Appendix to "Nonlinear theory of geostrophic adjustment. Part 2. Two-layer and continuously stratified primitive equations"

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A Detailed calculations and a proof of the results on the two-layer QG regime presented in §2.2.2

Below we use the following perturbative expansions of the PV's:

$$\Pi_i = 1 + \epsilon\Pi_i^{(1)} + \epsilon^2\Pi_i^{(2)} + \mathcal{O}(\epsilon^3) \tag{A1}$$

with

$$\Pi_i^{(1)} = \zeta_i^{(0)} + (-1)^{i+1}\bar{h}_{i+1}\eta_0, \;\; \Pi_i^{(2)} = \zeta_i^{(1)} + (-1)^{i+1}\bar{h}_{i+1}\eta^{(1)} + (-1)^{i+1}\bar{h}_{i+1}\eta^{(0)}\Pi_i^{(1)}. \tag{A2}$$

A1 Calculations in the lowest-order approximation

Equations (2.7) and (2.4), (2.9), (2.11) give:

$$\partial_t \mathbf{v}_i^{(0)} + \hat{\mathbf{z}} \wedge \mathbf{v}_i^{(0)} = -\nabla\pi_i^{(0)} \,, \; \partial_t\Pi_i^{(1)} = 0 \,. \tag{A3}$$

The corresponding initial conditions are[1]

$$\mathbf{v}_i^{(0)}\Big|_{t=0} = v_I \,, \, \eta^{(0)}\Big|_{t=0} = \eta_I \,. \tag{A4}$$

[1] here and below it is supposed that initial data have no dependence on ϵ

Hence,

$$\zeta_i^{(0)} + (-1)^{i+1}\bar{h}_{i+1}\eta^{(0)} = \Pi_i^{(1)}(x, y; t_1, ...). \tag{A5}$$

It is convenient to introduce the barotropic and the baroclinic components of velocity in each order α of the perturbation theory:

$$\begin{aligned} \mathbf{v}_{bt}^{(\alpha)} &= \bar{h}_1\mathbf{v}_1^{(\alpha)} + \bar{h}_2\mathbf{v}_2^{(\alpha)} \\ \mathbf{v}_{bc}^{(\alpha)} &= \mathbf{v}_1^{(\alpha)} - \mathbf{v}_2^{(\alpha)}, \quad \alpha = 0, 1, 2, ... \end{aligned} \tag{A6}$$

In the lowest order we have:

$$\partial_t \mathbf{v}_{bc}^{(0)} + \hat{\mathbf{z}} \wedge \mathbf{v}_{bc}^{(0)} = \nabla \eta^{(0)} \tag{A7}$$

$$\partial_t \mathbf{v}_{bt}^{(0)} + \hat{\mathbf{z}} \wedge \mathbf{v}_{bt}^{(0)} = -\nabla P^{(0)}, \tag{A8}$$

where $P^{(\alpha)}$ denotes the barotropic pressure component

$$P^{(\alpha)} = \bar{h}_1\pi_1^{(\alpha)} + \bar{h}_2\pi_2^{(\alpha)}. \tag{A9}$$

Accordingly, we obtain from (A5)

$$\begin{aligned} \zeta_{bc}^{(0)} + \eta^{(0)} &= \Pi_1^{(1)} - \Pi_2^{(1)}, \\ \zeta_{bt}^{(0)} &= \bar{h}_1\Pi_1^{(1)} + \bar{h}_2\Pi_2^{(1)}, \end{aligned} \tag{A10}$$

where

$$\zeta_{bc}^{(\alpha)} = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_{bc}^{(\alpha)}, \quad \zeta_{bt}^{(\alpha)} = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_{bt}^{(\alpha)}, \ \alpha = 0, 1, 2, ... \tag{A11}$$

As follows from the second equation in (A3) and the second equation in (A10), the barotropic relative vorticity is slow (t - independent). By rewriting (A8) in the form of vorticity and divergence equations we see that the barotropic velocity field is divergenceless and slow:

$$\mathbf{v}_{bt}^{(0)} = \bar{\mathbf{v}}_{bt}^{(0)} = \hat{\mathbf{z}} \wedge \nabla P^{(0)}, \tag{A12}$$

$$\nabla^2 P^{(0)} = \bar{h}_1\Pi_1^{(1)} + \bar{h}_2\Pi_2^{(1)}. \tag{A13}$$

The baroclinic component is split into fast and slow components denoted as usual by tilde and over-bar, respectively:

$$\begin{aligned} \mathbf{v}_{bc}^{(0)} &= \bar{\mathbf{v}}_{bc}^{(0)} + \tilde{\mathbf{v}}_{bc}^{(0)} \\ \eta^{(0)} &= \bar{\eta}^{(0)} + \tilde{\eta}^{(0)} \end{aligned} \tag{A14}$$

with

$$\bar{\mathbf{v}}_{bc}^{(0)} = -\hat{\mathbf{z}} \wedge \nabla \bar{\eta}^{(0)}, \tag{A15}$$

$$\nabla^2 \bar{\eta}^{(0)} - \bar{\eta}^{(0)} = -\left(\Pi_1^{(1)} - \Pi_2^{(1)}\right) \tag{A16}$$

and

$$\partial_t \tilde{\mathbf{v}}_{bc}^{(0)} + \hat{\mathbf{z}} \wedge \tilde{\mathbf{v}}_{bc}^{(0)} = \nabla \tilde{\eta}^{(0)}, \tag{A17}$$

$$\bar{\zeta}_{bc}^{(0)} + \bar{\eta}^{(0)} = 0. \tag{A18}$$

In order to get the initial conditions for both components consider (A16) at $t = 0$ and (A5):

$$-\bar{\eta}_I^{(0)} + \nabla^2 \bar{\eta}_I^{(0)} = -\left(\zeta_{1_I} - \zeta_{2_I} + \eta_I\right). \tag{A19}$$

This allows to find $\bar{\eta}_I^{(0)}$ and, hence, the initial conditions for the slow baroclinic component via

$$\bar{\mathbf{v}}_{bc_I}^{(0)} = -\hat{\mathbf{z}} \wedge \nabla \bar{\eta}_I^{(0)}. \tag{A20}$$

Initial conditions for the fast baroclinic component readily follow

$$\tilde{\mathbf{v}}_{bc}^{(0)} = \mathbf{v}_{1I} - \mathbf{v}_{2I} - \bar{\mathbf{v}}_{bc_I}^{(0)}, \;\; \tilde{\eta}_I^{(0)} = \eta_I - \bar{\eta}_I^{(0)}. \tag{A21}$$

The system (A17), (A18) is equivalent to a single Klein - Gordon equation for $\tilde{\eta}^{(0)}$:

$$-\frac{\partial^2 \tilde{\eta}^{(0)}}{\partial t^2} - \tilde{\eta}^{(0)} + \nabla^2 \tilde{\eta}^{(0)} = 0 \tag{A22}$$

with initial conditions

$$\tilde{\eta}^{(0)}\Big|_{t=0} = \tilde{\eta}_I^{(0)}, \;\; \partial_t \tilde{\eta}^{(0)}\Big|_{t=0} = \nabla \cdot \tilde{\mathbf{v}}_{bc_I}^{(0)}. \tag{A23}$$

These initial conditions allow to determine $\tilde{\eta}^{(0)}$ from (A22); if the initial conditions are localized the fast field decays as $\frac{1}{t}$ at $t \to \infty$ at a fixed spatial point (we do not repeat here the details of the calculations which follow those of P1, with obvious changes of notation).

Thus, as in the RSW case, in the zeroth order of the perturbation theory the motion is split into the fast and the slow components defined in a unique way starting from arbitrary initial conditions. Note that the procedure imposes no *à priori* limitations on the relative initial values of the fast and the slow components. The fast part of the flow is completely resolved while the slow part remains undetermined. Its evolution equation follows from the condition of absence of secular growth of the next order solution.

A2 Calculations in the first-order approximation

The horizontal momentum equations give at this order

$$\partial_t \mathbf{v}_i^{(1)} + \hat{\mathbf{z}} \wedge \mathbf{v}_i^{(1)} = -\nabla \pi_i^{(1)} + \mathcal{R}_{\mathbf{v}_i}^{(0)}, \;\; i = 1, 2, \tag{A24}$$

where we define

$$\mathcal{R}_{\mathbf{v}_i}^{(0)} = \left(\mathcal{R}_{u_i}^{(0)}, \mathcal{R}_{v_i}^{(0)}\right) = -\left(\partial_{t_1} + \mathbf{v}_i^{(0)} \cdot \nabla\right) \mathbf{v}_i^{(0)}. \tag{A25}$$

The first-order PV equations are

$$\partial_t \Pi_i^{(2)} + \partial_{t_1} \Pi_i^{(1)} + \mathbf{v}_i^{(0)} \cdot \nabla \Pi_i^{(1)} = 0, \;\; i = 1, 2 \tag{A26}$$

and (cf. 2.10)

$$\pi_2^{(1)} - \pi_1^{(1)} = \eta^{(1)}. \tag{A27}$$

A consistency condition for having bounded in the fast time solutions of (A26) is obtained by applying the fast-time averaging to (A26) and gives the standard quasigeostrophic PV (QGPV) equations

$$\left(\partial_{t_1} + \bar{\mathbf{v}}_i^{(0)} \cdot \nabla\right) \Pi_i^{(1)} = 0, \quad i = 1, 2 \tag{A28}$$

which may be rewritten in the form

$$\partial_{t_1} \Pi_i^{(1)} + J(\bar{\pi}_i^{(0)}, \Pi_i^{(1)}) = 0\,, \tag{A29}$$

(J, as usual, denotes the Jacobian) using the fact that

$$\bar{\mathbf{v}}_i^{(0)} = (-1)^{i+1} \bar{h}_{i+1} \bar{\mathbf{v}}_{bc}^{(0)} + \bar{\mathbf{v}}_{bt}^{(0)} = \hat{\mathbf{z}} \wedge \nabla \bar{\pi}_i^{(0)}. \tag{A30}$$

Recalling that

$$\Pi_i^{(1)} = \nabla^2 \bar{\pi}_i^{(0)} + (-1)^{i+1} \bar{h}_{i+1}\, \bar{\eta}^{(0)} \tag{A31}$$

we see that (A29), (A31) reproduce the standard QG equations in the two-layer model (cf. Pedlosky, 1982).

The first correction to the PV-fields obeys, thus, the following equation

$$\partial_t \Pi_i^{(2)} + \tilde{\mathbf{v}}_i^{(0)} \cdot \nabla \Pi_i^{(1)} = 0, \quad i = 1, 2 \tag{A32}$$

and using the fact that

$$\tilde{\mathbf{v}}_i^{(0)} = (-1)^{i+1} \bar{h}_{i+1} \tilde{\mathbf{v}}_{bc}^{(0)} \tag{A33}$$

and (A17) we get, by integrating (A32) in t :

$$\begin{aligned} \Pi_i^{(2)} &= \tilde{\Pi}_i^{(2)} + \bar{\Pi}_i^{(2)} = \\ & (-1)^{i+1} \bar{h}_{i+1} \left[J(\tilde{\mathcal{H}}_0 - < \tilde{\mathcal{H}}_0 >, \Pi_i^{(1)}) - \tilde{u}_{bc}^{(1)} \partial_y \Pi_i^{(1)} + \tilde{v}_{bc}^{(1)} \partial_x \Pi_i^{(1)} \right] + \bar{\Pi}_i^{(2)}, \end{aligned} \tag{A34}$$

where

$$\tilde{\mathcal{H}}_0 = \int_0^t \tilde{\eta}^{(0)}(t')\, dt' \tag{A35}$$

and the angle brackets denote the fast-time averaging. Here and below, by introducing $\tilde{\mathcal{H}}_0$ and $< \tilde{\mathcal{H}}_0 >$ we follow literally P1.

In order to get an equation for $\eta^{(1)}$ we introduce the baroclinic and barotropic components of the order-one velocity field (cf. (A6)) with the baroclinic one, $\mathbf{v}_{bc}^{(1)}$, obeying the following equation:

$$\partial_t \mathbf{v}_{bc}^{(1)} + \hat{\mathbf{z}} \wedge \mathbf{v}_{bc}^{(1)} = \nabla \eta^{(1)} + \mathcal{R}_{\mathbf{v}_{bc}}^{(0)}, \;\; \mathcal{R}_{\mathbf{v}_{bc}}^{(0)} = \mathcal{R}_{\mathbf{v}_1}^{(0)} - \mathcal{R}_{\mathbf{v}_2}^{(0)}. \tag{A36}$$

The equation for the baroclinic relative vorticity follows from (A34) and the second equation in (A2) (cf. (3.33) in P1)

$$\zeta_{bc}^{(1)} + \eta^{(1)} = \tilde{\Pi}_1^{(2)} - \tilde{\Pi}_2^{(2)} - \mathcal{R}_\zeta^{(0)}, \tag{A37}$$

where

$$\mathcal{R}_\zeta^{(0)} = \mathcal{R}_{bc}^{(1)}\eta^{(0)} - J\left(\bar{\mathcal{H}}_0 - <\bar{\mathcal{H}}_0>, \mathcal{R}_{bc}^{(1)}\right) + \hat{\mathbf{z}}\cdot\left(\bar{\mathbf{v}}_{bc}^{(0)} \wedge \nabla\mathcal{R}_{bc}^{(1)}\right) \qquad (A38)$$

and

$$\mathcal{R}_{bc}^{(1)}(x,y;t_1,...) = \bar{h}_2\Pi_1^{(1)} + \bar{h}_1\Pi_2^{(1)}. \qquad (A39)$$

Splitting the first-order baroclinic velocity into slow and fast parts, where the latter obeys the fast part of the equations (A36), and taking the fast part of (A37) we get:

$$\tilde{\zeta}_{bc}^{(1)} + \tilde{\eta}^{(1)} = -\mathcal{R}_{bc}^{(1)}\tilde{\eta}^{(0)} + J\left(\tilde{\mathcal{H}}_0 - <\tilde{\mathcal{H}}_0>, \mathcal{R}_{bc}^{(1)}\right) - \hat{\mathbf{z}}\cdot\left(\tilde{\mathbf{v}}_{bc}^{(0)} \wedge \nabla\mathcal{R}_{bc}^{(1)}\right) \equiv -\tilde{\mathcal{R}}_\zeta^{(0)}. \qquad (A40)$$

From this equation and equations of motion (A36) taken for the fast component we obtain the equation for the first-order fast interface displacement:

$$-\frac{\partial^2\tilde{\eta}^{(1)}}{\partial t^2} - \tilde{\eta}^{(1)} + \nabla^2\tilde{\eta}^{(1)} = -\frac{\partial^2\mathcal{R}_\zeta^{(0)}}{\partial t^2} - \tilde{\mathcal{R}}_\zeta^{(0)} - \tilde{\mathcal{D}} + \partial_t\tilde{\mathcal{Z}} \qquad (A41)$$

where

$$\tilde{\mathcal{Z}} = \partial_x\tilde{\mathcal{R}}_{v_{bc}}^{(0)} - \partial_y\tilde{\mathcal{R}}_{u_{bc}}^{(0)}, \qquad (A42)$$

$$\tilde{\mathcal{D}} = \partial_x\tilde{\mathcal{R}}_{u_{bc}}^{(0)} + \partial_y\tilde{\mathcal{R}}_{v_{bc}}^{(0)}. \qquad (A43)$$

The r.h.s. of (A41) is, thus, a known function of $\tilde{\mathbf{v}}_{bc}^{(0)}, \tilde{\eta}^{(0)}$ which are, in turn, known from the previous approximation.

The slow baroclinic velocity equations have the form:

$$\bar{\mathbf{v}}_{bc}^{(1)} = -\hat{\mathbf{z}} \wedge \left(\nabla\bar{\eta}^{(1)} + \bar{\mathcal{R}}_{\mathbf{v}_{bc}}\right). \qquad (A44)$$

The slow part of the baroclinic relative vorticity equation (A37) is

$$\bar{\zeta}_{bc}^{(1)} + \bar{\eta}^{(1)} = -\mathcal{R}_{bc}^{(1)}\bar{\eta}^{(0)} + \bar{\Pi}_1^{(2)} - \bar{\Pi}_2^{(2)}. \qquad (A45)$$

It follows from (A44) that

$$\bar{\zeta}_{bc}^{(1)} = \hat{\mathbf{z}}\cdot\left(\nabla \wedge \bar{\mathbf{v}}_{bc}^{(1)}\right) = -\nabla^2\bar{\eta}^{(1)} - \nabla\cdot\bar{\mathcal{R}}_{\mathbf{v}_{bc}} \qquad (A46)$$

and from this equation and (A45) one gets:

$$-\bar{\eta}_I^{(1)} + \nabla^2\bar{\eta}_I^{(1)} = \mathcal{R}_{bc}^{(1)}\bar{\eta}^{(0)} - \bar{\Pi}_1^{(2)} + \bar{\Pi}_2^{(2)} - \nabla\cdot\bar{\mathcal{R}}_{\mathbf{v}_{bc}}. \qquad (A47)$$

Considering this equation at $t = 0$, recalling that $\mathbf{v}_i^{(1)}$ and $\eta^{(1)}$ are zero at the initial moment and using (A37), (A38) we obtain

$$-\bar{\eta}_I^{(1)} + \nabla^2\bar{\eta}_I^{(1)} = \left[-\nabla\cdot\bar{\mathcal{R}}_{\mathbf{v}_{bc}} - \bar{\eta}_I^{(0)}\mathcal{R}_{bc}^{(1)} - J\left(<\bar{\mathcal{H}}_0>, \mathcal{R}_{bc}^{(1)}\right) + \left(\bar{v}_{bc_I}\partial_x\mathcal{R}_{bc}^{(1)} - \bar{u}_{bc_I}\partial_y\mathcal{R}_{bc}^{(1)}\right)\right]\Big|_{t=0}. \qquad (A48)$$

The r.h.s. of this equation is known. Therefore, (A48) determines $\bar{\eta}_I^{(1)}$ and, hence, $\bar{\eta}_I^{(1)}$, uniquely (if decaying at infinity boundary conditions are imposed) as $\bar{\eta}_I^{(1)} = -\bar{\eta}_I^{(1)}$. The second initial condition for the equation (A41) follows from (A36) and (A37):

$$\partial_t \tilde{\eta}^{(1)}\Big|_{t=0} = -\left.\mathcal{Z}\right|_{t=0} - \partial_t \tilde{\mathcal{R}}_\zeta^{(0)}\Big|_{t=0}, \tag{A49}$$

where

$$\mathcal{Z} = \partial_x \mathcal{R}_{v_{bc}}^{(0)} - \partial_y \mathcal{R}_{u_{bc}}^{(0)}. \tag{A50}$$

The r.h.s of (A49) may be expressed in terms of initial fields (cf. (A42)) by using, where necessary, the evolution equation for the slow component. Thus, for $\tilde{\eta}^{(1)}$ we get a linear initial-value problem with a source term (cf. (A41). The analysis showing that the source term is non-resonant is the same as in P1 and is not repeated here.

The first correction to the barotropic component may be easily determined, too. We write the evolution equation for the barotropic velocity field at this order:

$$\partial_t \mathbf{v}_{bt}^{(1)} + \hat{\mathbf{z}} \wedge \mathbf{v}_{bt}^{(1)} = -\nabla P^{(1)} + \mathcal{R}_{\mathbf{v}_{bt}}^{(0)}, \quad \mathcal{R}_{\mathbf{v}_{bt}}^{(0)} = \bar{h}_1 \mathcal{R}_{\mathbf{v}_1}^{(0)} + \bar{h}_2 \mathcal{R}_{\mathbf{v}_2}^{(0)} \tag{A51}$$

and get the barotropic relative vorticity using (A2), (A34):

$$\begin{aligned} \zeta_{bt}^{(1)} &= \bar{h}_1 \bar{\Pi}_1^{(2)} + \bar{h}_2 \bar{\Pi}_2^{(2)} + \bar{h}_1 \bar{h}_2 \left[-\eta^{(0)} \mathcal{R}_{bt}^{(1)} + \right. && \text{(A52)} \\ & \left. J(\tilde{\mathcal{H}}_0 - < \tilde{\mathcal{H}}_0 >, \mathcal{R}_{bt}^{(1)}) + \hat{\mathbf{z}} \cdot \left(\tilde{\mathbf{v}}_{bc}^{(0)} \wedge \nabla \mathcal{R}_{bt}^{(1)} \right) \right], && \text{(A53)} \end{aligned}$$

where we defined

$$\mathcal{R}_{bt}^{(1)} = \Pi_1^{(1)} - \Pi_2^{(1)}. \tag{A54}$$

The fast part of $\zeta_{bt}^{(1)}$ is:

$$\tilde{\zeta}_{bt}^{(1)} = \bar{h}_1 \bar{h}_2 \left[-\tilde{\eta}^{(0)} \mathcal{R}_{bt}^{(1)} + J(\tilde{\mathcal{H}}_0 - < \tilde{\mathcal{H}}_0 >, \Pi_1^{(1)} - \Pi_2^{(1)}) + \hat{\mathbf{z}} \cdot \left(\tilde{\mathbf{v}}_{bc}^{(0)} \wedge \nabla \mathcal{R}_{bt}^{(1)} \right) \right]. \tag{A55}$$

Taking curl and divergence of the fast part of the equations (A51) we get

$$\tilde{D}_{bt}^{(1)} = \nabla \cdot \tilde{\mathbf{v}}_{bt}^{(0)} = -\partial_t \tilde{\zeta}_{bt}^{(1)} + \partial_x \tilde{\mathcal{R}}_{v_{bt}}^{(0)} - \partial_y \tilde{\mathcal{R}}_{u_{bt}}^{(0)}, \tag{A56}$$

$$\nabla^2 \tilde{P}^{(1)} = -\partial_t \tilde{D}_{bt}^{(1)} + \tilde{\zeta}_{bt}^{(1)} + \partial_x \tilde{\mathcal{R}}_{u_{bt}}^{(0)} + \partial_y \tilde{\mathcal{R}}_{v_{bt}}^{(0)}. \tag{A57}$$

The last equation, together with (A55) and (A56), allows to determine, by inversion, the fast correction to the barotropic pressure and, via the fast part of the equations (A51), the fast barotropic velocity field at this order. Together, the fast barotropic and the fast baroclinic velocity fields allow to determine completely the fast velocity field in the model at the first order in Rossby number. The slow components evolve according to the standard QG equations (A29), (A31) at this order. It should be noticed that a fast correction to the slow zeroth-order barotropic fields appear at this order and that initial conditions at this order mix the lowest order fast and slow initial fields.

A3 Calculations in the second-order approximation

We limit ourselves at this order by calculating corrections to the slow geostrophic dynamics leaving apart the fast wave field. The PV equation (2.4) gives

$$\partial_t \Pi_i^{(3)} + \partial_{t_1} \Pi_i^{(2)} + \partial_{t_2} \Pi_i^{(1)} + \mathbf{v}_i^{(0)} \cdot \nabla \Pi_i^{(2)} + \mathbf{v}_i^{(1)} \cdot \nabla \Pi_i^{(1)} = 0\,,\ i = 1, 2. \quad \text{(A58)}$$

Taking the time-average of this equation we get:

$$\partial_{t_1} \bar{\Pi}_i^{(2)} + \partial_{t_2} \Pi_i^{(1)} + \bar{\mathbf{v}}_i^{(0)} \cdot \nabla \bar{\Pi}_i^{(2)} + \bar{\mathbf{v}}_i^{(1)} \cdot \nabla \Pi_i^{(1)} + \left\langle \tilde{\mathbf{v}}_i^{(0)} \cdot \nabla \tilde{\Pi}_i^{(2)} \right\rangle = 0\,,\ i = 1, 2. \quad \text{(A59)}$$

It is easy to show, using (A34) (cf. the analogous demonstration in P1) that $\tilde{\Pi}_i^{(2)} = \mathcal{O}\left(\frac{1}{t}\right)$ as $t \to \infty$ and, hence the two last terms in (A59) which represent the fast-component drag vanish. Hence, as in the RSW case we get splitting and the slow component of the flow evolves without being influenced by the fast one at this order. We, thus have

$$\left(\partial_{t_1} + \bar{\mathbf{v}}_i^{(0)} \cdot \nabla\right) \bar{\Pi}_i^{(2)} + \left(\partial_{t_2} + \bar{\mathbf{v}}_i^{(1)} \cdot \nabla\right) \Pi_i^{(1)} = 0, \ \ i = 1, 2 \quad \text{(A60)}$$

with, cf. (A2)

$$\bar{\Pi}_i^{(2)} = \bar{\zeta}_i^{(1)} + (-1)^{i+1} \bar{h}_{i+1} \left(\bar{\eta}^{(1)} + \bar{\eta}^{(0)} \Pi_i^{(1)}\right). \quad \text{(A61)}$$

Using the averaged equations (A24) and (A30) we find

$$\bar{\mathbf{v}}_i^{(1)} = \hat{\mathbf{z}} \wedge \nabla \bar{\pi}_i^{(1)} - \partial_{t_1} \nabla \bar{\pi}_i^{(0)} - J(\bar{\pi}_i^{(0)}, \nabla \bar{\pi}_i^{(0)}) \quad \text{(A62)}$$

and get

$$\bar{\zeta}_i^{(1)} = \nabla^2 \bar{\pi}_i^{(1)} - 2J\left(\partial_x \bar{\pi}_i^{(0)}, \partial_y \bar{\pi}_i^{(0)}\right) \quad \text{(A63)}$$

whence

$$\bar{\Pi}_i^{(2)} = \nabla^2 \bar{\pi}_i^{(1)} - 2J\left(\partial_x \bar{\pi}_i^{(0)}, \partial_y \bar{\pi}_i^{(0)}\right) + (-1)^{i+1} \bar{h}_{i+1} \left(\bar{\eta}^{(1)} + \bar{\eta}^{(0)} \Pi_i^{(1)}\right). \quad \text{(A64)}$$

By the same reasoning as in P1 we introduce a "full" slow pressure and interface displacement fields $\bar{\pi}_i = \bar{\pi}_i^{(0)} + \epsilon \bar{\pi}_i^{(1)}$, $\bar{\eta} = \eta^{(0)} + \epsilon \eta^{(1)}$ and get the "improved" QGPV equations of the two-layer model:

$$\begin{aligned} \frac{D_i}{Dt_1} \big[\nabla^2 \bar{\pi}_i + (-1)^{i+1} \bar{h}_{i+1} \bar{\eta} + \epsilon (-1)^{i+1} \bar{h}_{i+1} \bar{\eta} \left(\nabla^2 \bar{\pi}_i + (-1)^{i+1} \bar{h}_{i+1} \bar{\eta}\right) \quad &\text{(A65)} \\ - \epsilon \nabla \bar{\pi}_i \cdot \nabla \left(\nabla^2 \bar{\pi}_i + (-1)^{i+1} \bar{h}_{i+1} \bar{\eta}\right) - 2\epsilon J\left(\partial_x \bar{\pi}_i, \partial_y \bar{\pi}_i\right)\big] \ = \ 0, & \end{aligned}$$

where

$$\frac{D_i}{Dt_1}(\ldots) := \partial_{t_1}(\ldots) + J\left(\bar{\pi}_i - \epsilon \frac{(\nabla \bar{\pi}_i)^2}{2}, \ldots\right),\ i = 1, 2 \quad \text{(A66)}$$

and $\bar{\eta} = \bar{\pi}_2 - \bar{\pi}_1$.

B Detailed calculations and a proof of the results on the QG regime in continuously stratified fluid presented in §3.2.1

B1 Calculations in the lowest-order approximation

For zero-order fields we get from (3.10):

$$\begin{aligned} \partial_t \mathbf{v}_h^{(0)} + \hat{\mathbf{z}} \wedge \mathbf{v}_h^{(0)} &= -\nabla_h p^{(0)}, \\ \nabla \cdot \mathbf{v}^{(0)} = 0, \;\; \partial_z p^{(0)} + \rho^{(0)} &= 0. \\ \partial_t \rho^{(0)} - N^2 w^{(0)} &= 0. \end{aligned} \quad \text{(B67)}$$

The PV equation (3.11) gives

$$\partial_t \left[-\partial_z \left(\frac{\rho^{(0)}}{N^2} \right) + \zeta^{(0)} \right] = 0 \quad \text{(B68)}$$

whence it follows that

$$-\partial_z \left(\frac{\rho^{(0)}}{N^2} \right) + \zeta^{(0)} = \Omega^{(0)}(\mathbf{r}, t_1, ...). \quad \text{(B69)}$$

The horizontal momentum equations in (B67) may be rewritten in terms of vorticity $\zeta^{(0)}$ and divergence $D^{(0)}$ of $\mathbf{v}_h^{(0)}$:

$$\begin{aligned} \partial_t \zeta^{(0)} + D^{(0)} &= 0, \\ \partial_t D^{(0)} - \zeta^{(0)} + \nabla_h^2 p^{(0)} &= 0, \end{aligned} \quad \text{(B70)}$$

Excluding the divergence from (B70) we obtain the equation

$$-\partial_{tt}^2 \zeta^{(0)} - \zeta^{(0)} + \nabla_h^2 p^{(0)} = 0 \quad \text{(B71)}$$

which gives

$$\partial_{tt}^2 \partial_z \left(\frac{1}{N^2} \partial_z p^{(0)} \right) + \partial_z \left(\frac{1}{N^2} \partial_z p^{(0)} \right) + \nabla_h^2 p^{(0)} = \Omega^{(0)}(\mathbf{r}, t_1, ...). \quad \text{(B72)}$$

The zero-order vertical velocity $w^{(0)}$ is zero at the vertical boundaries, as follows from (3.2). Then from the density equation in (B67) it follows that density variations at these boundaries are slow. The hydrostatic balance implies that:

$$\partial_z p^{(0)} \Big|_{z=-1,0} = -\, \rho^{(0)}(x, y, t_1, ...) \Big|_{z=-1,0}. \quad \text{(B73)}$$

The initial conditions are:

$$\left(u^{(0)}, v^{(0)}, \rho^{(0)} \right)_{t=0} = (u_I, v_I, \rho_I). \quad \text{(B74)}$$

Representing the pressure as $p^{(0)} = \bar{p}^{(0)}(\mathbf{r}, t_1, ...) + \tilde{p}^{(0)}(\mathbf{r}, t, t_1, ...)$ where $\bar{p}$ and $\tilde{p}$ are the slow and the fast parts, respectively, we get the following equation for $\tilde{p}^{(0)}$:

$$\partial_{tt}^2 \partial_z \left(\frac{1}{N^2} \partial_z \tilde{p}^{(0)} \right) + \partial_z \left(\frac{1}{N^2} \partial_z \tilde{p}^{(0)} \right) + \nabla_h^2 \tilde{p}^{(0)} = 0, \tag{B75}$$

with the boundary condition

$$\partial_z \tilde{p}^{(0)} \Big|_{z=-1,0} = 0. \tag{B76}$$

For $\bar{p}^{(0)}$ we have the equation

$$\partial_z \left(\frac{1}{N^2} \partial_z \bar{p}^{(0)} \right) + \nabla_h^2 \bar{p}^{(0)} = \Omega^{(0)}(\mathbf{r}, t_1, ...), \tag{B77}$$

with the boundary condition

$$\partial_z \bar{p}^{(0)} \Big|_{z=-1,0} = - \rho^{(0)} \Big|_{z=-1,0}. \tag{B78}$$

The velocity and density fields are split into the fast and the slow components, too, with

$$\bar{\mathbf{v}}_h^{(0)} = \hat{\mathbf{z}} \wedge \bar{p}^{(0)}, \;\; \bar{w}^{(0)} = 0, \;\; \bar{\rho}^{(0)} = -\partial_z \bar{p}^{(0)} \tag{B79}$$

and

$$\partial_t \tilde{\mathbf{v}}_h^{(0)} + \hat{\mathbf{z}} \wedge \tilde{\mathbf{v}}_h^{(0)} = -\nabla_h p^{(0)}, \; \partial_z \tilde{p}^{(0)} + \tilde{\rho}^{(0)} = 0, \; \partial_t \tilde{\rho}^{(0)} - N^2 \tilde{w}^{(0)} = 0. \tag{B80}$$

The fields $\bar{\mathbf{v}}_h^{(0)}, \tilde{\mathbf{v}}_h^{(0)}, \bar{\rho}^{(0)}, \tilde{\rho}^{(0)}, \bar{w}^{(0)}, \tilde{w}^{(0)}$ may be easily found from (B79), (B80) once $\bar{p}^{(0)}$ and $\tilde{p}^{(0)}$ are given.

Equation (B77) allows us to initialize the slow and the fast part of the motion (cf. P1). We find that at the initial moment

$$\partial_z \left(\frac{1}{N^2} \partial_z \bar{p}_I^{(0)} \right) + \nabla_h^2 \bar{p}_I^{(0)} = \Omega_I^{(0)}(\mathbf{r}, t_1, ...) = \zeta_I - \partial_z \left(\frac{\rho_I}{N^2} \right) \tag{B81}$$

and

$$\partial_z \bar{p}_I^{(0)} \Big|_{z=-1,0} = - \rho_I|_{z=-1,0}. \tag{B82}$$

After finding $\bar{p}_I^{(0)}$ one can determine the initial slow velocity and density fields with the help of (B79)

$$\bar{\mathbf{v}}_{h_I}^{(0)} = \hat{\mathbf{z}} \wedge \nabla_h \bar{p}_I^{(0)}, \; \bar{\rho}_I^{(0)} = -\partial_z \bar{p}_I^{(0)}. \tag{B83}$$

Hence

$$\left(\tilde{u}_I^{(0)}, \tilde{v}_I^{(0)}, \tilde{\rho}_I^{(0)} \right) = \left(u_I - \bar{u}_I^{(0)}, v_I - \bar{v}_I^{(0)}, \rho_I - \bar{\rho}_I^{(0)} \right) \tag{B84}$$

and the first initial condition for $\tilde{p}^{(0)}$ readily follows

$$\partial_z \tilde{p}^{(0)} \Big|_{t=0} = -\tilde{\rho}_I^{(0)}. \tag{B85}$$

The second initial condition for $\tilde{p}^{(0)}$ is found from the first equation in (B70) by using (B69) and the hydrostatic equation in (B67):

$$\partial^2_{zt}\left(\frac{1}{N^2}\partial_z\tilde{p}^{(0)}\right)\Big|_{t=0} = D_I. \tag{B86}$$

The problem (B75), (B85), and (B86) is solved by use of Fourier-decomposition in the eigenfunctions Ψ_m of the following eigenproblem

$$\partial_z\left(\frac{1}{N^2}\partial_z\Psi_m\right) + \lambda_m^2\Psi_m = 0;\ \partial_z\Psi_m|_{z=-1,0} = 0;\ m = 0, 1, ..., \tag{B87}$$

where $\Psi_m(z)$ and λ_m are the eigenfunctions and the eigenvalues, respectively. As is well known, the eigenfunctions Ψ_m form a complete orthogonal basis. Thus, the fast pressure field is represented as

$$\tilde{p}^{(0)}(x, y, z; t, t_1, ...) = \sum_{m=0}^{\infty}\tilde{p}^{(0)}_m(x, y; t, t_1, ...)\Psi_m(z). \tag{B88}$$

In order to decompose the initial condition (B85) we write it in the following form:

$$\tilde{p}^{(0)}\Big|_{t=0} = \tilde{F}_0(x, y, z) + F_1(x, y), \tag{B89}$$

where

$$\tilde{F}_0 = -\int_{-1}^{z} dz\,\tilde{\rho}^{(0)}_I - \int_{-1}^{0} dz\,z\tilde{\rho}^{(0)}_I;\ \int_{-1}^{0} dz\,\tilde{F}_0 = 0 \tag{B90}$$

and F_1 is an arbitrary function. For each mode $m \neq 0$ we get the same Klein-Gordon equation as in the RSW case with the only difference that the coefficient in front of Laplacian is λ_m-dependent:

$$-\partial^2_{tt}\tilde{p}^{(0)}_m - \tilde{p}^{(0)}_m + \frac{1}{\lambda_m^2}\nabla^2_h\tilde{p}^{(0)}_n = 0;\ \left(\tilde{p}^{(0)}_m, \partial_t\tilde{p}^{(0)}_m\right)_{t=0} = \left(\tilde{F}_{0_m}, -\frac{1}{\lambda_m^2}\tilde{D}_{I_m}\right). \tag{B91}$$

For $m = 0$ we have

$$\nabla^2_h\tilde{p}^{(0)}_0 = 0,\ \tilde{p}^{(0)}_0\Big|_{t=0} = F_1. \tag{B92}$$

The functions $\tilde{F}_{0_m}, \tilde{D}_{I_m}$ are the coefficients of the corresponding Fourier-harmonics in the chosen basis. In order to have a localized solution we have to impose $\tilde{p}^{(0)}_0 = 0 \Leftrightarrow F_1 = 0$.

The solution of problem (B91) is conveniently written in the form of Fourier-integral:

$$\tilde{p}^{(0)}_m = \int d\mathbf{k}_h\left[\hat{c}^{(+)}_m e^{i(\mathbf{k}_h\cdot\mathbf{r}_h + \omega_m t)} + \hat{c}^{(-)}_m e^{i(\mathbf{k}_h\cdot\mathbf{r}_h - \omega_m t)}\right], \tag{B93}$$

where the modal frequencies are

$$\omega_m = \left(\frac{\mathbf{k}_h^2 + \lambda_m^2}{\lambda_m^2}\right)^{\frac{1}{2}} \tag{B94}$$

and the Fourier-coefficients are

$$\hat{c}_m^{(\pm)} = \frac{1}{2}\left(\hat{F}_{0_m}(\mathbf{k}_h) \mp \frac{\hat{D}_{I_m}(\mathbf{k}_h)}{i\lambda_m^2\omega_m}\right). \tag{B95}$$

Here $\hat{F}_{0_m}, \hat{D}_{I_m}$ are the Fourier-transforms of $\tilde{F}_{0_m}, \tilde{D}_{I_m}$, respectively.

In order to determine the horizontal velocity field we use the equation

$$\tilde{\zeta}^{(0)} = \partial_x \tilde{v}^{(0)} - \partial_y \tilde{u}^{(0)} = \partial_z \left(\frac{\tilde{p}^{(0)}}{N^2}\right) \tag{B96}$$

simply following from (B69). We have from the first equation in (B70) and (B96) that

$$\tilde{D}^{(0)} = \partial_x \tilde{u}^{(0)} + \partial_y \tilde{v}^{(0)} = \partial_z \left(\frac{1}{N^2}\partial_t \tilde{p}^{(0)}\right). \tag{B97}$$

From (B96), (B97) and the hydrostatic equation we obtain the equation

$$\nabla_h^2 \tilde{\mathcal{U}}^{(0)} = (\partial_t - i)\left[\partial_z\left(\frac{1}{N^2}\partial_z\right)\right]\left(\partial_x \tilde{p}^{(0)} + i\partial_y \tilde{p}^{(0)}\right) \tag{B98}$$

for the complex velocity $\tilde{\mathcal{U}}^{(0)} = \tilde{u}^{(0)} + i\tilde{v}^{(0)}$. Once $\tilde{p}^{(0)}$ is known (cf. (B93) - (B95)) we get for the coefficient $\tilde{\mathcal{U}}_m^{(0)}$ of the expression for $\tilde{\mathcal{U}}^{(0)}$ analogous to (B88):

$$\tilde{\mathcal{U}}_m^{(0)} = \int d\mathbf{r}_h\, e^{i\mathbf{k}\cdot\mathbf{r}_h} \hat{\mathcal{U}}_m^{(0)}(\mathbf{r}_h, t), \tag{B99}$$

$$\hat{\mathcal{U}}_m^{(0)} = \frac{\lambda_m^2}{\mathbf{k}^2}(k_1 + ik_2)\left[(1-\omega_m)\hat{c}_m^{(+)} e^{i\omega_m t} + (1+\omega_m)\hat{c}_m^{(-)} e^{-i\omega_m t}\right]. \tag{B100}$$

B2 The first-order solution

The second-order equations of motion are:

$$\begin{aligned}
\partial_t \mathbf{v}_h^{(1)} + \hat{\mathbf{z}} \wedge \mathbf{v}_h^{(1)} + \nabla_h p^{(1)} &= -\partial_{t_1}\mathbf{v}_h^{(0)} - \mathbf{v}^{(0)}\cdot\nabla\mathbf{v}_h^{(0)}, \qquad \text{(B101)}\\
\nabla\cdot\mathbf{v}^{(1)} &= 0,\ \ \partial_z p^{(1)} + \rho^{(1)} = 0,\\
\partial_t \rho^{(1)} - N^2 w^{(1)} &= -\partial_{t_1}\rho^{(0)} - \mathbf{v}^{(0)}\cdot\nabla\rho^{(0)}.
\end{aligned}$$

The first-order PV equation gives (cf. (3.13))

$$\partial_t\left(N^2\Omega^{(1)} - \Pi_2^{(0)} - \sigma{\rho^{(0)}}^2\right) + \left(\partial_{t_1} + \mathbf{v}^{(0)}\cdot\nabla\right)\left(N^2\Omega^{(0)}\right) = 0 \tag{B102}$$

whence by using the last equation in (B67) we have:

$$\partial_t\left(N^2\Omega^{(1)} - \Pi_2^{(0)} - \sigma{\rho^{(0)}}^2 + \frac{\rho^{(0)}}{N^2}\partial_z\left(N^2\Omega^{(0)}\right)\right) = -N^2\left[\left(\partial_{t_1} + \bar{\mathbf{v}}_h^{(0)}\cdot\nabla_h\right)\Omega^{(0)} + \tilde{\mathbf{v}}_h^{(0)}\Omega^{(0)}\right], \tag{B103}$$

where (cf. (3.12))

$$\Pi_2^{(0)} = -\partial_z v^{(0)}\partial_x \rho^{(0)} + \partial_z u^{(0)}\partial_y \rho^{(0)} + \zeta^{(0)}\partial_z \rho^{(0)}. \tag{B104}$$

Averaging (B103) over the fast time gives

$$\left(\partial_{t_1} + \bar{\mathbf{v}}_h^{(0)} \cdot \nabla_h\right)\Omega^{(0)} = 0 \tag{B105}$$

and, therefore,

$$\Omega^{(1)} = \frac{1}{N^2}\left[\Pi_2^{(0)} + \sigma\left(\tilde{\rho}^{(0)^2} + 2\tilde{\rho}^{(0)}\bar{\rho}^{(0)}\right) - \frac{\tilde{\rho}^{(0)}}{N^2}\partial_z\left(N^2\Omega^{(0)}\right)\right] - \tilde{U}_{01}\partial_x\Omega^{(0)} - \tilde{V}_{01}\partial_y\Omega^{(0)} + \bar{\Omega}^{(1)}(\mathbf{r}, t_1, ...). \tag{B106}$$

Here $\tilde{U}_{01}, \tilde{V}_{01}$ are defined via

$$\tilde{U}_{01} + i\tilde{V}_{01} = \int_0^t dt\,(\tilde{u}^{(0)} + i\tilde{v}^{(0)}) - \left\langle \int_0^t dt\,(\tilde{u}^{(0)} + i\tilde{v}^{(0)})\right\rangle \tag{B107}$$

and the angle brackets denote fast-time averaging, as usual. It follows from (B100) that the Fourier-transforms of $\tilde{U}_{01_m}$ in the decomposition $\tilde{U}_{01} = \sum_m \tilde{U}_{01_m}\Psi_m$ are given by

$$\hat{U}_{01_m} = \frac{\lambda_m^2}{\mathbf{k}^2}\frac{(k_1 + ik_2)}{i\omega_m}\left[(1-\omega_m)\hat{c}_m^{(+)}e^{i\omega_m t} - (1+\omega_m)\hat{c}_m^{(-)}e^{-i\omega_m t}\right]. \tag{B108}$$

Splitting all the fields into the slow and the fast components gives for the former:

$$\begin{aligned}
\bar{\mathbf{v}}_h^{(1)} &= \hat{\mathbf{z}} \wedge \left(\nabla_h \bar{p}^{(1)} - \bar{\mathcal{R}}_{\mathbf{v}_h}^{(0)}\right)\\
\bar{w}^{(1)} &= -\frac{\bar{\mathcal{R}}_\rho^{(0)}}{N^2};\\
\partial_z \bar{p}^{(1)} &+ \bar{\rho}^{(1)} = 0,\\
\bar{\zeta}^{(1)} - \partial_z\left(\frac{\bar{\rho}^{(1)}}{N^2}\right) &= \bar{\Omega}^{(1)}(\mathbf{r}, t_1, ...),
\end{aligned} \tag{B109}$$

where

$$\begin{aligned}
\bar{\mathcal{R}}_{\mathbf{v}_h}^{(0)} &= \left(\partial_{t_1} + \bar{\mathbf{v}}_h^{(0)} \cdot \nabla\right)\bar{\mathbf{v}}_h^{(0)}\\
\bar{\mathcal{R}}_\rho^{(0)} &= \left(\partial_{t_1} + \bar{\mathbf{v}}_h^{(0)} \cdot \nabla\right)\bar{\rho}^{(0)},
\end{aligned} \tag{B110}$$

and for the latter:

$$\begin{aligned}
\partial_t \tilde{\mathbf{v}}_h^{(1)} + \hat{\mathbf{z}} \wedge \tilde{\mathbf{v}}_h^{(1)} + \nabla_h \tilde{p}^{(1)}) &= \tilde{\mathcal{R}}_{\mathbf{v}_h}^{(0)}\\
\partial_t \tilde{\rho}^{(1)} - N^2 \tilde{w}^{(1)} &= \tilde{\mathcal{R}}_\rho^{(0)}\\
\partial_z \tilde{p}^{(1)} &+ \tilde{\rho}^{(1)} = 0,\\
\tilde{\zeta}^{(1)} - \partial_z\left(\frac{\tilde{\rho}^{(1)}}{N^2}\right) &= \tilde{\mathcal{R}}_\zeta^{(0)},
\end{aligned} \tag{B111}$$

where

$$\begin{aligned}
\tilde{\mathcal{R}}_{\mathbf{v}_h}^{(0)} &= \left(\partial_{t_1} + \left(\bar{\mathbf{v}}_h^{(0)} + \tilde{\mathbf{v}}_h^{(0)}\right)\cdot\nabla\right)\tilde{\mathbf{v}}_h^{(0)} + \tilde{\mathbf{v}}_h^{(0)}\cdot\nabla\bar{\mathbf{v}}_h^{(0)} + \tilde{w}^{(0)}\partial_z\left(\bar{\mathbf{v}}_h^{(0)} + \tilde{\mathbf{v}}_h^{(0)}\right) \quad \text{(B112)}\\
\tilde{\mathcal{R}}_{\rho}^{(0)} &= \left(\partial_{t_1} + \left(\bar{\mathbf{v}}_h^{(0)} + \tilde{\mathbf{v}}_h^{(0)}\right)\cdot\nabla\right)\tilde{\rho}^{(0)} + \tilde{\mathbf{v}}_h^{(0)}\cdot\nabla_h\bar{\rho}^{(0)} + \tilde{w}^{(0)}\partial_z\left(\bar{\rho}^{(0)} + \tilde{\rho}^{(0)}\right),\\
\tilde{\mathcal{R}}_{\zeta}^{(0)} &= \frac{1}{N^2}\left[\tilde{\Pi}_2^{(0)} + \sigma\left(\tilde{\rho}^{(0)^2} + 2\bar{\rho}^{(0)}\tilde{\rho}^{(0)}\right) - \frac{\tilde{\rho}^{(0)}}{N^2}\partial_z\left(N^2\Omega^{(0)}\right) - N^2\left(\tilde{\mathbf{U}}_{01}\cdot\nabla_h\Omega^{(0)}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Pi}_2^{(0)} &= \left(-\partial_z\bar{v}^{(0)} - \partial_z\tilde{v}^{(0)}\right)\partial_x\tilde{\rho}^{(0)} + \left(\partial_z\bar{u}^{(0)} + \partial_z\tilde{u}^{(0)}\right)\partial_y\tilde{\rho}^{(0)}\\
&+ \left(\bar{\zeta}^{(0)} + \tilde{\zeta}^{(0)}\right)\partial_z\tilde{\rho}^{(0)} - \partial_z\tilde{v}^{(0)}\partial_x\bar{\rho}^{(0)} + \partial_z\tilde{u}^{(0)}\partial_y\bar{\rho}^{(0)} + \tilde{\zeta}^{(0)}\partial_z\bar{\rho}^{(0)} \quad \text{(B113)}
\end{aligned}$$

The initial conditions for (B101) are

$$\mathbf{v}_I^{(1)} = 0,\ \rho_I^{(1)} = 0. \quad \text{(B114)}$$

Boundary conditions follow from the density equation in (B101):

$$\partial_t\rho^{(1)}\Big|_{z=-1,0} = -\left(\partial_{t_1}\rho^{(0)} + \mathbf{v}^{(0)}\cdot\nabla_h\rho^{(0)}\right)\Big|_{z=-1,0}. \quad \text{(B115)}$$

Since $\tilde{\rho}^{(0)}\big|_{z=-1,0} = 0$ (see (B76)), we have from (B115)

$$\left(\partial_{t_1}\bar{\rho}^{(0)} + \mathbf{v}^{(0)}\cdot\nabla_h\bar{\rho}^{(0)}\right)\Big|_{z=-1,0} = 0, \quad \text{(B116)}$$

$$\tilde{\rho}^{(1)}\Big|_{z=-1,0} = -\left(\tilde{\mathbf{U}}_{01}\cdot\nabla_h\bar{\rho}^{(0)}\right)\Big|_{z=-1,0}. \quad \text{(B117)}$$

The boundary conditions (B116) together with the PV equation (B105) and known $\bar{p}_I^{(0)} = \bar{p}^{(0)}\big|_{t_1=0}$ constitute the complete problem for the lowest-order slow component:

$$\partial_{t_1}\left(\nabla_h^2\bar{p}^{(0)} + \partial_z\left(\frac{1}{N^2}\partial_z\bar{p}^{(0)}\right)\right) + J\left(\bar{p}^{(0)}, \nabla_h^2\bar{p}^{(0)} + \partial_z\left(\frac{1}{N^2}\partial_z\bar{p}^{(0)}\right)\right) = 0, \quad \text{(B118)}$$

$$\begin{aligned}
\partial_z\partial_{t_1}\bar{p}^{(0)} + J\left(\bar{p}^{(0)}, \partial_z\bar{p}^{(0)}\right)\Big|_{z=-1,0} &= 0,\\
\bar{p}^{(0)}\Big|_{t_1=0} &= \bar{p}_I^{(0)}, \quad \text{(B119)}
\end{aligned}$$

where $\bar{p}_I^{(0)}$ is determined from the problem (B81, B82). From (B109) and (B111) one can obtain a single equation for $\bar{p}^{(1)}$ and a single equation for $\tilde{p}^{(1)}$, respectively:

$$\nabla_h^2\bar{p}^{(1)} + \partial_z\left(\frac{1}{N^2}\partial_z\bar{p}^{(1)}\right) = \bar{\Omega}^{(1)} + \nabla_h\cdot\bar{\mathcal{R}}_{\mathbf{v}_h}^{(0)} = \bar{\Omega}^{(1)} + 2J\left(\partial_x\bar{p}^{(0)}, \partial_y\bar{p}^{(0)}\right), \quad \text{(B120)}$$

$$\left(\partial_t^2+1\right)\partial_z\left(\frac{1}{N^2}\partial_z\tilde{p}^{(1)}\right)+\nabla_h^2\tilde{p}^{(1)}=\left(\partial_t^2+1\right)\tilde{\mathcal{R}}_\zeta^{(0)}-\partial_t\tilde{\mathcal{Z}}^{(0)}+\nabla_h\cdot\tilde{\mathcal{R}}_{\mathbf{v}_h}^{(0)}. \quad \text{(B121)}$$

Here $\tilde{\mathcal{Z}}^{(0)}=\partial_x\tilde{\mathcal{R}}_v^{(0)}-\partial_y\tilde{\mathcal{R}}_u^{(0)}$. Boundary conditions for (B121) follow from (B117) and the hydrostatic equation in (B111):

$$\partial_z\tilde{p}^{(1)}\Big|_{z=-1,0}=-\,\tilde{\rho}^{(1)}\Big|_{z=-1,0}=\left(\bar{\mathbf{U}}_{01}\cdot\bar{\rho}^{(0)}\right)\Big|_{z=-1,0}. \quad \text{(B122)}$$

In order to determine the initial conditions for (B121) we use (B120) at $t=0$:

$$\nabla_h^2\bar{p}_I^{(1)}+\partial_z\left(\frac{1}{N^2}\partial_z\bar{p}_I^{(1)}\right)=\bar{\Omega}_I^{(1)}+2J\left(\partial_x\bar{p}_I^{(0)},\partial_y\bar{p}_I^{(0)}\right). \quad \text{(B123)}$$

The function $\bar{\Omega}_I^{(1)}$is calculated from (B106), (B122):

$$\bar{\Omega}_I^{(1)}=-\frac{1}{N^2}\left[\Pi_{2_I}^{(0)}+\sigma\left(\bar{\rho}_I^{(0)^2}+2\bar{\rho}_I^{(0)}\tilde{\rho}_I^{(0)}\right)-\frac{1}{N^2}\bar{\rho}_I^{(0)}\partial_z\left(N^2\Omega_I^{(0)}\right)\right]+\bar{U}_{01_I}\partial_x\Omega_I^{(0)}+\bar{V}_{01_I}\partial_y\Omega_I^{(0)}. \quad \text{(B124)}$$

Boundary conditions for (B123) follow from (B114), (B122):

$$\partial_z\bar{p}_I^{(1)}\Big|_{z=-1,0}=-\left(\bar{\mathbf{U}}_{01_I}\cdot\nabla\bar{\rho}_I^{(0)}\right)\Big|_{z=-1,0}. \quad \text{(B125)}$$

Solution of (B123, B125) allows to find $\bar{p}_I^{(1)}$ and, therefore,

$$\tilde{p}_I^{(1)}=-\bar{p}_I^{(1)}. \quad \text{(B126)}$$

The second initial condition for (B121) is determined from (B111) and (B114). From the momentum equation in (B111) we get the vorticity equation

$$\partial_t\tilde{\zeta}^{(1)}+D^{(1)}=\partial_y\tilde{\mathcal{R}}_u^{(0)}-\partial_x\tilde{\mathcal{R}}_v^{(0)} \quad \text{(B127)}$$

and, hence, (cf. (B114))

$$\partial_t\tilde{\zeta}^{(1)}\Big|_{t=0}=\partial_y\tilde{\mathcal{R}}_{u_I}^{(0)}-\partial_x\tilde{\mathcal{R}}_{v_I}^{(0)}. \quad \text{(B128)}$$

From the last two equations in (B111) one obtains

$$\partial_t\tilde{\zeta}^{(1)}+\partial_t\partial_z\left(\frac{1}{N^2}\partial_z\tilde{p}^{(1)}\right)=\partial_t\tilde{\mathcal{R}}_\zeta^{(0)}. \quad \text{(B129)}$$

The second initial condition for $\tilde{p}^{(1)}$ follows from (B128, B129):

$$\partial_t\partial_z\left(\frac{1}{N^2}\partial_z\tilde{p}^{(1)}\right)_{t=0}=\partial_t\tilde{\mathcal{R}}_\zeta^{(0)}\Big|_{t=0}-\left(\partial_y\tilde{\mathcal{R}}_{u_I}^{(0)}-\partial_x\tilde{\mathcal{R}}_{v_I}^{(0)}\right). \quad \text{(B130)}$$

Thus we get for $\tilde{p}^{(1)}$ the closed problem (B121), (B122), (B126), and (B130).

B3 The second-order PV equation

At the third order we analyze only the PV equation which has the form (see (3.13)):

$$\partial_t\left(N^2\Omega^{(2)} - \Pi_2^{(1)} - 2\sigma\rho^{(0)}\rho^{(1)}\right) + \partial_{t_1}\left(N^2\Omega^{(1)} - \Pi_2^{(0)} - \sigma\rho^{(0)^2}\right) + \partial_{t_2}\left(N^2\Omega^{(0)}\right) \quad \text{(B131)}$$
$$+\mathbf{v}_h^{(0)}\cdot\nabla_h\left(N^2\Omega^{(1)} - \Pi_2^{(0)} - \sigma\rho^{(0)^2}\right) + \mathbf{v}^{(1)}\cdot\nabla\left(N^2\Omega^{(0)}\right) + \frac{d\sigma}{dz}w^{(0)}\rho^{(0)^2} = 0.$$

Averaging this equation over the fast time and using the last of the equations (B67) and the the fact that the fast-fast contributions vanish due to the radiation boundary conditions for the waves we get

$$\partial_{t_1}\left(N^2\bar{\Omega}^{(1)} - \bar{\Pi}_2^{(0)} - \sigma\bar{\rho}^{(0)^2}\right) + \partial_{t_2}\left(N^2\bar{\Omega}^{(0)}\right) + \quad \text{(B132)}$$
$$\bar{\mathbf{v}}_h^{(0)}\cdot\nabla_h\left(N^2\bar{\Omega}^{(1)} - \bar{\Pi}_2^{(0)} - \sigma\bar{\rho}^{(0)^2}\right) + \bar{\mathbf{v}}_h^{(1)}\cdot\nabla_h\left(N^2\bar{\Omega}_0\right) + \bar{w}^{(1)}\partial_z\left(N^2\bar{\Omega}_0\right) = 0.$$

We rewrite now this equation in terms of slow pressures $\bar{p}^{(0)}, \bar{p}^{(1)}$. From (B109) and (B110) we obtain the equation:

$$\bar{\mathbf{v}}_h^{(1)}\cdot\nabla_h\left(N^2\bar{\Omega}^{(0)}\right) = J\left(\bar{p}^{(1)}, N^2\Omega^{(0)}\right) - \left[\nabla_h\left(\partial_{t_1}\bar{p}^{(0)}\right)\cdot\nabla_h\left(N^2\Omega^{(0)}\right) + J\left(\bar{p}^{(0)}, \nabla_h\bar{p}^{(0)}\right)\cdot\nabla_h\left(N^2\Omega_0\right)\right] \quad \text{(B133)}$$

which can be represented in the following form with the help of (B105):

$$\bar{\mathbf{v}}_h^{(1)}\cdot\nabla_h\left(N^2\bar{\Omega}^{(0)}\right) = J\left(\bar{p}^{(1)} - \frac{\left(\nabla\bar{p}^{(0)}\right)^2}{2}, N^2\Omega^{(0)}\right) - \partial_{t_1}\left[\nabla_h\bar{p}^{(0)}\cdot\nabla_h\left(N^2\Omega^{(0)}\right)\right]$$
$$- J\left(\bar{p}^{(0)}, \nabla\bar{p}^{(0)}\cdot\nabla\left(N^2\Omega^{(0)}\right)\right). \quad \text{(B134)}$$

Analogously, using (B109), (B110), and (B105) we have

$$\bar{w}^{(1)}\partial_z\left(N^2\Omega^{(0)}\right) = -\partial_{t_1}\left(\frac{1}{N^2}\partial_z\bar{p}^{(0)}\partial_z\left(N^2\Omega^{(0)}\right)\right) - J\left(\bar{p}^{(0)}, \frac{1}{N^2}\partial_z\bar{p}^{(0)}\partial_z\left(N^2\Omega^{(0)}\right)\right)$$
$$- J\left(\frac{\partial_z\bar{p}^{(0)^2}}{2N^2}, N^2\Omega_0\right). \quad \text{(B135)}$$

From (B109) and (B79) we obtain

$$\bar{\Omega}^{(1)} = \bar{\zeta}^{(1)} - \partial_z\left(\frac{\bar{\rho}^{(1)}}{N^2}\right) = \nabla_h^2\bar{p}^{(1)} + \partial_z\left(\frac{1}{N^2}\partial_z\bar{p}^{(1)}\right) - 2J\left(\partial_x\bar{p}^{(0)}, \partial_y\bar{p}^{(0)}\right). \quad \text{(B136)}$$

By virtue of (B79) $\bar{\Pi}_2^{(0)}$ can be rewritten as

$$\bar{\Pi}_2^{(0)} = -\partial_z\bar{v}^{(0)}\partial_x\bar{p}^{(0)} + \partial_z\bar{u}^{(0)}\partial_y\bar{p}^{(0)} + \nabla_h^2\bar{p}^{(0)}\partial_z\bar{p}^{(0)} = -\nabla_h^2\bar{p}^{(0)}\partial_{zz}^2\bar{p}^{(0)} + \left(\partial_{xz}^2\bar{p}^{(0)}\right)^2 + \left(\partial_{yz}^2\bar{p}^{(0)}\right)^2. \quad \text{(B137)}$$

Substituting (B134), (B135), (B136), and (B137) into (B132) and combining the resulting equation with (B105) we finally get a single "improved" QG equation by introducing the "full" slow pressure field $\bar{p} = \bar{p}^{(0)} + \epsilon\bar{p}^{(1)}$:

$$\frac{D}{Dt_1}\left[\partial_z\left(\frac{1}{N^2}\partial_z\bar{p}\right) + \nabla_h^2\bar{p} - \epsilon 2J(\partial_x\bar{p}, \partial_y\bar{p}) + \frac{\epsilon}{N^2}\left(\partial_{zz}^2\bar{p}\nabla_h^2\bar{p} - \left(\partial_{zx}^2\bar{p}\right)^2 - \left(\partial_{zy}^2\bar{p}\right)^2 - \sigma\left(\partial_z\bar{p}\right)^2 - \nabla_N\bar{p}\cdot\nabla\left[N^2\left(\nabla_h^2\bar{p} + \partial_z\left(\frac{1}{N^2}\partial_z\bar{p}\right)\right)\right]\right)\right] = 0, \tag{B138}$$

where $\frac{D}{Dt_1}$ is the advective derivative corresponding to the "full" velocity field $\bar{\mathbf{v}}^{(0)} + \epsilon\bar{\mathbf{v}}^{(1)}$ given by

$$\frac{D}{Dt_1}\ldots = \partial_{t_1}\ldots + J\left(\bar{p} - \frac{\epsilon}{2}\nabla\bar{p}\cdot\nabla_N\bar{p}, \ldots\right) \tag{B139}$$

and we introduced a modified nabla $\nabla_N = \left(\nabla_h, \frac{1}{N^2}\partial_z\right)$.