Appendix to "Weakly nonlinear theory of shear-banding instability in granular plane Couette flow: analytical solution, comparison with numerics and bifurcation"

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Appendix A. Nonlinear terms $\mathcal{N}_2$ and $\mathcal{N}_3$

In the following, the subscripts ϕ and T indicate partial derivatives with respect to ϕ and T, respectively, and the superscript 0 indicates the values being evaluated at the base state. The quadratic, $\mathcal{N}_2$, and cubic, $\mathcal{N}_3$, nonlinear terms in disturbance equations (Eqn. 3.2 in main paper) can be written in vector forms:

$$\mathcal{N}_2 = \begin{pmatrix} \mathcal{N}_2^1 \\ \mathcal{N}_2^2 \\ \mathcal{N}_2^3 \\ \mathcal{N}_2^4 \end{pmatrix} \quad \text{and} \quad \mathcal{N}_3 = \begin{pmatrix} \mathcal{N}_3^1 \\ \mathcal{N}_3^2 \\ \mathcal{N}_3^3 \\ \mathcal{N}_3^4 \end{pmatrix} \tag{A 1}$$

where the superscripts 1, 2, 3 and 4 on nonlinear terms correspond to terms originating from the mass balance, x-momentum, y-momentum and energy balance equations, respectively.

Mass balance equation

$$\mathcal{N}_2^1 = -\frac{\partial(\phi' v')}{\partial y} \quad \text{and} \quad \mathcal{N}_3^1 = 0. \tag{A 2}$$

x-momentum equation

$$\mathcal{N}_2^2 = -\frac{1}{\phi^0}\left[\phi^0 v'\frac{\partial u'}{\partial y} + \phi'\left(\frac{\partial u'}{\partial t} + v'\right)\right] + \frac{1}{\phi^0 H^2}\left[(\mu_\phi^0\phi' + \mu_T^0 T')\frac{\partial^2 u'}{\partial y^2}\right.$$
$$\left. + \frac{1}{2}\left(\mu_{\phi\phi}^0\frac{\partial \phi'^2}{\partial y} + \mu_{TT}^0\frac{\partial T'^2}{\partial y} + 2\mu_{\phi T}^0\frac{\partial \phi' T'}{\partial y}\right) + \left(\mu_\phi^0\frac{\partial \phi'}{\partial y} + \mu_T^0\frac{\partial T'}{\partial y}\right)\frac{\partial u'}{\partial y}\right] \tag{A 3}$$

$$\mathcal{N}_3^2 = -\frac{1}{\phi^0}\phi' v'\frac{\partial u'}{\partial y} + \frac{1}{\phi^0 H^2}\left[\frac{1}{2}\left(\mu_{\phi\phi}^0\phi'^2 + \mu_{TT}^0 T'^2 + 2\mu_{\phi T}^0\phi' T'\right)\frac{\partial^2 u'}{\partial y^2}\right.$$
$$+\frac{1}{6}\left(\mu_{\phi\phi\phi}^0\frac{\partial \phi'^3}{\partial y} + \mu_{TTT}^0\frac{\partial T'^3}{\partial y} + 3\mu_{\phi\phi T}^0\frac{\partial \phi'^2 T'}{\partial y} + 3\mu_{\phi TT}^0\frac{\partial \phi' T'^2}{\partial y}\right)$$
$$\left. + \frac{1}{2}\left(\mu_{\phi\phi}^0\frac{\partial \phi'^2}{\partial y} + \mu_{TT}^0\frac{\partial T'^2}{\partial y} + 2\mu_{\phi T}^0\frac{\partial \phi' T'}{\partial y}\right)\frac{\partial u'}{\partial y}\right] \tag{A 4}$$

y-momentum equation

$$\mathcal{N}_2^3 = -\frac{1}{\phi^0}\left(\phi^0 v'\frac{\partial v'}{\partial y} + \phi'\frac{\partial v'}{\partial t}\right)$$
$$+\frac{1}{\phi^0 H^2}\left[-\frac{1}{2}\left(p_{\phi\phi}^0\frac{\partial \phi'^2}{\partial y} + p_{TT}^0\frac{\partial T'^2}{\partial y} + 2p_{\phi T}^0\frac{\partial \phi' T'}{\partial y}\right) + 2\left(\mu_\phi^0\phi' + \mu_T^0 T'\right)\frac{\partial^2 v'}{\partial y^2}\right.$$

$$+\left((2\mu_\phi^0+\lambda_\phi^0)\frac{\partial\phi'}{\partial y}+(2\mu_T^0+\lambda_T^0)\frac{\partial T'}{\partial y}\right)\frac{\partial v'}{\partial y}+\left(\lambda_\phi^0\phi'+\lambda_T^0 T'\right)\frac{\partial^2 v'}{\partial y^2}\Bigg] \qquad \text{(A 5)}$$

$$\begin{aligned}\mathcal{N}_3^3 = &-\frac{1}{\phi^0}\phi' v'\frac{\partial v'}{\partial y}+\frac{1}{\phi^0 H^2}\left[-\frac{1}{6}\left(p^0_{\phi\phi\phi}\frac{\partial\phi^{'3}}{\partial y}+p^0_{TTT}\frac{\partial T^{'3}}{\partial y}+3p^0_{\phi\phi T}\frac{\partial\phi^{'2}T'}{\partial y}+3p^0_{\phi TT}\frac{\partial\phi' T^{'2}}{\partial y}\right)\right.\\
&+\left(\mu^0_{\phi\phi}\phi^{'2}+\mu^0_{TT}T^{'2}+2\mu^0_{\phi T}\phi' T'\right)\frac{\partial^2 v'}{\partial y^2}+\left(\mu^0_{\phi\phi}\frac{\partial\phi^{'2}}{\partial y}+\mu^0_{TT}\frac{\partial T^{'2}}{\partial y}+2\mu^0_{\phi T}\frac{\partial\phi' T'}{\partial y}\right)\frac{\partial v'}{\partial y}\\
&+\frac{1}{2}\left(\lambda^0_{\phi\phi}\frac{\partial\phi^{'2}}{\partial y}+\lambda^0_{TT}\frac{\partial T^{'2}}{\partial y}+2\lambda^0_{\phi T}\frac{\partial\phi' T'}{\partial y}\right)\frac{\partial v'}{\partial y}\\
&\left.+\frac{1}{2}\left(\lambda^0_{\phi\phi}\phi^{'2}+\lambda^0_{TT}T^{'2}+2\lambda^0_{\phi T}\phi' T'\right)\frac{\partial^2 v'}{\partial y^2}\right] \qquad \text{(A 6)}\end{aligned}$$

Granular energy equation

$$\begin{aligned}\mathcal{N}_2^4 = &-\frac{1}{\phi^0}\left(\phi^0 v'\frac{\partial T'}{\partial y}+\phi'\frac{\partial T'}{\partial t}\right)\\
&+\left[\frac{1}{H^2}\left[\frac{\partial T'}{\partial y}\left(\kappa_\phi^0\frac{\partial\phi'}{\partial y}+\kappa_T^0\frac{\partial T'}{\partial y}\right)+\frac{\partial^2 T'}{\partial y^2}\left(\kappa_\phi^0\phi'+\kappa_T^0 T'\right)\right]\right.\\
&-(p_\phi^0\phi'+p_T^0 T')\frac{\partial v'}{\partial y}+2\mu^0\left[\left(\frac{\partial v'}{\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial u'}{\partial y}\right)^2\right]\\
&+\left(\frac{1}{2}\mu^0_{\phi\phi}\phi'^2+\frac{1}{2}\mu^0_{TT}T'^2+\mu^0_{\phi T}\phi' T'\right)+2(\mu_\phi^0\phi'+\mu_T^0 T')\left(\frac{\partial u'}{\partial y}\right)\\
&\left.+\lambda^0\left(\frac{\partial v'}{\partial y}\right)^2-\left(\frac{1}{2}\mathcal{D}^0_{\phi\phi}\phi'^2+\frac{1}{2}\mathcal{D}^0_{TT}T^{'2}+\mathcal{D}^0_{\phi T}\phi' T'\right)\right]\left(\frac{2}{\phi^0 dim}\right) \qquad \text{(A 7)}\end{aligned}$$

$$\begin{aligned}\mathcal{N}_3^4 = &-\frac{1}{\phi^0}\left(\phi' v'\frac{\partial T'}{\partial y}\right)\\
&+\left[\frac{1}{H^2}\left[\frac{\partial T'}{\partial y}\left(\frac{1}{2}\kappa^0_{\phi\phi}\frac{\partial\phi^{'2}}{\partial y}+\frac{1}{2}\kappa^0_{TT}\frac{\partial T^{'2}}{\partial y}+\kappa^0_{\phi T}\frac{\partial\phi' T'}{\partial y}\right)\right.\right.\\
&\left.+\frac{\partial^2 T'}{\partial y^2}\left(\frac{1}{2}\kappa^0_{\phi\phi}\phi^{'2}+\frac{1}{2}\kappa^0_{TT}T^{'2}+\kappa^0_{\phi T}\phi' T'\right)\right]\\
&-\left(\frac{1}{2}p^0_{\phi\phi}\phi^{'2}+\frac{1}{2}p^0_{TT}T^{'2}+p^0_{\phi T}\phi' T'\right)\frac{\partial v'}{\partial y}+2(\mu_\phi^0\phi'+\mu_T^0 T')\left[\left(\frac{\partial v'}{\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial u'}{\partial y}\right)^2\right]\\
&+\left(\frac{1}{6}\mu^0_{\phi\phi\phi}\phi'^3+\frac{1}{6}\mu^0_{TTT}T'^3+\frac{1}{2}\mu^0_{\phi\phi T}\phi'^2 T'+\frac{1}{2}\mu^0_{\phi TT}\phi' T^{'2}\right)\\
&+\left(\mu^0_{\phi\phi}\phi^{'2}+\mu^0_{TT}T^{'2}+2\mu^0_{\phi T}\phi^{'} T'\right)\left(\frac{\partial u'}{\partial y}\right)+\left(\lambda_\phi^0\phi'+\lambda_T^0 T'\right)\left(\frac{\partial v'}{\partial y}\right)^2\\
&\left.-\left(\frac{1}{6}\mathcal{D}^0_{\phi\phi\phi}\phi^{'3}++\frac{1}{6}\mathcal{D}^0_{TTT}T^{'3}+\frac{1}{2}\mathcal{D}^0_{\phi\phi T}\phi^{'2}T'+\frac{1}{2}\mathcal{D}^0_{\phi TT}\phi' T^{'2}\right)\right]\left(\frac{2}{\phi^0 dim}\right) \qquad \text{(A 8)}\end{aligned}$$

Appendix B. Inhomogeneous terms G_{13}

The y-independent terms of $G_{13} = (G^1_{13}, G^2_{13}, G^3_{13}, G^4_{13})$ in (7.13-7.14) are

$$G^{1\beta3}_{13} = -\frac{1}{2}\left[(k_{2\beta} + k_\beta)(\phi_1 v_2 + \phi_2 v_1)\right]$$
$$G^{1\beta1}_{13} = -\frac{1}{2}\left[(k_{2\beta} - k_\beta)(\phi_1 v_2 - \phi_2 v_1)\right]$$

$$\begin{aligned}
G^{2\beta3}_{13} = &-\frac{1}{2}v_1 u_2 k_{2\beta} + \frac{1}{2}\Bigg[-\frac{1}{\phi^0}\phi_1 v_2 - 2c^{(0)}\frac{\phi_1 u_2}{\phi^0} \\
&+\frac{1}{\phi^0 H^2}\left(-k^2_{2\beta}(\mu^0_\phi \phi_1 + \mu^0_T T_1)u_2 - k_{2\beta}\left(\mu^0_{\phi\phi}\phi_1\phi_2 + \mu^0_{TT}T_1T_2 + \mu^0_{\phi T}\phi_1 T_2 + \mu^0_{\phi T}T_1\phi_2\right)\right. \\
&\left.-k_\beta k_{2\beta} u_1\left(\mu^0_\phi\phi_2 + \mu^0_T T_2\right)\right)\Bigg] - \frac{1}{2}v_2 u_1 k_\beta + \frac{1}{2}\Bigg[-\frac{1}{\phi^0}\phi_2 v_1 - c^{(0)}\frac{\phi_2 u_1}{\phi^0} \\
&+\frac{1}{\phi^0 H^2}\left(-k^2_\beta(\mu^0_\phi\phi_2 + \mu^0_T T_2)u_1 - k_\beta\left(\mu^0_{\phi\phi}\phi_2\phi_1 + \mu^0_{TT}T_2T_1 + \mu^0_{\phi T}\phi_2 T_1 + \mu^0_{\phi T}T_2\phi_1\right)\right. \\
&\left.-k_\beta k_{2\beta} u_2(\mu^0_\phi\phi_1 + \mu^0_T T_1)\right)\Bigg] \\
&+\frac{1}{4}\left[-\frac{1}{\phi^0}k_\beta\phi_1 v_1 u_1\right] + \frac{1}{4\phi^0 H^2}\Bigg[-k^2_\beta\left(\frac{1}{2}\mu^0_{\phi\phi}\phi_1^2 + \frac{1}{2}\mu^0_{TT}T_1^2 + \mu^0_{\phi T}\phi_1 T_1\right)u_1 \\
&-k_\beta\left(\frac{1}{2}\mu^0_{\phi\phi\phi}\phi_1^3 + \frac{1}{2}\mu^0_{TTT}T_1^3 + \frac{3}{2}\mu^0_{\phi\phi T}\phi_1^2 T_1 + \frac{3}{2}\mu^0_{TT\phi}T_1^2\phi_1\right) \\
&-k^2_\beta u_1\left(\mu^0_{\phi\phi}\phi_1^2 + \mu^0_{TT}T_1^2 + 2\mu^0_{\phi T}\phi_1 T_1\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
G^{2\beta1}_{13} = &\frac{1}{2}v_1 u_2 k_{2\beta} + \frac{1}{2}\Bigg[-\frac{1}{\phi^0}\phi_1 v_2 - 2c^{(0)}\frac{\phi_1 u_2}{\phi^0} \\
&+\frac{1}{\phi^0 H^2}\left(-k^2_{2\beta}(\mu^0_\phi \phi_1 + \mu^0_T T_1)u_2 - k_{2\beta}\left(\mu^0_{\phi\phi}\phi_1\phi_2 + \mu^0_{TT}T_1T_2 + \mu^0_{\phi T}\phi_1 T_2 + \mu^0_{\phi T}T_1\phi_2\right)\right. \\
&\left.-k_\beta k_{2\beta} u_1\left(\mu^0_\phi\phi_2 + \mu^0_T T_2\right)\right)\Bigg] - \frac{1}{2}v_2 u_1 k_\beta - \frac{1}{2}\Bigg[-\frac{1}{\phi^0}\phi_2 v_1 - c^{(0)}\frac{\phi_2 u_1}{\phi^0} \\
&+\frac{1}{\phi^0 H^2}\left(-k^2_\beta(\mu^0_\phi\phi_2 + \mu^0_T T_2)u_1 - k_\beta\left(\mu^0_{\phi\phi}\phi_2\phi_1 + \mu^0_{TT}T_2T_1 + \mu^0_{\phi T}\phi_2 T_1 + \mu^0_{\phi T}T_2\phi_1\right)\right. \\
&\left.-k_\beta k_{2\beta} u_2(\mu^0_\phi\phi_1 + \mu^0_T T_1)\right)\Bigg] + T^{mean}_{k_{2\beta}}\frac{1}{\phi^0 H^2}\left[-k_\beta\left(\mu^0_{TT}T_1 + \mu^0_{\phi T}\phi_1\right) - k^2_\beta\mu^0_T u_1\right] \\
&+\frac{1}{4}\left[-\frac{1}{\phi^0}k_\beta\phi_1 v_1 u_1\right] + \frac{1}{4\phi^0 H^2}\Bigg[-k^2_\beta\left(\frac{1}{2}\mu^0_{\phi\phi}\phi_1^2 + \frac{1}{2}\mu^0_{TT}T_1^2 + \mu^0_{\phi T}\phi_1 T_1\right)u_1 \\
&-k_\beta\left(\frac{1}{2}\mu^0_{\phi\phi\phi}\phi_1^3 + \frac{1}{2}\mu^0_{TTT}T_1^3 + \frac{3}{2}\mu^0_{\phi\phi T}\phi_1^2 T_1 + \frac{3}{2}\mu^0_{TT\phi}T_1^2\phi_1\right) \\
&-k^2_\beta u_1\left(\mu^0_{\phi\phi}\phi_1^2 + \mu^0_{TT}T_1^2 + 2\mu^0_{\phi T}\phi_1 T_1\right)\Bigg]
\end{aligned}$$

$$\begin{aligned}
G^{3\beta3}_{13} = &\frac{1}{2}\left[-k_{2\beta}v_1 v_2 - \frac{k_\beta k_{2\beta} v_2}{\phi^0 H^2}\left(2(\mu^0_\phi\phi_1 + \mu^0_T T_1) + (\lambda^0_\phi\phi_1 + \lambda^0_T T_1)\right)\right] - \frac{1}{\phi^0}c^{(0)}\phi_1 v_2 \\
&+\frac{1}{2\phi^0 H^2}\left[k_{2\beta}\left(p^0_{\phi\phi}\phi_1\phi_2 + p^0_{TT}T_1T_2 + p^0_{\phi T}\phi_1 T_2 + p^0_{\phi T}T_1\phi_2\right)\right. \\
&\left.-k^2_{2\beta}2\left(\mu^0_\phi\phi_1 + \mu^0_T T_1\right)v_2 - k^2_{2\beta}\left(\lambda^0_\phi\phi_1 + \lambda^0_T T_1\right)v_2\right]
\end{aligned}$$

$$+\frac{1}{2}\left[-k_\beta v_1 v_2 - \frac{k_\beta k_{2\beta} v_1}{\phi^0 H^2}\left(2(\mu^0_\phi \phi_2 + \mu^0_T T_2) + (\lambda^0_\phi \phi_2 + \lambda^0_T T_2)\right)\right] - \frac{1}{2\phi^0} c^{(0)} \phi_2 v_1$$
$$+\frac{1}{2\phi^0 H^2}\left[k_\beta\left(p^0_{\phi\phi}\phi_2\phi_1 + p^0_{TT} T_2 T_1 + p^0_{\phi T}\phi_2 T_1 + p^0_{\phi T} T_2 \phi_1\right)\right.$$
$$\left.-k_\beta^2 2\left(\mu^0_\phi \phi_2 + \mu^0_T T_2\right) v_1 - k_\beta^2\left(\lambda^0_\phi \phi_2 + \lambda^0_T T_2\right) v_1\right]$$
$$+\frac{1}{4}\left[-\frac{1}{\phi^0} k_\beta \phi_1 v_1^2\right] + \frac{1}{4\phi^0 H^2}\left[k_\beta\left(\frac{1}{2} p^0_{\phi\phi\phi}\phi_1^3 + \frac{1}{2} p^0_{TTT} T_1^3 + \frac{3}{2} p^0_{\phi\phi T}\phi_1^2 T_1 + \frac{3}{2} p^0_{\phi TT} T_1^2 \phi_1\right)\right.$$
$$\left.-2k_\beta^2\left(\frac{3}{2}\mu^0_{\phi\phi}\phi_1^2 + \frac{3}{2}\mu^0_{TT} T_1^2 + 3\mu^0_{\phi T}\phi_1 T_1\right) v_1 - k_\beta^2\left(\frac{3}{2}\lambda^0_{\phi\phi}\phi_1^2 + \frac{3}{2}\lambda^0_{TT} T_1^2 + 3\lambda^0_{\phi T}\phi_1 T_1\right) v_1\right]$$

$$G^{3\beta 1}_{13} = -\frac{1}{2}\left[-k_{2\beta} v_1 v_2 - \frac{k_\beta k_{2\beta} v_2}{\phi^0 H^2}\left(2(\mu^0_\phi \phi_1 + \mu^0_T T_1) + (\lambda^0_\phi \phi_1 + \lambda^0_T T_1)\right)\right] - \frac{1}{\phi^0} c^{(0)} \phi_1 v_2$$
$$+\frac{1}{2\phi^0 H^2}\left[k_{2\beta}\left(p^0_{\phi\phi}\phi_1\phi_2 + p^0_{TT} T_1 T_2 + p^0_{\phi T}\phi_1 T_2 + p^0_{\phi T} T_1 \phi_2\right)\right.$$
$$\left.-2k_{2\beta}^2\left(\mu^0_\phi \phi_1 + \mu^0_T T_1\right) v_2 - k_{2\beta}^2\left(\lambda^0_\phi \phi_1 + \lambda^0_T T_1\right) v_2\right]$$
$$+\frac{1}{2}\left[-k_\beta v_1 v_2 - \frac{k_\beta k_{2\beta} v_1}{\phi^0 H^2}\left(2(\mu^0_\phi \phi_2 + \mu^0_T T_2) + (\lambda^0_\phi \phi_2 + \lambda^0_T T_2)\right)\right] + \frac{1}{2\phi^0} c^{(0)} \phi_2 v_1$$
$$-\frac{1}{2\phi^0 H^2}\left[k_\beta\left(p^0_{\phi\phi}\phi_2\phi_1 + p^0_{TT} T_2 T_1 + p^0_{\phi T}\phi_2 T_1 + p^0_{\phi T} T_2 \phi_1\right)\right.$$
$$\left.-2k_\beta^2\left(\mu^0_\phi \phi_2 + \mu^0_T T_2\right) v_1 - k_\beta^2\left(\lambda^0_\phi \phi_2 + \lambda^0_T T_2\right) v_1\right]$$
$$+\frac{1}{\phi^0 H^2} T^{mean}_{k_{2\beta}}\left[k_\beta\left(p^0_{TT} T_1 + p^0_{\phi T}\phi_1\right) - 2\mu^0_T k_\beta^2 v_1 - \lambda^0_T k_\beta^2 v_1\right]$$
$$+\frac{1}{4}\left[-\frac{1}{\phi^0} k_\beta \phi_1 v_1^2\right] + \frac{1}{4\phi^0 H^2}\left[k_\beta\left(\frac{1}{2} p^0_{\phi\phi\phi}\phi_1^3 + \frac{1}{2} p^0_{TTT} T_1^3 + \frac{3}{2} p^0_{\phi\phi T}\phi_1^2 T_1 + \frac{3}{2} p^0_{\phi TT} T_1^2 \phi_1\right)\right.$$
$$\left.-2k_\beta^2\left(\frac{3}{2}\mu^0_{\phi\phi}\phi_1^2 + \frac{3}{2}\mu^0_{TT} T_1^2 + 3\mu^0_{\phi T}\phi_1 T_1\right) v_1 - k_\beta^2\left(\frac{3}{2}\lambda^0_{\phi\phi}\phi_1^2 + \frac{3}{2}\lambda^0_{TT} T_1^2 + 3\lambda^0_{\phi T}\phi_1 T_1\right) v_1\right]$$

$$G^{4\beta 3}_{13} = -\frac{1}{2}\left[k_{2\beta} v_1 T_2 + \frac{2}{\phi^0 H^2 dim} k_\beta k_{2\beta} T_1\left(\kappa^0_\phi \phi_2 + \kappa^0_T T_2\right)\right]$$
$$+\frac{1}{2}\left[-\frac{2}{\phi^0} c^{(0)} \phi_1 T_2 + \frac{2}{\phi^0 dim}\left(-\frac{1}{H^2} k_\beta^2 T_1\left(\kappa^0_\phi \phi_2 + \kappa^0_T T_2\right) - k_{2\beta}\left(p^0_\phi \phi_1 + p^0_T T_1\right) v_2\right.\right.$$
$$+2\mu^0 k_\beta k_{2\beta}\left(v_1 v_2 + \frac{1}{2} u_1 u_2\right) + \left(\frac{1}{2}\mu^0_{\phi\phi}\phi_1\phi_2 + \frac{1}{2}\mu^0_{TT} T_1 T_2 + \mu^0_{\phi T}\phi_1 T_2\right)$$
$$\left.\left.+2k_{2\beta}\left(\mu^0_\phi \phi_1 + \mu^0_T T_1\right) u_2 + \lambda^0 k_\beta k_{2\beta} v_1 v_2 - \left(\frac{1}{2}\mathcal{D}^0_{\phi\phi}\phi_1\phi_2 + \frac{1}{2}\mathcal{D}^0_{TT} T_1 T_2 + \mathcal{D}^0_{\phi T}\phi_1 T_2\right)\right)\right]$$
$$-\frac{1}{2}\left[k_\beta v_2 T_1 + \frac{2}{\phi^0 H^2 dim} k_\beta k_{2\beta} T_2\left(\kappa^0_\phi \phi_1 + \kappa^0_T T_1\right)\right]$$
$$+\frac{1}{2}\left[-\frac{1}{\phi^0} c^{(0)} \phi_2 T_1 + \frac{2}{\phi^0 dim}\left(-\frac{1}{H^2} k_{2\beta}^2 T_2\left(\kappa^0_\phi \phi_1 + \kappa^0_T T_1\right) - k_\beta\left(p^0_\phi \phi_2 + p^0_T T_2\right) v_1\right.\right.$$
$$+2\mu^0 k_\beta k_{2\beta}\left(v_2 v_1 + \frac{1}{2} u_2 u_1\right) + \left(\frac{1}{2}\mu^0_{\phi\phi}\phi_2\phi_1 + \frac{1}{2}\mu^0_{TT} T_2 T_1 + \mu^0_{\phi T}\phi_2 T_1\right)$$
$$\left.\left.+2k_\beta\left(\mu^0_\phi \phi_2 + \mu^0_T T_2\right) u_1 + \lambda^0 k_\beta k_{2\beta} v_2 v_1 - \left(\frac{1}{2}\mathcal{D}^0_{\phi\phi}\phi_2\phi_1 + \frac{1}{2}\mathcal{D}^0_{TT} T_2 T_1 + \mathcal{D}^0_{\phi T}\phi_2 T_1\right)\right)\right]$$

$$
\begin{aligned}
&+\frac{1}{4}\left[\frac{k_\beta}{\phi^0}\phi_1 v_1 T_1+\frac{2}{\phi^0 dim}\left(\frac{k_\beta^2}{H^2}T_1\left(\kappa^0_{\phi\phi}\phi_1^2+\kappa^0_{TT}T_1^2+2\kappa^0_{\phi T}\phi_1T_1\right)\ \right)\right]\\
&+\left(\frac{1}{2\phi^0 dim}\right)\left[-\frac{k_\beta^2}{H^2}T_1\left(\frac{1}{2}\kappa^0_{\phi\phi}\phi_1^2+\frac{1}{2}\kappa^0_{TT}T_1^2+\kappa^0_{\phi T}\phi_1T_1\right)\right.\\
&-k_\beta\left(\frac{1}{2}p^0_{\phi\phi}\phi_1^2+\frac{1}{2}p^0_{TT}T_1^2+p^0_{\phi T}\phi_1T_1\right)v_1+2k_\beta^2\left(\mu^0_\phi\phi_1+\mu^0_TT_1\right)\left(v_1^2+\frac{1}{2}u_1^2\right)\\
&+\left(\frac{1}{6}\mu^0_{\phi\phi\phi}\phi_1^3+\frac{1}{6}\mu^0_{TTT}T_1^3+\frac{1}{2}\mu^0_{\phi\phi T}\phi_1^2T_1+\frac{1}{2}\mu^0_{\phi TT}\phi_1T_1^2\right)\\
&+2k_\beta\left(\frac{1}{2}\mu^0_{\phi\phi}\phi_1^2+\frac{1}{2}\mu^0_{TT}T_1^2+\mu^0_{\phi T}\phi_1T_1\right)u_1+k_\beta^2\left(\lambda^0_\phi\phi_1+\lambda^0_TT_1\right)v_1^2\\
&\left.-\left(\frac{1}{6}\mathcal{D}^0_{\phi\phi\phi}\phi_1^3+\frac{1}{6}\mathcal{D}^0_{TTT}T_1^3+\frac{1}{2}\mathcal{D}^0_{\phi\phi T}\phi_1^2T_1+\frac{1}{2}\mathcal{D}^0_{\phi TT}\phi_1T_1^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
G^{4\beta 1}_{13}=&\frac{1}{2}\left[k_{2\beta}v_1T_2+\frac{2}{\phi^0H^2dim}k_\beta k_{2\beta}T_1\left(\kappa^0_\phi\phi_2+\kappa^0_TT_2\right)\right]\\
&+\frac{1}{2}\left[-\frac{2}{\phi^0}c^{(0)}\phi_1T_2+\frac{2}{\phi^0 dim}\left(-\frac{1}{H^2}k_\beta^2T_1\left(\kappa^0_\phi\phi_2+\kappa^0_TT_2\right)-k_{2\beta}\left(p^0_\phi\phi_1+p^0_TT_1\right)v_2\right.\right.\\
&+2\mu^0k_\beta k_{2\beta}\left(v_1v_2+\frac{1}{2}u_1u_2\right)+\left(\frac{1}{2}\mu^0_{\phi\phi}\phi_1\phi_2+\frac{1}{2}\mu^0_{TT}T_1T_2+\mu^0_{\phi T}\phi_1T_2\right)\\
&\left.\left.+2k_{2\beta}\left(\mu^0_\phi\phi_1+\mu^0_TT_1\right)u_2+\lambda^0k_\beta k_{2\beta}v_1v_2-\ \left(\frac{1}{2}\mathcal{D}^0_{\phi\phi}\phi_1\phi_2+\frac{1}{2}\mathcal{D}^0_{TT}T_1T_2+\mathcal{D}^0_{\phi T}\phi_1T_2\right)\right)\right]\\
&+T^{mean}_{k_{2\beta}}\left[-\frac{2c^{(0)}}{\phi^0}\phi_1+\frac{2}{\phi^0 dim}\left(-\frac{k_\beta^2}{H^2}T_1\kappa^0_T+\left(\frac{1}{2}\mu^0_{TT}T_1+\mu^0_{\phi T}\phi_1\right)-\left(\frac{1}{2}\mathcal{D}^0_{TT}T_1+\mathcal{D}^0_{\phi T}\phi_1\right)\right)\right]\\
&+\frac{1}{2}\left[k_\beta v_2T_1+\frac{2}{\phi^0H^2dim}k_\beta k_{2\beta}T_2\left(\kappa^0_\phi\phi_1+\kappa^0_TT_1\right)\right]\\
&+\frac{1}{2}\left[-\frac{1}{\phi^0}c^{(0)}\phi_2T_1+\frac{2}{\phi^0 dim}\left(-\frac{1}{H^2}k_{2\beta}^2T_2\left(\kappa^0_\phi\phi_1+\kappa^0_TT_1\right)-k_\beta\left(p^0_\phi\phi_2+p^0_TT_2\right)v_1\right.\right.\\
&+2\mu^0k_\beta k_{2\beta}\left(v_2v_1+\frac{1}{2}u_2u_1\right)+\left(\frac{1}{2}\mu^0_{\phi\phi}\phi_2\phi_1+\frac{1}{2}\mu^0_{TT}T_2T_1+\mu^0_{\phi T}\phi_2T_1\right)\\
&\left.\left.+2k_\beta\left(\mu^0_\phi\phi_2+\mu^0_TT_2\right)u_1+\lambda^0k_\beta k_{2\beta}v_2v_1-\ \left(\frac{1}{2}\mathcal{D}^0_{\phi\phi}\phi_2\phi_1+\frac{1}{2}\mathcal{D}^0_{TT}T_2T_1+\mathcal{D}^0_{\phi T}\phi_2T_1\right)\right)\right]\\
&+T^{mean}_{k_{2\beta}}\frac{2}{\phi^0 dim}\left[-k_\beta p^0_Tv_1+\frac{1}{2}\mu^0_{TT}T_1+2\mu^0_Tk_\beta u_1-\frac{1}{2}\mathcal{D}^0_{TT}T_1\right]\\
&+\frac{1}{4}\left[\frac{k_\beta}{\phi^0}\phi_1v_1T_1+\frac{2}{\phi^0 dim}\left(\frac{k_\beta^2}{H^2}T_1\left(\kappa^0_{\phi\phi}\phi_1^2+\kappa^0_{TT}T_1^2+2\kappa^0_{\phi T}\phi_1T_1\right)\right)\right]\\
&+\left(\frac{3}{2\phi^0 dim}\right)\left[-\frac{k_\beta^2}{H^2}T_1\left(\frac{1}{2}\kappa^0_{\phi\phi}\phi_1^2+\frac{1}{2}\kappa^0_{TT}T_1^2+\kappa^0_{\phi T}\phi_1T_1\right)\right.\\
&-k_\beta\left(\frac{1}{2}p^0_{\phi\phi}\phi_1^2+\frac{1}{2}p^0_{TT}T_1^2+p^0_{\phi T}\phi_1T_1\right)v_1+2k_\beta^2\left(\mu^0_\phi\phi_1+\mu^0_TT_1\right)\left(v_1^2+\frac{1}{2}u_1^2\right)\\
&+\left(\frac{1}{6}\mu^0_{\phi\phi\phi}\phi_1^3+\frac{1}{6}\mu^0_{TTT}T_1^3+\frac{1}{2}\mu^0_{\phi\phi T}\phi_1^2T_1+\frac{1}{2}\mu^0_{\phi TT}\phi_1T_1^2\right)
\end{aligned}
$$

$$+2k_\beta\left(\frac{1}{2}\mu^0_{\phi\phi}\phi_1^2+\frac{1}{2}\mu^0_{TT}T_1^2+\mu^0_{\phi T}\phi_1T_1\right)u_1+k_\beta^2\left(\lambda^0_\phi\phi_1+\lambda^0_TT_1\right)v_1^2$$
$$-\left(\frac{1}{6}\mathcal{D}^0_{\phi\phi\phi}\phi_1^3+\frac{1}{6}\mathcal{D}^0_{TTT}T_1^3+\frac{1}{2}\mathcal{D}^0_{\phi\phi T}\phi_1^2T_1+\frac{1}{2}\mathcal{D}^0_{\phi TT}\phi_1T_1^2\right)\Bigg]$$

Appendix C. Locus of $a^{(2)}=0$

The condition for 'zero' first Landau coefficient, $a^{(2)}=0$, is

$$\phi_1^\dagger G_{13}^{1\beta 1}+T_1^\dagger G_{13}^{4\beta 1}+u_1^\dagger G_{13}^{2\beta 1}+v_1^\dagger G_{13}^{3\beta 1}=0, \tag{C 1}$$

where $G_{13}^{1\beta1}, G_{13}^{4\beta1},\ldots$ can be written as

$$G_{13}^{1\beta1}=\frac{1}{H^2}C_{12}+C_{10},\quad G_{13}^{2\beta1}=\frac{1}{H^2}C_{22}+C_{20}$$
$$G_{13}^{3\beta1}=\frac{1}{H^2}C_{32}+C_{30},\quad G_{13}^{4\beta1}=\frac{1}{H^2}C_{42}+C_{40}.$$

In terms of $C_{12}, C_{22},\ldots$ the zero-loci of the first Landau coefficient can be written as

$$H^2=\frac{-\left(\phi_1^\dagger C_{12}+u_1^\dagger C_{22}+v_1^\dagger C_{32}+T_1^\dagger C_{42}\right)}{\left(\phi_1^\dagger C_{10}+u_1^\dagger C_{20}+v_1^\dagger C_{30}+T_1^\dagger C_{40}\right)}. \tag{C 2}$$

$C_{12}, C_{22},\ldots$ can be represented in powers of $k_\beta=\beta\pi$ (see table 1 for definition), if we write the mean-terms of second harmonic in terms of a series in k_β as

$$T^{mean}_{k_{2\beta}}=\frac{f_{nl}}{f_l}=\frac{k_\beta^2 f_{nl}^{(2)}+k_\beta f_{nl}^{(1)}+f_{nl}^{(0)}}{f_l^0} \tag{C 3}$$

$$f_{nl}^{(2)}=\frac{1}{\phi^0\,dim}\left[2\mu^0\left(v_1^2+\frac{u_1^2}{2}\right)+\lambda^0v_1^2\right] \tag{C 4}$$

$$f_{nl}^{(1)}=\frac{v_1T_1}{2\phi^0}+\frac{1}{\phi^0\,dim}\left[-\left(p^0_\phi\phi_1+p^0_TT_1\right)v_1+2u_1\left(\mu^0_\phi\phi_1+\mu^0_TT_1\right)\right] \tag{C 5}$$

$$f_{nl}^{(0)}=\frac{-c^{(0)}}{2\phi^0}\phi_1T_1+\frac{1}{\phi^0\,dim}\left[\frac{1}{2}\mu^0_{\phi\phi}\phi_1^2+\frac{1}{2}\mu^0_{TT}T_1^2+\mu^0_{\phi T}\phi_1T_1\right.$$
$$\left.-\left(\frac{1}{2}\mathcal{D}^0_{\phi\phi}\phi_1^2+\frac{1}{2}\mathcal{D}^0_{TT}T_1^2+\mathcal{D}^0_{\phi T}\phi_1T_1\right)\right] \tag{C 6}$$

$$f_l=f_l^{(0)}. \tag{C 7}$$

Hence the expressions for C_{ij} become

$$\left.\begin{array}{lllllll}
C_{12} & = & 0, & & C_{10} & = & k_\beta\,C_{10}^{(1)}\\
C_{22} & = & k_\beta^4\,C_{22}^{(4)}+k_\beta^3\,C_{22}^{(3)}+k_\beta^2\,C_{22}^{(2)}+k_\beta\,C_{22}^{(1)}, & & C_{20} & = & k_\beta\,C_{20}^{(1)}+C_{20}^{(0)}\\
C_{32} & = & k_\beta^4\,C_{32}^{(4)}+k_\beta^3\,C_{32}^{(3)}+k_\beta^2\,C_{32}^{(2)}+k_\beta\,C_{32}^{(1)}, & & C_{30} & = & k_\beta\,C_{30}^{(1)}+C_{30}^{(0)}\\
C_{42} & = & k_\beta^4\,C_{42}^{(4)}+k_\beta^3\,C_{42}^{(3)}+k_\beta^2\,C_{42}^{(2)}, & & C_{40} & = & k_\beta^3\,C_{40}^{(3)}+k_\beta^2\,C_{40}^{(2)}+k_\beta\,C_{40}^{(1)}+C_{40}^{(0)}
\end{array}\right\} \tag{C 8}$$

Inserting these expressions of C_{ij} into (C.2), we can further simplify (C.2) as

$$H^2=\frac{-\left(k_\beta^4K_1+k_\beta^3K_2+k_\beta^2K_3+k_\beta K_4\right)}{\left(k_\beta^3K_5+k_\beta^2K_6+k_\beta K_7+K_0\right)} \tag{C 9}$$

where $K_1, K_2, K_3 \ldots$ are

$$\begin{aligned}
K_1 &= u_1^\dagger C_{22}^{(4)} + v_1^\dagger C_{32}^{(4)} + T_1^\dagger C_{42}^{(4)},\\
K_2 &= u_1^\dagger C_{22}^{(3)} + v_1^\dagger C_{32}^{(3)} + T_1^\dagger C_{42}^{(3)},\\
K_3 &= u_1^\dagger C_{22}^{(2)} + v_1^\dagger C_{32}^{(2)} + T_1^\dagger C_{42}^{(2)},\\
K_4 &= u_1^\dagger C_{22}^{(1)} + v_1^\dagger C_{32}^{(1)},\\
K_5 &= T_1^\dagger C_{40}^{(3)},\\
K_6 &= T_1^\dagger C_{40}^{(2)},\\
K_7 &= \phi_1^\dagger C_{10}^{(1)} + u_1^\dagger C_{20}^{(1)} + v_1^\dagger C_{30}^{(1)} + T_1^\dagger C_{40}^{(1)},\\
K_0 &= u_1^\dagger C_{20}^{(0)} + v_1^\dagger C_{30}^{(0)} + T_1^\dagger C_{40}^{(0)}
\end{aligned}$$

and the expressions for $C_{ij}^{(k)}$ are

$$\begin{aligned}
C_{10}^{(1)} &= \frac{1}{2}\left(\phi_2 v_1 - \phi_1 v_2\right)\\
C_{22}^{(4)} &= \frac{-\mu_T^0 u_1 f_{nl}^{(2)}}{\phi^0 f_l^{(0)}}, \qquad C_{22}^{(3)} = \frac{-\mu_T^0 u_1 f_{nl}^{(1)}}{\phi^0 f_l^{(0)}} - \left(\mu_{TT}^0 T_1 + \mu_{\phi T}^0 \phi_1\right)\frac{f_{nl}^{(2)}}{\phi^0 f_l^{(0)}}\\
C_{22}^{(2)} &= -\frac{1}{2\phi^0}\left[2u_2\left(\mu_\phi^0\phi_1 + \mu_T^0 T_1\right) + u_1\left(\mu_\phi^0\phi_2 + \mu_T^0 T_2\right)\right] - \frac{u_1}{4\phi^0}\left(\frac{3}{2}\mu_{\phi\phi}^0\phi_1^2 + \frac{3}{2}\mu_{TT}^0 T_1^2 + 3\mu_{\phi T}^0\phi_1 T_1\right)\\
&\quad \frac{-\mu_T^0 u_1 f_{nl}^{(0)}}{\phi^0 f_l^{(0)}} - \left(\mu_{TT}^0 T_1 + \mu_{\phi T}^0\phi_1\right)\frac{f_{nl}^{(1)}}{\phi^0 f_l^{(0)}}\\
C_{22}^{(1)} &= -\frac{1}{2\phi^0}\left(\mu_{\phi\phi}^0\phi_1\phi_2 + \mu_{TT}^0 T_1 T_2 + \mu_{\phi T}^0\phi_1 T_2 + \mu_{\phi T}^0 T_1\phi_2\right)\\
&\quad -\frac{1}{4\phi^0}\left(\frac{1}{2}\mu_{\phi\phi\phi}^0\phi_1^3 + \frac{1}{2}\mu_{TTT}^0 T_1^3 + \frac{3}{2}\mu_{\phi\phi T}^0\phi_1^2 T_1 + \frac{3}{2}\mu_{\phi TT}^0 T_1^2\phi_1\right) - \left(\mu_{TT}^0 T_1 + \mu_{\phi T}^0\phi_1\right)\frac{f_{nl}^{(0)}}{\phi^0 f_l^{(0)}}\\
C_{20}^{(1)} &= \frac{1}{4}\left(4u_2 v_1 - 2u_1 v_2\right) - \frac{u_1 v_1\phi_1}{4\phi^0}\\
C_{20}^{(0)} &= \frac{1}{2\phi^0}\left[c^{(0)}\left(-2u_2\phi_1 + u_1\phi_2\right) + \left(-v_2\phi_1 + v_1\phi_2\right)\right]\\
C_{32}^{(4)} &= -\left(2\mu_T^0 + \lambda_T^0\right)\frac{v_1 f_{nl}^{(2)}}{\phi^0 f_l^{(0)}}, \quad C_{32}^{(3)} = -\left(2\mu_T^0 + \lambda_T^0\right)\frac{v_1 f_{nl}^{(1)}}{\phi^0 f_l^{(0)}} + \left(p_{TT}^0 T_1 + p_{\phi T}^0\phi_1\right)\frac{f_{nl}^{(2)}}{\phi^0 f_l^{(0)}}\\
C_{32}^{(2)} &= -\frac{1}{\phi^0}\left[2\left(\mu_\phi^0\phi_1 + \mu_T^0 T_1\right)v_2 + \left(\lambda_\phi^0\phi_1 + \lambda_T^0 T_1\right)v_2 + \left(\mu_\phi^0\phi_2 + \mu_T^0 T_2\right)v_1 + \frac{1}{2}\left(\lambda_\phi^0\phi_1 + \lambda_T^0 T_2\right)v_1\right]\\
&\quad -\frac{1}{4\phi^0}\left[2\left(\frac{3}{2}\mu_{\phi\phi}^0\phi_1^2 + \frac{3}{2}\mu_{TT}^0 T_1^2 + 3\mu_{\phi T}^0\phi_1 T_1\right)v_1 + \left(\frac{3}{2}\lambda_{\phi\phi}^0\phi_1^2 + \frac{3}{2}\lambda_{TT}^0 T_1^2 + 3\lambda_{\phi T}^0\phi_1 T_1\right)v_1\right]\\
&\quad -\left(2\mu_T^0 + \lambda_T^0\right)\frac{v_1 f_{nl}^{(0)}}{\phi^0 f_l^{(0)}} + \left(p_{TT}^0 T_1 + p_{\phi T}^0\phi_1\right)\frac{f_{nl}^{(1)}}{\phi^0 f_l^{(0)}}\\
C_{32}^{(1)} &= \frac{1}{2\phi^0}\left(p_{\phi\phi}^0\phi_1\phi_2 + p_{TT}^0 T_1 T_2 + p_{\phi T}^0\phi_1 T_2 + p_{\phi T}^0 T_1\phi_2\right) + \left(p_{TT}^0 T_1 + p_{\phi T}^0\phi_1\right)\frac{f_{nl}^{(0)}}{\phi^0 f_l^{(0)}}\\
&\quad +\frac{1}{4\phi^0}\left(\frac{1}{2}p_{\phi\phi\phi}^0\phi_1^3 + \frac{1}{2}p_{TTT}^0 T_1^3 + \frac{3}{2}p_{\phi\phi T}^0\phi_1^2 T_1 + \frac{3}{2}p_{\phi TT}^0 T_1^2\phi_1\right)
\end{aligned}$$

$$C_{30}^{(1)} = \frac{v_1}{4\phi^0}\left(2v_2\phi^0 - v_1\phi_1\right), \quad C_{30}^{(0)} = \frac{c^{(0)}}{2\phi^0}\left(-2v_2\phi_1 + v_1\phi_2\right)$$

$$C_{42}^{(4)} = \frac{-2T_1\kappa_T^0 f_{nl}^{(2)}}{\phi^0 f_l^{(0)} dim}, \quad C_{42}^{(3)} = \frac{-2T_1\kappa_T^0 f_{nl}^{(1)}}{\phi^0 f_l^{(0)} dim}$$

$$C_{42}^{(2)} = \frac{1}{\phi^0 dim}\left[\kappa_\phi^0(T_1\phi_2 - 2\phi_1 T_2) - \kappa_T^0 T_1\left(T_2 + \frac{2f_{nl}^{(0)}}{f_l^{(0)}}\right) - \frac{T_1}{2}\left(\frac{1}{2}\kappa_{\phi\phi}^0\phi_1^2 + \frac{1}{2}\kappa_{TT}^0 T_1^2 + \kappa_{\phi T}^0\phi_1 T_1\right)\right]$$

$$C_{40}^{(3)} = \left(-p_T^0 v_1 + 2\mu_T^0 u_1\right)\frac{2f_{nl}^{(2)}}{\phi^0 f_l^{(0)} dim}$$

$$\begin{aligned}
C_{40}^{(2)} = & \frac{1}{\phi^0 dim}\left[8\mu^0\left(v_1 v_2 + \frac{1}{2}u_1 u_2\right) + 4\lambda^0 v_2 v_1\right] + \frac{3}{2\phi^0 dim}\left[2\left(\mu_\phi^0\phi_1 + \mu_T^0 T_1\right)\left(v_1^2 + \frac{1}{2}u_1^2\right)\right. \\
& \left. + \left(\lambda_\phi^0\phi_1 + \lambda_T^0 T_1\right)v_1^2\right] + \left(-p_T^0 v_1 + 2\mu_T^0 u_1\right)\frac{2f_{nl}^{(1)}}{\phi^0 f_l^{(0)} dim} - \frac{2c^{(0)} f_{nl}^{(2)}}{\phi^0 f_l^{(0)}} \\
& + \left(\mu_{TT}^0 T_1 - \mathcal{D}_{TT}^0 T_1 + \mu_{\phi T}^0\phi_1 - \mathcal{D}_{\phi T}^0\right)\frac{2f_{nl}^{(2)}}{\phi^0 f_l^{(0)} dim}
\end{aligned}$$

$$\begin{aligned}
C_{40}^{(1)} = & \left(v_1 T_2 + \frac{1}{2}v_2 T_1\right) + \frac{1}{\phi^0 dim}\left[-2(p_\phi^0\phi_1 + p_T^0 T_1)v_2 - (p_\phi^0\phi_2 + p_T^0 T_2)v_1\right. \\
& \left. + 4(\mu_\phi^0\phi_1 + \mu_T^0 T_1)u_2 + 2(\mu_\phi^0\phi_2 + \mu_T^0 T_2)u_1\right] + \frac{1}{4\phi^0}\phi_1 v_1 T_1 \\
& + \frac{3}{2\phi^0 dim}\left[-\left(\frac{1}{2}p_{\phi\phi}^0\phi_1^2 + \frac{1}{2}p_{TT}^0 T_1^2 + p_{\phi T}^0\phi_1 T_1\right)v_1 + 2\left(\frac{1}{2}\mu_{\phi\phi}^0\phi_1^2 + \frac{1}{2}\mu_{TT}^0 T_1^2 + \mu_{\phi T}^0\phi_1 T_1\right)u_1\right] \\
& + \left(-p_T^0 v_1 + 2\mu_T^0 u_1\right)\frac{2f_{nl}^{(0)}}{\phi^0 f_l^{(0)} dim} - \frac{2c^{(0)} f_{nl}^{(1)}}{\phi^0 f_l^{(0)}} + \left(\mu_{TT}^0 T_1 + \mu_{\phi T}^0\phi_1 - \mathcal{D}_{TT}^0 T_1 - \mathcal{D}_{\phi T}^0\phi_1\right)\frac{2f_{nl}^{(1)}}{\phi^0 f_l^{(0)} dim}
\end{aligned}$$

$$\begin{aligned}
C_{40}^{(0)} = & -\frac{c^{(0)}}{\phi^0}\left(\phi_1 T_2 + \frac{1}{2}\phi_2 T_1\right) + \frac{2f_{nl}^{(0)}}{\phi^0 f_l^{(0)} dim}(\mu_{TT}^0 T_1 + \mu_{\phi T}^0\phi_1 - \mathcal{D}_{TT}^0 T_1 - \mathcal{D}_{\phi T}^0\phi_1) \\
& - \frac{2c^{(0)} f_{nl}^{(0)}}{\phi^0 f_l^{(0)}} + \frac{1}{\phi^0 dim}\left[\left(\mu_{\phi\phi}^0\phi_1\phi_2 + \mu_{TT}^0 T_1 T_2 + \mu_{\phi T}^0\phi_1 T_2 + \mu_{\phi T}^0\phi_2 T_1\right)\right. \\
& \left. - \left(\mathcal{D}_{\phi\phi}^0\phi_1\phi_2 + \mathcal{D}_{TT}^0 T_1 T_2 + \mathcal{D}_{\phi T}^0\phi_1 T_2 + \mathcal{D}_{\phi T}^0\phi_2 T_1\right)\right] \\
& + \frac{3}{2\phi^0 dim}\left[\left(\frac{1}{6}\mu_{\phi\phi\phi}^0\phi_1^3 + +\frac{1}{6}\mu_{TTT}^0 T_1^3 + \frac{1}{2}\mu_{\phi\phi T}^0\phi_1^2 T_1 + \frac{1}{2}\mu_{\phi TT}^0\phi_1 T_1^2\right)\right. \\
& \left. - \left(\frac{1}{6}\mathcal{D}_{\phi\phi\phi}^0\phi_1^3 + +\frac{1}{6}\mathcal{D}_{TTT}^0 T_1^3 + \frac{1}{2}\mathcal{D}_{\phi\phi T}^0\phi_1^2 T_1 + \frac{1}{2}\mathcal{D}_{\phi TT}^0\phi_1 T_1^2\right)\right]
\end{aligned}$$