf{x}}+2\mathbf{e}_3) + (\bar{x}_3+2)\bar{\tau}^2\mathbf{e}_3\right)$$

$$\bar{\mathbf{u}}_{\mathrm{Dip}} = \bar{\mathbf{u}}_3 + \frac{\mathbf{e}_3}{\bar{r}^3} - 3\frac{\bar{x}_3\bar{\mathbf{x}}}{\bar{r}^5}$$

Then, the "outer" integrals are

$$E_0^1 = \int_{\bar{V}_\mathrm{O}\cup\bar{V}_\mathrm{I}\cup\bar{V}_\mathrm{B}} \left[\mathbf{e}_1.\bar{\nabla}\bar{\mathbf{U}}_{\mathrm{Sto}}\right].\bar{\mathbf{u}}_{\mathrm{Sto}}\mathrm{d}^3\bar{\mathbf{x}}\ ,\quad E_1^1 = \int_{\bar{V}_\mathrm{O}\cup\bar{V}_\mathrm{I}\cup\bar{V}_\mathrm{B}} \left[\bar{\mathbf{U}}_{\mathrm{Sto}}.\bar{\nabla}\bar{\mathbf{U}}_{\mathrm{Sto}}\right].\bar{\mathbf{u}}_{\mathrm{Sto}}\mathrm{d}^3\bar{\mathbf{x}},$$

$$E_2^1 = \int_{\bar{V}_\mathrm{O}\cup\bar{V}_\mathrm{I}\cup\bar{V}_\mathrm{B}} \left\{\left[\mathbf{e}_1.\bar{\nabla}\bar{\mathbf{U}}_{\mathrm{Dip}}\right].\bar{\mathbf{u}}_{\mathrm{Sto}} + \left[\mathbf{e}_1.\bar{\nabla}\bar{\mathbf{U}}_{\mathrm{Sto}}\right].\bar{\mathbf{u}}_{\mathrm{Dip}}\right\}\mathrm{d}^3\bar{\mathbf{x}}\ .$$

Similarly, to express the "inner" integrals, we define the elementary velocity fields

$$\mathbf{U}_{\mathrm{Sto}} = \frac{1}{2}\left(\frac{\mathbf{e}_1}{r} + \frac{x_1\mathbf{x}}{r^3}\right)\ ,\ \mathbf{U}_{\mathrm{Dip}} = \frac{\mathbf{e}_1}{r^3} - 3\frac{x_1\mathbf{x}}{r^5}\ ,\ \ \mathbf{U}_\mathrm{L} = x_3\mathbf{e}_1 + x_1\mathbf{e}_3\ ,$$

$$\mathbf{U}_{\mathrm{Qua}} = \frac{x_3\mathbf{e}_1}{r^5} + \frac{x_1\mathbf{e}_3}{r^5} - 5\frac{x_1x_3\mathbf{x}}{r^7},\ \ \mathbf{U}_{\mathrm{Str}} = \frac{x_1x_3\mathbf{x}}{r^5}\ ,$$

$$\tilde{\mathbf{U}}_{\mathrm{Sti}} = \left(1-2r^2\right)\mathbf{e}_1 + x_1\mathbf{x}\ ,\ \ \tilde{\mathbf{U}}_{\mathrm{Sri}} = \left(5r^2-3\right)\left(x_3\mathbf{e}_1 + x_1\mathbf{e}_3\right) - 4x_1x_3\mathbf{x}$$

$$\mathbf{u}_{\mathrm{Sto}} = \frac{1}{2}\left(\frac{\mathbf{e}_3}{r} + \frac{x_3\mathbf{x}}{r^3}\right)\ ,\ \mathbf{u}_{\mathrm{Dip}} = \frac{\mathbf{e}_3}{r^3} - 3\frac{x_3\mathbf{x}}{r^5}\ ,\ \ \mathbf{u}_\mathrm{L} = \mathbf{x} - 3x_3\mathbf{e}_3\ ,$$

$$\mathbf{u}_{\mathrm{Str}} = \frac{\mathbf{x}}{r^3} - 3\frac{x_3^2\mathbf{x}}{r^5}\ ,\ \ \mathbf{u}_{\mathrm{Qua}} = \frac{\mathbf{x}}{r^5} + 2\frac{x_3\mathbf{e}_3}{r^5} + -5\frac{x_3^2\mathbf{x}}{r^7}\ ,$$

$$\tilde{\mathbf{u}}_{\mathrm{Sti}} = \left(1-2r^2\right)\mathbf{e}_3 + x_3\mathbf{x}\ ,\ \ \tilde{\mathbf{u}}_{\mathrm{Sri}} = \left(r^2 + 2x_3^2 - 1\right)\mathbf{x} + \left(3-5r^2\right)x_3\mathbf{e}_3.$$

With these definitions, the "inner" integrals involved in Eq. (D1) are

$$I_1 = \int_{V_\mathrm{I}\cup V_\mathrm{O}} \left[\mathbf{e}_1.\nabla\mathbf{U}_{\mathrm{Str}}\right].\mathbf{u}_{\mathrm{Sto}}d^3\mathbf{x} = \frac{2}{15}\pi\ ,$$

$$I_2 = \int_{V_\mathrm{I}\cup V_\mathrm{O}} \left[\mathbf{U}_{\mathrm{Sto}}.\nabla\mathbf{U}_{\mathrm{Str}} + \mathbf{U}_{\mathrm{Str}}.\nabla\mathbf{U}_{\mathrm{Sto}}\right].\mathbf{u}_{\mathrm{Sto}}d^3\mathbf{x} = -\frac{1}{5}\pi\ ,$$

$$I_3 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{e}_1.\nabla \mathbf{U}_{\mathrm{Sto}}].\mathbf{u}_{\mathrm{Str}} d^3\mathbf{x} = -\frac{4}{5}\pi \ , \quad I_4 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Sto}}.\nabla \mathbf{U}_{\mathrm{Sto}}].\mathbf{u}_{\mathrm{Str}} d^3\mathbf{x} = -\frac{1}{3}\pi \ ,$$

$$I_5 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{e}_1.\nabla \mathbf{U}_{\mathrm{Str}}].\mathbf{u}_{\mathrm{Dip}} d^3\mathbf{x} = -\frac{4}{15}\pi \ ,$$

$$I_6 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Str}} + \mathbf{U}_{\mathrm{Str}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Dip}} d^3\mathbf{x} = -\frac{16}{15}\pi \ ,$$

$$I_7 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} \left\{ [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Str}} + \mathbf{U}_{\mathrm{Str}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Sto}} + [\mathbf{U}_{\mathrm{Sto}}.\nabla \mathbf{U}_{\mathrm{Str}} + \mathbf{U}_{\mathrm{Str}}.\nabla \mathbf{U}_{\mathrm{Sto}}].\mathbf{u}_{\mathrm{Dip}} \right\} d^3\mathbf{x} = \frac{17}{15}\pi \ ,$$

$$I_8 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Str}} d^3\mathbf{x} = -\frac{4}{5}\pi \ , \quad I_9 = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{e}_1.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Str}} d^3\mathbf{x} = \frac{8}{5}\pi \ ,$$

$$I_{10} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Sto}} + \mathbf{U}_{\mathrm{Sto}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Str}} d^3\mathbf{x} = \frac{6}{5}\pi \ ,$$

$$I_{11} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Sto}}.\nabla \mathbf{U}_{\mathrm{Sto}}].\mathbf{u}_{\mathrm{Qua}} d^3\mathbf{x} = -\frac{2}{15}\pi \ ,$$

$$I_{12} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{e}_1.\nabla \mathbf{U}_{\mathrm{Sto}}].\mathbf{u}_{\mathrm{Qua}} d^3\mathbf{x} = -\frac{4}{15}\pi \ , \quad I_{13} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Qua}} d^3\mathbf{x} = -\frac{4}{5}\pi \ ,$$

$$I_{14} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} \left\{ [\mathbf{e}_1.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Qua}} - \frac{1}{6}[\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{L}} \right\} d^3\mathbf{x} = \frac{26}{15}\pi \ ,$$

$$I_{15} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Sto}} + \mathbf{U}_{\mathrm{Sto}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Qua}} d^3\mathbf{x} = \frac{16}{15}\pi \ ,$$

$$I_{16} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Qua}}.\nabla \mathbf{U}_{\mathrm{Sto}} + \mathbf{U}_{\mathrm{Sto}}.\nabla \mathbf{U}_{\mathrm{Qua}}].\mathbf{u}_{\mathrm{Sto}} d^3\mathbf{x} = \frac{\pi}{2} \ ,$$

$$I_{17} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} \left\{ [\mathbf{e}_1.\nabla \mathbf{U}_{\mathrm{Qua}}].\mathbf{u}_{\mathrm{Dip}} + \frac{1}{4}[\mathbf{U}_{\mathrm{L}}.\nabla \mathbf{U}_{\mathrm{Dip}} + \mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{L}}].\mathbf{u}_{\mathrm{Dip}} \right\} d^3\mathbf{x} = -\frac{2}{15}\pi \ ,$$

$$I_{18} = \int\limits_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} [\mathbf{U}_{\mathrm{Dip}}.\nabla \mathbf{U}_{\mathrm{Qua}} + \mathbf{U}_{\mathrm{Qua}}.\nabla \mathbf{U}_{\mathrm{Dip}}].\mathbf{u}_{\mathrm{Dip}} d^3\mathbf{x} = \frac{24}{5}\pi \ ,$$

$$I_{19} = \int_{V_I \cup V_O} \left\{ \left[\mathbf{U}_{Qua} . \nabla \mathbf{U}_{Dip} + \mathbf{U}_{Dip} . \nabla \mathbf{U}_{Qua} \right] . \mathbf{u}_{Sto} + \left[\mathbf{U}_{Qua} . \nabla \mathbf{U}_{Sto} + \mathbf{U}_{Sto} . \nabla \mathbf{U}_{Qua} \right] . \mathbf{u}_{Dip} \right\} d^3\mathbf{x}$$

$$= -\frac{18}{5}\pi \ ,$$

$$I_{20} = \int_{V_I \cup V_O} \left\{ \left[\mathbf{U}_{Dip} . \nabla \mathbf{U}_{L} + \mathbf{U}_{L} . \nabla \mathbf{U}_{Dip} \right] . \mathbf{u}_{Sto} + \left[\mathbf{U}_{Sto} . \nabla \mathbf{U}_{L} + \mathbf{U}_{L} . \nabla \mathbf{U}_{Sto} \right] . \mathbf{u}_{Dip} \right\} d^3\mathbf{x} = \frac{16}{15}\pi,$$

$$I_{21} = \int_{V_I \cup V_O} \left[\mathbf{U}_{Dip} . \nabla \mathbf{U}_{Sto} + \mathbf{U}_{Sto} . \nabla \mathbf{U}_{Dip} \right] . \mathbf{u}_{L} \, d^3\mathbf{x} = \frac{12}{5}\pi \ ,$$

$$J_1 = \int_{V_B} \left[\mathbf{U}_{Sto} . \nabla \mathbf{U}_{Sto} \right] . \mathbf{u}_{L} \, d^3\mathbf{x} = -\frac{13}{15}\pi \ , \quad J_2 = \int_{V_B} \left[\mathbf{e}_1 . \nabla \mathbf{U}_{Sto} \right] . \mathbf{u}_{L} \, d^3\mathbf{x} = -\frac{2}{5}\pi \ ,$$

$$J_3 = \int_{V_B} \left[\mathbf{U}_{Sto} . \nabla \mathbf{U}_{L} + \mathbf{U}_{L} . \nabla \mathbf{U}_{Sto} \right] . \mathbf{u}_{Sto} d^3\mathbf{x} = 2\pi \ , \quad J_4 = \int_{V_B} \left[\mathbf{e}_1 . \nabla \mathbf{U}_{L} \right] . \mathbf{u}_{Sto} d^3\mathbf{x} = \frac{4}{3}\pi \ ,$$

$$H_1 = \int_{V_B} \left[\tilde{\mathbf{U}}_{Sti} . \nabla \tilde{\mathbf{U}}_{Sti} \right] . \tilde{\mathbf{u}}_{Sri} d^3\mathbf{x} = 0 \ , \quad H_2 = \int_{V_B} \left[\tilde{\mathbf{U}}_{Sti} . \nabla \tilde{\mathbf{U}}_{Sri} + \tilde{\mathbf{U}}_{Sri} . \nabla \tilde{\mathbf{U}}_{Sti} \right] . \tilde{\mathbf{u}}_{Sti} d^3\mathbf{x} = 0.$$

To evaluate the right-hand side of Eq. (D1), the final step consists in calculating the "outer" integrals E_i^j. Cox & Hsu (1977) Fourier-transformed the integral equivalent to E_0^1 and evaluated it analytically by using contour integration and polar coordinates. To save time we have evaluated all three integrals E_i^j numerically after having reduced them to double integrals by setting $\overline{x_1} = \bar{\rho}\cos\psi$, $\overline{x_2} = \bar{\rho}\sin\psi$ in the integrands and integrating analytically with respect to ψ. Note that while these integrals are still singular at the origin, the overall contribution of the remaining singularity is always zero. Consequently, there is no particular difficulty in their evaluation, provided the collocation points belonging to the interval $x_3 \in \left]-1,1\right[$ are distributed symmetrically with respect to the origin. Numerical integration yielded[1] $4\,E_0^1/\pi \approx 1.000$, $8\,E_1^1/\pi = C \approx 2.000$, $2E_2^1/\pi = \mathcal{D} \approx 0.864$. Since the numerical value of E_0^1

[1] The prefactors $\pi/2$, $\pi/4$ and $\pi/8$ result from the analytical integration with respect to ψ.

agrees with the exact value $\pi/4$ obtained by Cox & Hsu (1977) up to three decimals, we infer that all three values determined numerically have a similar accuracy. Inserting all intermediate results into Eq. (D1), we finally obtain the inertial lift force in the form

$$\mathbf{F}_{\mathrm{M}}^{(G)} = \frac{\pi}{4}\left(V_{B1}^{(0)}\right)^2 R_\mu^2 \left\{1 + \frac{R_\mu}{8}\kappa + C_2(\lambda)\kappa^2\right\}\mathbf{e}_3 + O\left(\kappa^3\right) \tag{D2}$$

with

$$C_2(\lambda) = -\frac{33}{64}R_\mu^2 + \frac{D_\mu}{R_\mu}\left(2\mathcal{D} + \frac{160 + 338\lambda + 169\lambda^2}{40\ (1+\lambda)^2}\right) \tag{D3}$$

In the case of a linear shear flow, we evaluate the right-hand side of Eq. (42) truncated at $O(\kappa)$ in order to limit the volume of the calculations. We then obtain

$$\begin{aligned}
&4\pi R_\mu(1-J_D)V_{B3}^{(G)} = \\
&\left(V_{B1}^{(0)}\right)^2\left(R_\mu^2(1-I_D-J_D)E_0^1 - \kappa R_\mu^3 E_1^1\right) - \alpha^2\left(\kappa^{-1}R_\mu^2 R_S I_{SD} F_{-1}^1 - R_\mu R_S(1-J_D)F_0^1\right) \\
&-\alpha V_{B1}^{(0)}\Big\{\kappa^{-1}R_\mu^2 K_0 F_{-1}^1 + \kappa R_\mu\left(R_S F_1^1 + D_\mu F_1^2\right) \\
&\quad + K_2\Big[R_\mu\left(R_S\left(I_1 - R_\mu I_2\right) + D_\mu\left(I_{27} - R_S I_7\right) - R_\mu J_5\right) + \frac{\overline{\rho}}{2}M_\mu^2\left(M_\mu H_2 + H_3\right) \\
&\quad + D_\mu\left(R_S\left(I_5 - D_\mu I_6\right) + D_\mu I_{26} - 2\left(R_\mu^2 I_{16} + R_\mu D_\mu I_{19} + D_\mu^2 I_{18}\right)\right)\Big] \\
&\quad - J_D\left(R_S\left(R_\mu I_{24} + D_\mu I_{25}\right) + 2D_\mu\left(R_\mu I_{22} + D_\mu I_{23}\right) + R_\mu J_6\right)\Big\}
\end{aligned} \tag{D4}$$

with $K_2 = 1 - I_D - J_D$. The outer integrals involved in Eq. (D4) are

$$F_{-1}^1 = \int_{\overline{V_\mathrm{O}}\cup\overline{V_\mathrm{I}}\cup\overline{V_\mathrm{B}}}\left[\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{U}}_{\mathrm{Sto}} + \mathbf{e}_1\left(\overline{\mathbf{U}}_{\mathrm{Sto}}.\mathbf{e}_3\right)\right].\overline{\mathbf{u}}_{\mathrm{Sto}}\,d^3\overline{\mathbf{x}}\ ,$$

$$F_0^1 = \int_{\overline{V_\mathrm{O}}\cup\overline{V_\mathrm{I}}\cup\overline{V_\mathrm{B}}}\left[\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{U}}_{\mathrm{Str}} + \mathbf{e}_1\left(\overline{\mathbf{U}}_{\mathrm{Str}}.\mathbf{e}_3\right)\right].\overline{\mathbf{u}}_{\mathrm{Sto}}\,d^3\overline{\mathbf{x}}\ ,$$

$$F_1^1 = \int_{\overline{V_\mathrm{O}}\cup\overline{V_\mathrm{I}}\cup\overline{V_\mathrm{B}}}\left\{\left[\mathbf{e}_1.\overline{\nabla}\overline{\mathbf{U}}_2\right].\overline{\mathbf{u}}_{\mathrm{Sto}} + \left[\mathbf{e}_1.\overline{\nabla}\left(\frac{\overline{x_1}\,\overline{x_3}\,\overline{\mathbf{x}}}{\overline{r}^5}\right)\right].\overline{\mathbf{u}}_1\right\}d^3\overline{\mathbf{x}}\ ,$$

$$F_1^2 = \int_{\overline{V_O}\cup\overline{V_I}\cup\overline{V_B}} \Big\{\overline{\mathbf{u}}_{Sto}.\left(\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{U}}_3 + \mathbf{e}_1\left(\overline{\mathbf{U}}_3.\mathbf{e}_3\right)\right) + \overline{\mathbf{u}}_3.\left(\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{U}}_{Sto} + \mathbf{e}_1\left(\overline{\mathbf{U}}_{Sto}.\mathbf{e}_3\right)\right)$$

$$+ \overline{\mathbf{u}}_1.\left(\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{U}}_{Dip} + \mathbf{e}_1\left(\overline{\mathbf{U}}_{Dip}.\mathbf{e}_3\right)\right) + \overline{\mathbf{u}}_{Dip}.\left(\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{U}}_1 + \mathbf{e}_1\left(\overline{\mathbf{U}}_1.\mathbf{e}_3\right)\right)\Big\} d^3\overline{\mathbf{x}}$$

with

$$\overline{\mathbf{U}}_2 = \frac{2+\overline{x_3}}{\overline{\tau}^5}\overline{x_1}\overline{\mathbf{x}} + 2\frac{1+\overline{x_3}}{\overline{\tau}^7}\left[\overline{\tau}^2\left(\left(2+\overline{x_3}\right)\mathbf{e}_1 + \overline{x_1}\mathbf{e}_3\right) - 5\overline{x_1}\left(2+\overline{x_3}\right)\left(\overline{\mathbf{x}}+2\mathbf{e}_3\right)\right],\ \overline{\mathbf{U}}_{Str} = \overline{\mathbf{U}}_2 + \frac{\overline{x_1}\,\overline{x_3}\,\overline{\mathbf{x}}}{\overline{r}^5}.$$

Similarly, the inner integrals to be evaluated are

$$I_{22} = \int_{V_I\cup V_O}\left[\mathbf{U}_{Sto}.\nabla\mathbf{U}_{Qua} + \mathbf{U}_{Qua}.\nabla\mathbf{U}_{Sto}\right].\mathbf{e}_3 d^3\mathbf{x} = \frac{2}{5}\pi\ ,$$

$$I_{23} = \int_{V_I\cup V_O}\left[\mathbf{U}_{Dip}.\nabla\mathbf{U}_{Qua} + \mathbf{U}_{Qua}.\nabla\mathbf{U}_{Dip}\right].\mathbf{e}_3 d^3\mathbf{x} = -\frac{12}{5}\pi\ ,$$

$$I_{24} = \int_{V_I\cup V_O}\left[\mathbf{U}_{Sto}.\nabla\mathbf{U}_{Str} + \mathbf{U}_{Str}.\nabla\mathbf{U}_{Sto}\right].\mathbf{e}_3 d^3\mathbf{x} = -\frac{2}{5}\pi\ ,$$

$$I_{25} = \int_{V_I\cup V_O}\left[\mathbf{U}_{Dip}.\nabla\mathbf{U}_{Str} + \mathbf{U}_{Str}.\nabla\mathbf{U}_{Dip}\right].\mathbf{e}_3 d^3\mathbf{x} = \frac{4}{3}\pi$$

$$I_{26} = \int_{V_I\cup V_O}\left\{\left[\mathrm{x}_3\mathbf{e}_1.\nabla\mathbf{U}_{Dip} + \mathbf{e}_1\left(\mathbf{e}_3.\mathbf{U}_{Dip}\right)\right].\mathbf{u}_{Dip}\right\} d^3\mathbf{x} = \frac{12}{5}\pi\ ,$$

$$I_{27} = \int_{V_I\cup V_O}\Big\{x_3\left(\mathbf{u}_{Sto}.\left(\mathbf{e}_1.\nabla\right)\mathbf{U}_{Dip} + \mathbf{u}_{Dip}.\left(\mathbf{e}_1.\nabla\right)\mathbf{U}_{Sto}\right)$$

$$+\left(\mathbf{u}_{Sto}.\mathbf{e}_1\right)\left(\mathbf{U}_{Dip}.\mathbf{e}_3\right) + \left(\mathbf{u}_{Dip}.\mathbf{e}_1\right)\left(\mathbf{U}_{Sto}.\mathbf{e}_3\right)\Big\} d^3\mathbf{x} = -\frac{32}{15}\pi\ ,$$

$$J_5 = \int_{V_B}\left\{x_3\mathbf{u}_{Sto}.\left(\mathbf{e}_1.\nabla\right)\mathbf{U}_{Sto} + \left(\mathbf{u}_{Sto}.\mathbf{e}_1\right)\left(\mathbf{U}_{Sto}.\mathbf{e}_3\right)\right\} d^3\mathbf{x} = \frac{1}{3}\pi\ ,$$

$$J_6 = \int_{V_B}\left\{x_3\mathbf{e}_3.\left(\mathbf{e}_1.\nabla\right)\mathbf{U}_{Sto}\right\} d^3\mathbf{x} = \frac{2}{15}\pi\ ,$$

$$H_3 = \int_{V_B}\left\{\tilde{\mathbf{U}}_{Sti}\left(\mathbf{e}_3\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_3\right) + \left(x_3\mathbf{e}_1 - x_1\mathbf{e}_3\right).\nabla\tilde{\mathbf{U}}_{Sti}\right\} d^3\mathbf{x} = 0.$$

Again, the four outer integrals were evaluated numerically after changing to polar coordinates and integrating analytically with respect to Ψ. We obtained $4\,F_{-1}^{1}/\pi \approx 1.833$, $2\,F_{0}^{1}/\pi \approx 0.305$, $2\,F_{1}^{1}/\pi = \mathcal{E}_1 \approx 0.683$ and $2\,F_{1}^{2}/\pi = \mathcal{F}_1 \approx -0.634$. The first two values agree within three digits with the analytical values 11/6 and 11/36 determined by Cherukat & McLaughlin (1994). Taking into account the contribution of the "inner" integrals, the shear-induced contribution $\mathbf{F}_{\mathrm{MS}}^{(G)}$ to the inertial lift force takes the form

$$\mathbf{F}_{\mathrm{MS}}^{(G)} = -\frac{11}{24}\pi\alpha V_{S1}^{(0)} R_\mu^2\left(\kappa^{-1} + D_0(\lambda) + \kappa D_1(\lambda)\right) + \frac{11}{72}\pi R_\mu R_S \alpha^2\left(1 + \frac{3}{8}R_\mu \kappa\right) + O\left(\kappa^2\right) \qquad \text{(D5)}$$

with $D_0(\lambda) = \frac{9}{8}R_\mu + G_1(\lambda)$, $D_1(\lambda) = \frac{63}{64}R_\mu^2 + \frac{12}{11}\left(\frac{R_S}{R_\mu}\mathcal{E}_1 + \frac{D_\mu}{R_\mu}\mathcal{F}_1\right) + \frac{9}{8}R_\mu G_1(\lambda) + G_2(\lambda)$

$$G_1(\lambda) = -\frac{2}{55}\frac{\lambda\left(168 + 372\lambda + 210\lambda^2\right)}{(1+\lambda)(2+3\lambda)^2} \ , \quad G_2(\lambda) = -\frac{3}{55}\frac{26 + 79\lambda + 53\lambda^2}{(1+\lambda)(2+3\lambda)} \qquad \text{(D6)}$$

In the case of a buoyant drop moving in a linear shear flow near a horizontal wall, the expression of the total horizontal inertial force truncated at $O(\kappa^0)$ is

$$\begin{aligned} 4\pi R_\mu V_{B1}^{(G)} = -\alpha V_{B3}^{(0)}\Big\{&\kappa^{-1}R_\mu^2 K_2 G_{-1}^{1} + \overline{\rho}\left[\frac{M_\mu^2}{2}\left(M_\mu H_2 - H_3\right) - \frac{4}{3}\pi\right] \\ &+ R_\mu\left(R_S\left(I_1 - R_\mu I_2\right) + D_\mu\left(I_{29} - R_S I_7\right) - R_\mu J_7\right) \\ &+ D_\mu\left(R_S\left(I_5 - D_\mu I_6\right) + D_\mu I_{28} - 2\left(R_\mu^2 I_{16} + R_\mu D_\mu I_{19} + D_\mu^2 I_{18}\right)\right)\Big\} \end{aligned} \qquad \text{(D7)}$$

Note that most of the integrals involved in the $O(\kappa^0)$-term are identical to those of Eq. (D4), as can be shown by interchanging x_1 and x_3, $\mathbf{e}_1$ and $\mathbf{e}_3$.

The new integrals to be evaluated are

$$G_{-1}^{1} = \int_{\overline{V_{\mathrm{O}}} \cup \overline{V_{\mathrm{I}}} \cup \overline{V_{\mathrm{B}}}} \left[\overline{x_3}\left(\mathbf{e}_1.\overline{\nabla}\right)\overline{\mathbf{u}}_{\mathrm{Sto}} + \mathbf{e}_1\left(\overline{\mathbf{u}}_{\mathrm{Sto}}.\mathbf{e}_3\right)\right].\overline{\mathbf{U}}_{\mathrm{Sto}} d^3\overline{\mathbf{x}}$$

$$I_{28} = \int_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} \left\{\left[x_3 \mathbf{e}_1.\nabla \mathbf{u}_{\mathrm{Dip}} + \mathbf{e}_1\left(\mathbf{e}_3.\mathbf{u}_{\mathrm{Dip}}\right)\right].\mathbf{U}_{\mathrm{Dip}}\right\} d^3\mathbf{x} = -\frac{44}{15}\pi ,$$

$$I_{29} = \int_{V_{\mathrm{I}} \cup V_{\mathrm{O}}} \left\{x_3\left(\mathbf{U}_{\mathrm{Sto}}.\left(\mathbf{e}_1.\nabla\right)\mathbf{u}_{\mathrm{Dip}} + \mathbf{U}_{\mathrm{Dip}}.\left(\mathbf{e}_1.\nabla\right)\mathbf{u}_{\mathrm{Sto}}\right)\right.$$
$$\left. + \left(\mathbf{U}_{\mathrm{Sto}}.\mathbf{e}_1\right)\left(\mathbf{u}_{\mathrm{Dip}}.\mathbf{e}_3\right) + \left(\mathbf{U}_{\mathrm{Dip}}.\mathbf{e}_1\right)\left(\mathbf{u}_{\mathrm{Sto}}.\mathbf{e}_3\right)\right\} d^3\mathbf{x} = 2\pi ,$$

$$J_7 = \int_{V_{\mathrm{B}}} \left\{x_3 \mathbf{U}_{\mathrm{Sto}}.\left(\mathbf{e}_1.\nabla\right)\mathbf{u}_{\mathrm{Sto}} + \left(\mathbf{U}_{\mathrm{Sto}}.\mathbf{e}_1\right)\left(\mathbf{u}_{\mathrm{Sto}}.\mathbf{e}_3\right)\right\} d^3\mathbf{x} = \frac{5}{3}\pi .$$

We found numerically $4\,G_{-1}^{1}/\pi = 2.500$ up to three digits, from which we inferred that the exact value of the outer integral is $G_{-1}^{1} = \frac{5}{8}\pi$. Using these results, Eq. (D7) yields

$$\mathbf{F}_{\mathrm{M}}^{(G)}.\mathbf{e}_1 = -\frac{5}{8}\pi\alpha V_{B3}^{(0)} R_{\mu}^{2}\left(\kappa^{-1} - E_0(\lambda)\right) + O(\kappa) \tag{D8}$$

with $$E_0(\lambda) = -\frac{9}{8} R_{\mu} + \frac{4}{75}\frac{\left(160 + 600\lambda + 756\lambda^2 + 319\lambda^3\right)}{(1+\lambda)(2+3\lambda)^2} \tag{D9}$$