Appendices to "Short-wave instability due to wall slip and numerical observation of wall-slip instability for microchannel flows"

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A Explicit form of the determinant for the short-wave approximation

The determinant in the left hand side of (25) is computed as

$$D_R = \begin{vmatrix} e^{-\alpha}\cos\theta_1 & e^{-\alpha q}\cos\theta_3 & e^{\alpha}\cos\theta_1 & e^{\alpha q}\cos\theta_3 \\ e^{\alpha}\cos\theta_2 & e^{\alpha q}\cos\theta_4 & e^{-\alpha}\cos\theta_2 & e^{-\alpha q}\cos\theta_4 \\ K_{31}e^{-\alpha} & K_{32}e^{-\alpha q} & K_{33}e^{\alpha} & K_{34}e^{\alpha q} \\ K_{41}e^{\alpha} & K_{42}e^{\alpha q} & K_{43}e^{-\alpha} & K_{44}e^{-\alpha q} \end{vmatrix}$$

$$= K_{31}K_{41}\cos\theta_1\cos\theta_2 \times \begin{vmatrix} 1 & e^{-\alpha(q-1)}\frac{\cos\theta_3}{\cos\theta_1} & e^{2\alpha} & e^{\alpha(q+1)}\frac{\cos\theta_3}{\cos\theta_1} \\ 1 & e^{\alpha(q-1)}\frac{\cos\theta_4}{\cos\theta_2} & e^{-2\alpha} & e^{-\alpha(q+1)}\frac{\cos\theta_4}{\cos\theta_2} \\ 1 & e^{-\alpha(q-1)}\frac{K_{32}}{K_{31}} & e^{2\alpha}\frac{K_{33}}{K_{31}} & e^{\alpha(q+1)}\frac{K_{34}}{K_{31}} \\ 1 & e^{\alpha(q-1)}\frac{K_{42}}{K_{41}} & e^{-2\alpha}\frac{K_{43}}{K_{41}} & e^{-\alpha(q+1)}\frac{K_{44}}{K_{41}} \end{vmatrix}$$

$$= K_{31}K_{41}\cos\theta_1\cos\theta_2 \times$$
$$\begin{vmatrix} 1 & e^{-\alpha(q-1)}\frac{\cos\theta_3}{\cos\theta_1} & e^{2\alpha} & e^{\alpha(q+1)}\frac{\cos\theta_3}{\cos\theta_1} \\ 0 & e^{\alpha(q-1)}\frac{\cos\theta_4}{\cos\theta_2} - e^{-\alpha(q-1)}\frac{\cos\theta_3}{\cos\theta_1} & 2\sinh(-2\alpha) & e^{-\alpha(q+1)}\frac{\cos\theta_4}{\cos\theta_2} - e^{\alpha(q+1)}\frac{\cos\theta_3}{\cos\theta_1} \\ 0 & e^{-\alpha(q-1)}(\frac{K_{32}}{K_{31}} - \frac{\cos\theta_3}{\cos\theta_1}) & e^{2\alpha}(\frac{K_{33}}{K_{31}} - 1) & e^{\alpha(q+1)}(\frac{K_{34}}{K_{31}} - \frac{\cos\theta_3}{\cos\theta_1}) \\ 0 & e^{\alpha(q-1)}\frac{K_{42}}{K_{41}} - e^{-\alpha(q-1)}\frac{\cos\theta_3}{\cos\theta_1} & e^{-2\alpha}\frac{K_{43}}{K_{41}} - e^{2\alpha} & e^{-\alpha(q+1)}\frac{K_{44}}{K_{41}} - e^{\alpha(q+1)}\frac{\cos\theta_3}{\cos\theta_1} \end{vmatrix}$$

$$= K_{31}K_{41}\cos\theta_1\cos\theta_2 \times$$
$$\begin{vmatrix} e^{\alpha(q-1)}\frac{\cos\theta_4}{\cos\theta_2} - e^{-\alpha(q-1)}\frac{\cos\theta_3}{\cos\theta_1} & 2\sinh(-2\alpha) & e^{-\alpha(q+1)}\frac{\cos\theta_4}{\cos\theta_2} - e^{\alpha(q+1)}\frac{\cos\theta_3}{\cos\theta_1} \\ e^{-\alpha(q-1)}(\frac{K_{32}}{K_{31}} - \frac{\cos\theta_3}{\cos\theta_1}) & e^{2\alpha}(\frac{K_{33}}{K_{31}} - 1) & e^{\alpha(q+1)}(\frac{K_{34}}{K_{31}} - \frac{\cos\theta_3}{\cos\theta_1}) \\ e^{\alpha(q-1)}\frac{K_{42}}{K_{41}} - e^{-\alpha(q-1)}\frac{\cos\theta_3}{\cos\theta_1} & e^{-2\alpha}\frac{K_{43}}{K_{41}} - e^{2\alpha} & e^{-\alpha(q+1)}\frac{K_{44}}{K_{41}} - e^{\alpha(q+1)}\frac{\cos\theta_3}{\cos\theta_1} \end{vmatrix}$$

or

$$
\begin{aligned}
D_R = & e^{2\alpha q+2\alpha}\{(K_{34}\cos\theta_1 - K_{33}\cos\theta_3)(K_{42}\cos\theta_2 - K_{41}\cos\theta_4)\} + \\
& e^{2\alpha q-2\alpha}\{(K_{31}\cos\theta_3 - K_{34}\cos\theta_1)(K_{42}\cos\theta_2 - K_{43}\cos\theta_4)\} + \\
& e^{-2\alpha q+2\alpha}\{(K_{32}\cos\theta_1 - K_{33}\cos\theta_3)(K_{41}\cos\theta_4 - K_{44}\cos\theta_2)\} + \\
& e^{-2\alpha q-2\alpha}\{(K_{32}\cos\theta_1 - K_{31}\cos\theta_3)(K_{44}\cos\theta_2 - K_{43}\cos\theta_4)\} + \\
& \{(K_{32} - K_{34})(K_{43} - K_{41})\cos\theta_2\cos\theta_3 - \\
& (K_{33} - K_{31})(K_{44} - K_{42})\cos\theta_1\cos\theta_4\}.
\end{aligned}
$$

By use of the following transformation

$$
\begin{aligned}
&\cos\theta_1 \rightarrow \sin\theta_1, \cos\theta_2 \rightarrow \sin\theta_2, \cos\theta_3 \rightarrow \sin\theta_3, \cos\theta_4 \rightarrow \sin\theta_4, \\
&K_{31} \rightarrow -G_{31}, K_{32} \rightarrow -G_{32}, K_{33} \rightarrow -G_{33}, K_{34} \rightarrow -G_{34}, \\
&K_{41} \rightarrow -G_{41}, K_{42} \rightarrow -G_{42}, K_{43} \rightarrow -G_{43}, K_{44} \rightarrow -G_{44},
\end{aligned}
$$

we obtain the expression of (26) as follows:

$$
\begin{aligned}
D_I = & e^{2\alpha q+2\alpha}\{(-G_{34}\sin\theta_1 + G_{33}\sin\theta_3)(-G_{42}\sin\theta_2 + G_{41}\sin\theta_4)\} + \\
& e^{2\alpha q-2\alpha}\{(-G_{31}\text{sin}\theta_3 + G_{34}\text{sin}\theta_1)(-G_{42}\text{sin}\theta_2 + G_{43}\text{sin}\theta_4)\} + \\
& e^{-2\alpha q+2\alpha}\{(-G_{32}\text{sin}\theta_1 + G_{33}\text{sin}\theta_3)(-G_{41}\text{sin}\theta_4 + G_{44}\text{sin}\theta_2)\} + \\
& e^{-2\alpha q-2\alpha}\{(-G_{32}\text{sin}\theta_1 + G_{31}\text{sin}\theta_3)(-G_{44}\text{sin}\theta_2 + G_{43}\text{sin}\theta_4)\} + \\
& \{(-G_{32} + G_{34})(-G_{43} + G_{41})\text{sin}\theta_2\text{sin}\theta_3 - \\
& (-G_{33} + G_{31})(-G_{44} + G_{42})\text{sin}\theta_1\text{sin}\theta_4\}.
\end{aligned}
$$

B Explicit form of the determinant for the long-wave approximation

The determinant for the long-wave approximation and $q > 1$ in the left hand side of (45) is computed as

$$
\begin{aligned}
D = & \begin{vmatrix}
e^{-\alpha} & e^{-q\alpha} & e^{\alpha} & e^{q\alpha} \\
e^{\alpha} & e^{q\alpha} & e^{-\alpha} & e^{-q\alpha} \\
(-\alpha+\alpha^2 b)e^{-\alpha} & (-q\alpha+q^2\alpha^2 b)e^{-q\alpha} & (\alpha+\alpha^2 b)e^{\alpha} & (q\alpha+q^2\alpha^2 b)e^{q\alpha} \\
(-\alpha-\alpha^2 b)e^{\alpha} & (-q\alpha-q^2\alpha^2 b)e^{q\alpha} & (\alpha-\alpha^2 b)e^{-\alpha} & (q\alpha-q^2\alpha^2 b)e^{-q\alpha}
\end{vmatrix} \\
= & (\text{-}\alpha+\alpha^2\text{b}_1)(-\alpha-\alpha^2 b)\cdot \\
& \begin{vmatrix}
1 & \text{e}^{(\text{1-q})\alpha} & \text{e}^{2\alpha} & \text{e}^{(\text{q+1})\alpha} \\
1 & \text{e}^{(\text{q-1})\alpha} & \text{e}^{\text{-}2\alpha} & \text{e}^{\text{-}(\text{q+1})\alpha} \\
1 & \frac{(\text{-q}\alpha+\text{q}^2\alpha^2\text{b}_1)}{-\alpha+\alpha^2\text{b}_1}\text{e}^{(\text{1-q})\alpha} & \frac{(\alpha+\alpha^2\text{b}_1)}{-\alpha+\alpha^2\text{b}_1}\text{e}^{2\alpha} & \frac{\text{q}\alpha+\text{q}^2\alpha^2\text{b}_1}{-\alpha+\alpha^2\text{b}_1}\text{e}^{(\text{q+1})\alpha} \\
1 & \frac{(\text{q}\alpha+\text{q}^2\alpha^2\text{b}_1)}{\alpha+\alpha^2\text{b}_1}\text{e}^{(\text{q-1})\alpha} & \text{-}\frac{(\alpha\text{-}\alpha^2\text{b}_1)}{\alpha+\alpha^2\text{b}_1}\text{e}^{\text{-}2\alpha} & \text{-}\frac{\text{q}\alpha\text{-q}^2\alpha^2\text{b}_1}{\alpha+\alpha^2\text{b}_1}\text{e}^{-(\text{q+1})\alpha}
\end{vmatrix}
\end{aligned}
$$

$$= (\text{-}\alpha+\alpha^2 b_1)(-\alpha-\alpha^2 b)\cdot$$

$$\begin{vmatrix} 1 & e^{(1\text{-}q)\alpha} & e^{2\alpha} & e^{(q+1)\alpha} \\ 0 & 2\sinh[(q\text{-}1)\alpha] & 2\sinh(\text{-}2\alpha) & 2\sinh[\text{-}(q+1)\alpha] \\ 0 & (\frac{\text{-}q\alpha+q^2\alpha^2 b_1}{-\alpha+\alpha^2 b_1}\text{-}1)e^{(1\text{-}q)\alpha} & (\frac{\alpha+\alpha^2 b_1}{-\alpha+\alpha^2 b_1}-1)e^{2\alpha} & (\frac{q\alpha+q^2\alpha^2 b_1}{-\alpha+\alpha^2 b_1}-1)e^{(q+1)\alpha} \\ 0 & \frac{(q\alpha+q^2\alpha^2 b_1)}{\alpha+\alpha^2 b_1}e^{(q\text{-}1)\alpha}\text{-}e^{(1\text{-}q)\alpha} & \text{-}\frac{(\alpha\text{-}\alpha^2 b_1)}{\alpha+\alpha^2 b_1}e^{\text{-}2\alpha}\text{-}e^{2\alpha} & \text{-}\frac{q\alpha\text{-}q^2\alpha^2 b_1}{\alpha+\alpha^2 b_1}e^{\text{-}(q+1)\alpha}\text{-}e^{(q+1)\alpha} \end{vmatrix}$$

$$= (\text{-}\alpha+\alpha^2 b_1)(-\alpha-\alpha^2 b)\cdot$$

$$\begin{vmatrix} 2\sinh[(q\text{-}1)\alpha] & 2\sinh(\text{-}2\alpha) & 2\sinh[\text{-}(q+1)\alpha] \\ (\frac{\text{-}q\alpha+q^2\alpha^2 b_1}{-\alpha+\alpha^2 b_1}\text{-}1)e^{(1\text{-}q)\alpha} & (\frac{\alpha+\alpha^2 b_1}{-\alpha+\alpha^2 b_1}\text{-}1)e^{2\alpha} & (\frac{q\alpha+q^2\alpha^2 b_1}{-\alpha+\alpha^2 b_1}-1)e^{(q+1)\alpha} \\ \frac{(q\alpha+q^2\alpha^2 b_1)}{\alpha+\alpha^2 b_1}e^{(q\text{-}1)\alpha}\text{-}e^{(1\text{-}q)\alpha} & \text{-}\frac{(\alpha\text{-}\alpha^2 b_1)}{\alpha+\alpha^2 b_1}e^{\text{-}2\alpha}\text{-}e^{2\alpha} & \text{-}\frac{q\alpha\text{-}q^2\alpha^2 b_1}{\alpha+\alpha^2 b_1}e^{\text{-}(q+1)\alpha}\text{-}e^{(q+1)\alpha} \end{vmatrix},$$

hence we have

$$\begin{aligned} D = & -2(\eta_1^2+\eta_2^2)[\cosh(2q\alpha)\cosh(2\alpha)+\sinh(2q\alpha)\sinh(2\alpha)] \\ & -4\eta_1\eta_2[\sinh(2q\alpha)\cosh(2\alpha)+\cosh(2q\alpha)\sinh(2\alpha)] \\ & +2(\eta_1^2+\eta_3^2)[\cosh(2q\alpha)\cosh(2\alpha)-\sinh(2q\alpha)\sinh(2\alpha)] \\ & +4\eta_1\eta_3[\sinh(2q\alpha)\cosh(2\alpha)-\cosh(2q\alpha)\sinh(2\alpha)]-8q\alpha^2 \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= (q^2-1)\alpha^2 b \\ \eta_2 &= (q-1)\alpha \\ \eta_3 &= (q+1)\alpha \end{aligned}$$

C Proof of $D \neq 0$ for $q > 1$

The expression for the long-wave approximation with $q > 1$ is expressed as

$$\begin{aligned} D = & -2(\eta_1^2+\eta_2^2)[\cosh(2q\alpha)\cosh(2\alpha)+\sinh(2q\alpha)\sinh(2\alpha)] \\ & -4\eta_1\eta_2[\sinh(2q\alpha)\cosh(2\alpha)+\cosh(2q\alpha)\sinh(2\alpha)] \\ & +2(\eta_1^2+\eta_3^2)[\cosh(2q\alpha)\cosh(2\alpha)-\sinh(2q\alpha)\sinh(2\alpha)] \\ & +4\eta_1\eta_3[\sinh(2q\alpha)\cosh(2\alpha)-\cosh(2q\alpha)\sinh(2\alpha)]-8q\alpha^2 \end{aligned}$$

or

$$\begin{aligned} D = & -4\eta_1^2\sinh(2q\alpha)\sinh(2\alpha)] \\ & -2\eta_2^2[\cosh(2q\alpha+2\alpha)]+2\eta_3^2[\cosh(2q\alpha-2\alpha)]-8q\alpha^2 \\ & -4\eta_1\eta_2[\sinh(2q\alpha+2\alpha)]+4\eta_1\eta_3\sinh(2q\alpha-2\alpha) \end{aligned}$$

where

$$\begin{aligned}\eta_1 &= (q^2-1)\alpha^2 b\\ \eta_2 &= (q-1)\alpha\\ \eta_3 &= (q+1)\alpha.\end{aligned}$$

Obviously the expression

$$\begin{aligned}&-2\eta_2^2[\cosh(2q\alpha+2\alpha)] + 2\eta_3^2[\cosh(2q\alpha-2\alpha)] - 8q\alpha^2\\ &= -2\eta_2^2\{1+\frac{2^2\eta_3^2}{2!}+\frac{2^4\eta_3^4}{4!}+\cdots\} + 2\eta_3^2\{1+\frac{2^2\eta_2^2}{2!}+\frac{2^4\eta_2^4}{4!}+\cdots\} - 8q\alpha^2\end{aligned}$$

and the expression

$$\begin{aligned}&-4\eta_1\eta_2[\sinh(2q\alpha+2\alpha)] + 4\eta_1\eta_3\sinh(2q\alpha-2\alpha)\\ &= -4\eta_1\{\eta_2\sinh(2\eta_3) - \eta_3\sinh(2\eta_2)\}\\ &= -4\eta_1\{\eta_2[2\eta_3+\frac{(2\eta_3)^3}{3!}+\frac{(2\eta_3)^5}{5!}+\cdots] - \eta_3[2\eta_2+\frac{(2\eta_2)^3}{3!}+\frac{(2\eta_2)^5}{5!}+\cdots]\}\end{aligned}$$

are always less than 0, therefore for any $b > 0$ and $\alpha > 0$, we can obtain that the expression D is always less than 0, i.e.

$$D \neq 0$$

with $q > 1$.

D Proof of $D \neq 0$ for $q = 1$

The expression for the long-wave approximation with $q = 1$ is described as follows:

$$\begin{aligned}D(q=1) &= (1+2\alpha b)^2 e^{4\alpha} + (1-2\alpha b)^2 e^{-4\alpha} - 8\alpha^2(1+2b+\alpha b+b) - 2\\ &= [1+(2\alpha b)^2]\cosh(4\alpha) + 8\alpha b\sinh(4\alpha) - 8\alpha^2(1+2b+\alpha b+b) - 2\\ &> 8\left(8\alpha^4 b + (2\alpha^2+\frac{32}{3}\alpha^4-\alpha^3)b+\alpha^2\right)\end{aligned}$$

Setting

$$F = 8\alpha^4 b + (2\alpha^2+\frac{32}{3}\alpha^4-\alpha^3)b+\alpha^2,$$

the derivative of F with respect to α can be expressed as

$$\frac{dF}{d\alpha} = 32\alpha^3 b + (4\alpha+\frac{128}{3}\alpha^3-3\alpha^2)b + 2\alpha.$$

i) If $(4\alpha + \frac{128}{3}\alpha^{\ 3} - 3\alpha^2)^2 - 4 \cdot 32\alpha^3 \cdot (2\alpha) < 0$, then the inequality $\frac{dF}{d\alpha} > 0$ always holds.

ii) If $(4\alpha + \frac{128}{3}\alpha^3 - 3\alpha^2)^2 - 4 \cdot 32\alpha^3 \cdot (2\alpha) \geq 0$ and setting $\frac{dF}{d\alpha} = 0$, due to the slip length $b > 0$, hence the coefficient of b should be less than 0, i.e. $(4\alpha + \frac{128}{3}\alpha^3 - 3\alpha^2) < 0$. However there is no wave number α (> 0) leading to the inequality $(4\alpha + \frac{128}{3}\alpha^3 - 3\alpha^2) < 0$ holding, so for any $b > 0$ and $\alpha > 0$, the value of the derivative $\frac{dF}{d\alpha}$ is always either larger than 0 or less than 0. Additionally for $\alpha = 0.1$, we have $\frac{dF}{d\alpha} = 0.263 > 0$, therefore the inequality $\frac{dF}{d\alpha} > 0$ holds.

Hence $\frac{dF}{d\alpha} > 0$ holds for all cases. This implies that $F(\alpha) \geq F(0) = 0$ and then $D(q = 1) \neq 0$.

E Numerical method

The Navier-Stokes equations in Cartesian coordinates for a perfect gas obeying the γ-law are

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} + \frac{\partial g(w)}{\partial y} = \frac{\partial f^{(V)}(w)}{\partial x} + \frac{\partial g^{(V)}(w)}{\partial y} \tag{1}$$

where

$$w = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho e \end{pmatrix}, f(w) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho ue \end{pmatrix}, g = \begin{pmatrix} \rho v \\ \rho vu \\ \rho v^2 + p \\ \rho ve \end{pmatrix}$$

$$f^{(V)}(w) = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} - F_x \end{pmatrix}, g^{(V)}(w) = \begin{pmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ u\tau_{yx} + v\tau_{yy} - F_y \end{pmatrix}$$

Here ρ, p and e are the density, pressure and specific total energy of the gas, with

$$e = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho},$$

$\tau_{xx}, \tau_{xy}, \tau_{yx}$ and τ_{yy} are viscous stresses defined, for a Newtonian fluid, by

$$\begin{aligned}
\tau_{xx} &= -\frac{2\mu}{3}(u_x + v_y) + 2\mu u_x \\
\tau_{xy} &= \mu(u_y + v_x) = \tau_{yx} \\
\tau_{yy} &= -\frac{2\mu}{3}(u_x + v_y) + 2\mu v_y
\end{aligned}$$

where μ is the dynamic viscosity coefficient, which is related to the temperature

$$T = \frac{1}{c_V}\frac{1}{\gamma - 1}\frac{p}{\rho}$$

by

$$\frac{\mu}{\mu_\infty} = \left(\frac{T}{T_\infty}\right)^{\frac{2}{3}} \tag{2}$$

where $\mu_\infty = \mu_\infty(T_\infty)$.

The heat flux $\digamma_x$ and $\digamma_y$, when assuming a constant specific heat, are defined by

$$\digamma_x = -\frac{\gamma\mu}{P_r}\left(\frac{p}{(\gamma-1)\rho}\right)_x, \digamma_y = -\frac{\gamma\mu}{P_r}\left(\frac{p}{(\gamma-1)\rho}\right)_y$$

where P_r is the Prandtl number, taken to be 0.72 in this paper.

For computational purpose, one often uses the nondimensional form. When using nondimensional parameters, the equations remain unchanged, except that μ is now replaced by $1/R_e$, where R_e is the Reynolds number.

As the system of Navier-Stokes equations (1) is in conservation form, any well-established numerical method for compressible flows can be used to solve the inviscid part. We shall use the Godunov method with the MUSCL update to higher resolution to solve system (1). The viscous part is discretized by using central differences.

Let $\mathbf{E}_{i,j}^k$ be the numerical solution at point (i, j) and time step k. Applying the divergence theorem to (1) over the cuboid cell (i, j, k) results in

$$\mathbf{E}_{i,j}^{k+1} = \mathbf{E}_{i,j}^k - \frac{\Delta t^k}{\Delta x}\left(\overline{\mathbf{F}}_{i+1/2,j}^{k+1/2} - \overline{\mathbf{F}}_{i-1/2,j}^{k+1/2}\right) - \frac{\Delta t^k}{\Delta y}\left(\overline{\mathbf{G}}_{i,j+1/2}^{k+1/2} - \overline{\mathbf{G}}_{i,j-1/2}^{k+1/2}\right),$$
$$i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n, \tag{3}$$

where

$$\overline{\mathbf{F}}_{i+1/2,j}^{k+1/2} = \mathbf{F}_{i+1/2,j}^{k+1/2} + \underline{\mathbf{F}}_{i+1/2,j}^k, \quad \overline{\mathbf{G}}_{i,j+1/2}^{k+1/2} = \mathbf{G}_{i,j+1/2}^{k+1/2} + \underline{\mathbf{G}}_{i,j+1/2}^k$$

where $\mathbf{F}_{i+1/2,j}^{k+1/2}$ and $\mathbf{G}_{i,j+1/2}^{k+1/2}$ are the inviscid fluxes computed by the Godunov scheme (Godunov(1959), Toro (1999)) with MUSCL treatment (van Leer(1976), Toro (1999)) to achieve second order of accuracy, and $\underline{\mathbf{F}}_{i+1/2,j}^k$,$\underline{\mathbf{G}}_{i,j+1/2}^k$ are viscous contributions defined simply by second order central differences. The symbols Δt, Δx, Δy denote, respectively, the time step, the mesh size in x and the mesh size in y. Strang splitting (Strang,1968) is used to render the two-dimensional problem to a sequential solution of three one-dimensional problem. Let $\mathcal{L}_{\Delta t}^x$ be the difference operator for the 1-D equation (simply obtained from (1) by dropping the y terms)

$$\widehat{\mathbf{E}}_{i,j}^{k+1} = \mathbf{E}_{i,j}^{k} - \frac{\Delta t^k}{\Delta x_i}\left(\overline{\mathbf{F}}_{i+1/2,j}^{k+1/2} - \overline{\mathbf{F}}_{i-1/2,j}^{k+1/2}\right)$$

whose solution, after one time step Δt, is denoted $\widehat{\mathbf{E}}^{k+1} = \mathcal{L}_{\Delta t}^{x}\mathbf{E}^{k}$. Using $\widehat{\mathbf{E}}^{k+1}$ as initial solution and compute the final solution by

$$\mathbf{E}^{k+1} = \mathcal{L}_{\Delta t}^{y}\widehat{\mathbf{E}}^{k+1} = \mathcal{L}_{\Delta t}^{y}\mathcal{L}_{\Delta t}^{x}\mathbf{E}^{k} \tag{4}$$

where $\mathcal{L}_{\Delta t}^{y}$ is the operator for the 1-D equation (simply obtained from (1) by dropping the x terms)

$$\mathbf{E}_{i,j}^{k+1} = \widehat{\mathbf{E}}^{k+1} - \frac{\Delta \lambda^k}{\Delta \eta_j}\left(\overline{\mathbf{G}}_{i,j+1/2}^{k+1/2}\left(\widehat{\mathbf{E}}^{k+1}\right) - \overline{\mathbf{G}}_{i,j-1/2}^{k+1/2}\left(\widehat{\mathbf{E}}^{k+1}\right)\right)$$

The splitting method (4) is not very accurate. The Strang splitting, which is more accurate than (4), is defined as

$$\mathbf{E}^{k+1} = \mathcal{L}_{\frac{\Delta t}{2}}^{x}\mathcal{L}_{\Delta t}^{y}\mathcal{L}_{\frac{\Delta t}{2}}^{x}\mathbf{E}^{k} \tag{5}$$

for marching from t^k to $t^{k+1} = t^k + \Delta t, k = 0, 1, 2, \cdots$.