Nonlinear energy transfer between fluid sloshing and vessel motion

Supplementary material

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This supplementary material contains a more complete derivation of the perturbation solution to the nonlinear coupled problem of a fluid sloshing in a rectangular vessel, along with more details on the normal form derivation. For full details of the coordinate system, the parameter definition and the 1 : 1 resonance in the linear solution, see the main paper.

1. Nonlinear equations governing the coupled motion.

Assuming the fluid motion is irrotational so it can be written in terms of a velocity potential $\phi(x, y, t)$, the nonlinear equations governing the coupled fluid and vessel motion are

$$\begin{aligned}
\phi_{xx} + \phi_{yy} &= 0 \quad &&\text{in} \quad 0 < y < h(x,t) \quad 0 < x < L,\\
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) - \dot{q}_1\phi_x - \dot{q}_2\phi_y + g(y - h_0) &= Be \quad &&\text{on} \quad y = h(x,t),\\
h_t + (\phi_x - \dot{q}_1)h_x &= \phi_y - \dot{q}_2 \quad &&\text{on} \quad y = h(x,t),\\
\phi_y &= \dot{q}_2 \quad &&\text{on} \quad y = 0,\\
\phi_x &= \dot{q}_1 \quad &&\text{on} \quad x = 0, L,
\end{aligned}$$

for the fluid and

$$m_v \frac{d^2\theta}{dt^2} + \frac{g}{l}(m_v + m_f)\sin\theta = -\frac{W}{l}\cos\theta \left(\int_0^L \int_0^h \rho\phi_{xt}\, dydx + \int_0^L \rho h_t \phi_x(x, h(x,t), t)\, dx\right)$$
$$-\frac{W}{l}\sin\theta \left(\int_0^L \int_0^h \rho\phi_{yt}\, dydx + \int_0^L \rho h_t \phi_y(x, h(x,t), t)\, dx\right),$$

for the vessel motion.

Here

$$q_1 = l\sin\theta(t) \quad \text{and} \quad q_2 = -l\cos\theta(t),$$

which parametrize the constraint as $q_1^2 + q_2^2 = l^2$, and Be is the Bernoulli constant.

We look for a solution to this system of equations in the form of a perturbation expansion in powers of $\epsilon \ll 1$ about the quiescent solution $\theta = 0$, $h = h_0$ and $\phi =$ constant. The parameter ϵ is a measure of the wave amplitude induced on the free-surface during the vessel motion. Hence we introduce

$$\begin{aligned}
\phi(x,y,t) &= \Phi_0 + \epsilon\phi_1(x,y,t) + \epsilon^2\phi_2(x,y,t) + O(\epsilon^3),\\
h(x,t) &= h_0 + \epsilon h_1(x,t) + \epsilon^2 h_2(x,t) + O(\epsilon^3),\\
\theta(t) &= \epsilon\theta_1(t) + \epsilon^2\theta_2(t) + O(\epsilon^3),\\
Be &= \epsilon Be_1 + \epsilon^2 Be_2 + O(\epsilon^3),
\end{aligned}$$

where Φ_0 is a constant. At this stage it is convenient to remark that previous studies have shown that the linear solution of the above system of equations has frequency ω, and so it is convenient to introduce a new time variable $\tilde{t} = \omega t$ so the linear solution is now 2π periodic. As it will be clear from the context, we drop the $\tilde{}$ for the rest of this work and just consider the solutions to be 2π periodic in time.

In order to derive the normal form equations in §3 of this report we also need to expand the frequency ω in powers of ϵ, so

$$\omega = \omega_n + \epsilon\omega_{n,1} + \epsilon^2\omega_{n,2}.$$

Therefore, the derivative of $h(x,t)$ for example, with respect to the original unscaled time becomes

$$h_t = \epsilon\omega_n h_{1t} + \epsilon^2(\omega_n h_{2t} + \omega_{n,1}h_{1t}) + O(\epsilon^3).$$

Substituting the above expressions into the nonlinear equations and grouping terms of the same order in ϵ leads to the systems of equations given in the subsections below.

1.1. *The $O(\epsilon)$ equations*

At $O(\epsilon)$ we obtain the linear equations

$$\begin{array}{ccc} \phi_{1xx} + \phi_{1yy} = 0 & \text{in} & 0 < y < h_0 \quad 0 < x < L, \\ \omega_n\phi_{1t} + gh_1 = Be_1 & \text{on} & y = h_0, \\ \omega_n h_{1t} = \phi_{1y} & \text{on} & y = h_0, \\ \phi_{1y} = 0 & \text{on} & y = 0, \\ \phi_{1x} = l\omega_n\theta_{1t} & \text{on} & x = 0, L, \end{array} \tag{1.1}$$

for the fluid and

$$m_v\omega_n^2\theta_{1tt} + \frac{g}{l}(m_v + m_f)\theta_1 = -\frac{\omega_n W}{l}\int_0^L\int_0^{h_0}\rho\phi_{1xt}\,dydx,$$

for the vessel motion. It is easy to show that $Be_1 = 0$. This system can be reduced to a system for ϕ_1 and θ_1 only, by combining the two free surface conditions together to give

$$\omega_n^2\phi_{1tt} + g\phi_{1y} = 0, \quad \text{on} \quad y = h_0.$$

These equations have been solved using two contrasting approaches, namely expanding the solutions in terms of different eigenfunctions. Alemi Ardakani *et al.* (2012*a*) investigates the two different expansions, the cosine and the vertical eigenfunction expansions, and proves that both methods are equivalent. In this work we use the infinite cosine expansion so as to overcome the numerical issues found when using the vertical eigenfunction expansion highlighted by Alemi Ardakani *et al.* (2012*b*).

At the 1 : 1 resonance the dispersion relation for the linear problem, given by (1.1) of the main paper has a double root, and the velocity potential has the solution

$$\phi_1(x,y,t) = \hat{\phi}_1\cos(t) = (AS_0(x,y) + BS_1(x,y))\cos(t). \tag{1.2}$$

Here A and B are arbitrary constants and

$$S_0 = \frac{\cosh\beta_n y}{\cosh\beta_n h_0}\cos\beta_n x, \tag{1.3}$$

$$S_1 = l\omega_n\left(x - \frac{1}{2}L - \frac{4\omega_n^2}{L}\sum_{m=0}^{\infty}\frac{1}{\alpha_m^2\sigma_{m,n}}\frac{\cosh\alpha_m y}{\cosh\alpha_m h_0}\cos\alpha_m x\right), \tag{1.4}$$

where $\alpha_m = (2m+1)\pi/L$, $\beta_n = 2n\pi/L$ and $\sigma_{m,n} = h_0 g\alpha_m^2\tanh\alpha_m h_0/(\alpha_m h_0) - \omega_n^2$. Alternatively, we can also write

$$S_1 = l\omega_n\sum_{m=0}^{\infty}\left(p_m - \frac{4\omega_n^2}{L}\frac{1}{\alpha_m^2\sigma_{m,n}}\frac{\cosh\alpha_m y}{\cosh\alpha_m h_0}\right)\cos\alpha_m x,$$

where the constants $p_m = -4/(\alpha_m^2 L)$ come from expanding $x - L/2$ as cosine expansion,

$$x - L/2 = \sum_0^{\infty}p_m\cos(\alpha_m x). \tag{1.5}$$

Note that at the 1 : 1 resonance, ω_n and β_n are related by $\omega_n^2 = g\beta_n\tanh(\beta_n h_0)$.

Also, the solution for $\theta_1(t)$ is

$$\theta_1(t) = B\sin(t).$$

The linearized problem is solved numerically with the parameter values given in table 1, which correspond to the series of experiments of Cooker (1994).

Parameter	Value
m_v	0.552 kg
W	0.13 m
L	0.525 m
ρ	$1000\,\mathrm{kgm}^{-3}$ (water)

TABLE 1. Values of the parameters used in this paper, based on the experiments in Cooker (1994).

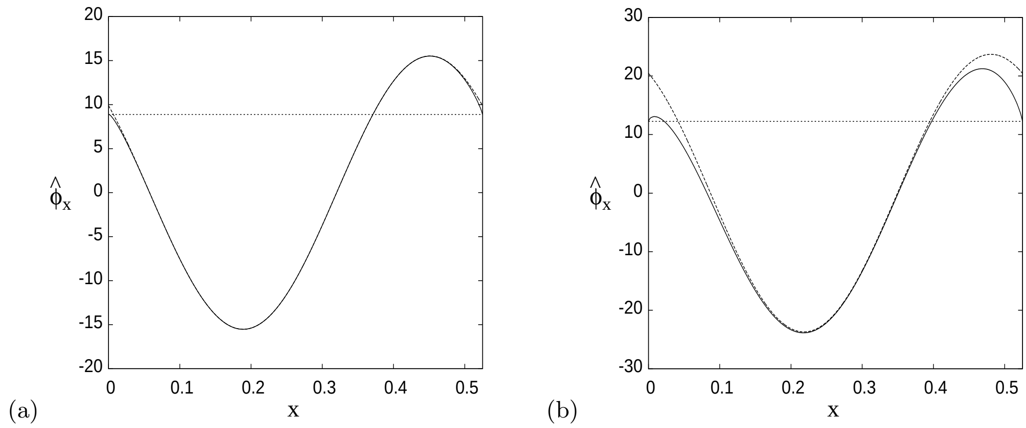

FIGURE 1. Plot of $\hat{\phi}_x(x, h_0)$ with $\widetilde{A} = 1\,\mathrm{m}^2\mathrm{s}^{-1}$ and $\widetilde{B} = 1$ for the lowest frequency 1:1 resonance for (a) $h_0 = 0.05\,\mathrm{m}$ and (b) $h_0 = 0.15\,\mathrm{m}$. In each case the solid line gives the result using the cosine expansion formulation, while the dashed line gives the same result from the vertical eigenfunction expansion. The horizontal dotted lines correspond to the value $l\omega_n\widetilde{B}$ which indicates the required boundary conditions of $\hat{\phi}_x$ at $x = 0$ and L.

In figure 1 we plot $\hat{\phi}_x(x, h_0)$ from (3.11) of the main paper with $\widetilde{A} = 1\,\mathrm{m}^2\mathrm{s}^{-1}$ and $\widetilde{B} = 1$, plus the corresponding form of $\hat{\phi}_x(x, h_0)$ using the vertical eigenmode expression, for two separate values of h_0. The vertical eigenfunction expansion for $\hat{\phi}$ at resonance is given by the equation before (5.23) in Alemi Ardakani *et al.* (2012*a*). In the cosine expansion we use 500 terms in the sum and in the vertical eigenfunction expansion we use 50 terms in the sum. Both these values are sufficiently large so that the numerical solutions have converged. The linear boundary condition $\phi_x = l\dot{\theta}$ at $x = 0$ and L states that $\hat{\phi}_x = l\omega_n\widetilde{B}$, but for larger values of h_0, the vertical eigenvalue expression fails to give this value (denoted by the horizontal dotted line), while the cosine expansion still accurately gives this value. The reason for this discrepancy at $x = 0, L$ for larger values of h_0 is due to the Gibbs phenomenon causing oscillations in the vertical eigenfunction series expansion close to $y = h_0$. This issue is addressed in §6.4 of Alemi Ardakani *et al.* (2012*b*) and explains why we choose to use the cosine expansion approach in this paper even though it has slower convergence than the vertical eigenfunction expansion. Although we highlight this convergence problem with the vertical eigenfunctions here for a value of ω_n at resonance, this convergence problem at $y = h_0$ is also observed at frequencies away from resonance. This convergence issue does not affect the calculation of the sloshing frequency from the dispersion relation (1.1) from the main paper, because in this case the x integrals occurring in the solution procedure are evaluated exactly. However, when calculating the nonlinear normal form in §3 we need to calculate x integrals involving the function $\hat{\phi}_x$ numerically, and it is here that any inaccuracies in the function at $x = 0, L$ would contribute to errors in the solution.

1.2. The $O(\epsilon^2)$ equations

Retaining only the terms of $O(\epsilon^2)$ we obtain the following system of equations for ϕ_2, h_2 and θ_2:

$$\begin{aligned}
\phi_{2xx} + \phi_{2yy} &= 0 && \text{in} \quad 0 < y < h_0 \quad 0 < x < L,\\
\omega_n \phi_{2t} + g h_2 &= Be_2 - \omega_n h_1 \phi_{1yt} - \tfrac{1}{2}(\phi_{1x}^2 + \phi_{1y}^2) + l\omega_n \theta_{1t}\phi_{1x} - \omega_{n,1}\phi_{1t} && \text{on} \quad y = h_0,\\
\omega_n h_{2t} &= \phi_{2y} - (\phi_{1x} - l\omega_n \theta_{1t}) h_{1x} + h_1 \phi_{1yy} - l\omega_n \theta_1 \theta_{1t} - \omega_{n,1} h_{1t} && \text{on} \quad y = h_0,\\
\phi_{2y} &= l\omega_n \theta_1 \theta_{1t} && \text{on} \quad y = 0,\\
\phi_{2x} &= l\omega_n \theta_{2t} + l\omega_{n,1}\theta_{1t} && \text{on} \quad x = 0, L,
\end{aligned} \tag{1.6}$$

for the fluid and

$$m_v \omega_n^2 \theta_{2tt} + \frac{g}{l}(m_v + m_f)\theta_2 = -2 m_v \omega_n \omega_{n,1}\theta_{1tt} - \frac{\omega_n W}{l}\int_0^L \int_0^{h_0} \rho \phi_{2xt}\, dydx - \frac{\omega_n \theta_1 W}{l}\int_0^L \int_0^{h_0} \rho \phi_{1yt}\, dydx$$
$$- \frac{\omega_n W}{l}\int_0^L \rho h_{1t}\phi_{1x}(x, y = h_0, t)\, dx - \frac{\omega_n W}{l}\int_0^L \rho h_1 \phi_{1xt}(x, y = h_0, t)\, dx - \frac{\omega_{n,1} W}{l}\int_0^L \int_0^{h_0} \rho \phi_{1xt}\, dydx,$$

for the vessel.

In order to solve this system of equations, the RHS of each equation needs to be expressed fully in order to determine the time dependence of the second order solution.

2. Solution of $O(\epsilon^2)$ equations

As for the $O(\epsilon)$ equations, we can eliminate h_2 (and h_1) and consider the problem solely as a system of equations for ϕ_2 and θ_2. Combining the two free surface boundary conditions above, and using $h_1 = -\omega_0 \phi_{1t}/g$ to eliminate h_1, we generate the free surface condition

$$\begin{aligned}
\omega_n^2 \phi_{2tt} + g\phi_{2y} = &-2\omega_n \phi_{1x}\phi_{1tx} + \omega_n \phi_{1t}\phi_{1yy} - 2\omega_n \phi_{1y}\phi_{1ty} + \omega_n^3 g^{-1}\phi_{1t}\phi_{1ytt}\\
&+ 2l\omega_n^2 \theta_{1t}\phi_{1xt} + l\omega_n^2 \theta_{1tt}\phi_{1x} + gl\omega_n \theta_1 \theta_{1t} - 2\omega_n \omega_{n,1}\phi_{1tt}.
\end{aligned}$$

Introducing the forms of ϕ_1 and θ_1 from the leading order solution we find

$$\omega_n^2 \phi_{2tt} + g\phi_{2y} = \left(\omega_n \left(2\hat{\phi}_{1x}^2 + 3\hat{\phi}_{1y}^2 - \hat{\phi}_1 \hat{\phi}_{1yy}\right) - 3l\omega_n^2 B\hat{\phi}_{1x} + gl\omega_n B^2\right)\frac{1}{2}\sin(2t) + 2\omega_n \omega_{n,1}\hat{\phi}_1 \cos(t), \tag{2.1}$$

where we have left ϕ_1 in terms of $\hat{\phi}_1$ at this stage, and we have also used the leading order free surface condition ($\hat{\phi}_{1y} = \frac{\omega_n^2}{g}\hat{\phi}_1$) to remove first order derivatives with respect to y. This is useful for simplifying the final form of the solution.

Looking at equation (2.1) it is clear that if $\omega_{n,1} = 0$ then only second harmonics appear in the equations and the solution procedure is simplified. This is also the case in the vessel equation and the side wall boundary conditions. In fact, $\omega_{n,1} = 0$ is the only value $\omega_{n,1}$ can take, and this can be shown formally using a solvability condition at this order which can be found in appendix A. This also agrees with the works of Tadjbakhsh & Keller (1960) and Feng & Sethna (1990) who consider similar problems.

Thus, with $\omega_{n,1} = 0$ we can look for a solution to the $O(\epsilon^2)$ system of equations with

$$\phi_2(x, y, t) = \hat{\phi}_2(x, y)\sin(2t) \quad \text{and} \quad \theta_2(t) = \hat{\theta}_2 \cos(2t).$$

Substituting these into the $O(\epsilon^2)$ system of equations reduces them to

$$\begin{aligned}
\hat{\phi}_{2xx} + \hat{\phi}_{2yy} &= 0 && \text{in} \quad 0 < y < h_0 \quad 0 < x < L,\\
-4\omega_n^2 \hat{\phi}_2 + g\hat{\phi}_{2y} &= \frac{1}{2}\omega_n \left(2\hat{\phi}_{1x}^2 + 3\hat{\phi}_{1y}^2 - \hat{\phi}_1 \hat{\phi}_{1yy}\right) - \frac{3}{2}l\omega_n^2 B\hat{\phi}_{1x} + \frac{1}{2}gl\omega_n B^2 && \text{on} \quad y = h_0,\\
\hat{\phi}_{2y} &= \frac{1}{2}l\omega_n B^2 && \text{on} \quad y = 0,\\
\hat{\phi}_{2x} &= -2\omega_n l\hat{\theta}_2 && \text{on} \quad x = 0, L,
\end{aligned}$$

for the fluid and

$$-4\omega_n^2 m_v \hat{\theta}_2 + \frac{g}{l}(m_v + m_f)\hat{\theta}_2 = -\frac{2\omega_n W}{l}\int_0^L \int_0^{h_0} \rho \hat{\phi}_{2x}\, dydx - \frac{\omega_n^2 W}{gl}\int_0^L \rho \hat{\phi}_1 \hat{\phi}_{1x}(x, y = h_0)\, dx,$$

for the vessel. We have simplified the vessel equation by again using $h_1 = -\omega_n \phi_{1t}(x, h_0)/g$, and by noting that

$$\int_0^L \int_0^{h_0} \rho \hat{\phi}_{1y}\, dy dx = 0.$$

The equations can be further simplified by removing the constant terms in the free surface boundary condition and the bottom boundary condition. This is achieved by introducing the new velocity potential

$$\tilde{\phi}_2 = \hat{\phi}_2 - \frac{1}{2} l\omega_n B^2 y + \frac{l\omega_n h_0 B^2}{2},$$

and thus the equations become

$$\begin{aligned}
\tilde{\phi}_{2xx} + \tilde{\phi}_{2yy} &= 0 \quad \text{in} \quad 0 < y < h_0 \quad 0 < x < L,\\
-4\omega_n^2 \tilde{\phi}_2 + g\tilde{\phi}_{2y} = \frac{1}{2}\omega_n \left(2\hat{\phi}_{1x}^2 + 3\hat{\phi}_{1y}^2 - \hat{\phi}_1 \hat{\phi}_{1yy}\right) - \frac{3}{2} l\omega_n^2 B\hat{\phi}_{1x} &\quad \text{on} \quad y = h_0,\\
\tilde{\phi}_{2y} &= 0 \quad \text{on} \quad y = 0,\\
\tilde{\phi}_{2x} &= -2\omega_n l\hat{\theta}_2 \quad \text{on} \quad x = 0, L,
\end{aligned}$$

for the fluid and

$$-4\omega_n^2 m_v \hat{\theta}_2 + \frac{g}{l}(m_v + m_f)\hat{\theta}_2 = -\frac{2\omega_n W}{l} \int_0^L \int_0^{h_0} \rho \tilde{\phi}_{2x}\, dy dx - \frac{\omega_n^2 W}{gl} \int_0^L \rho \hat{\phi}_1 \hat{\phi}_{1x}(x, y = h_0)\, dx,$$

for the vessel.

In the free surface condition, we now need to determine the form of the RHS. To do this we substitute the form of $\hat{\phi}_1$ from (1.2) using the forms of $S_0(x, h_0)$ and $S_1(x, h_0)$ given in (1.3) and (1.4). Thus the terms on the RHS can be written as

$$\begin{aligned}
\hat{\phi}_{1x}^2 &= \frac{1}{2} A^2 \beta_n^2 (1 - \cos 2\beta_n x) - 2ABl\omega_n \beta_n \sum_{r=0}^{\infty} \frac{4\beta_n}{L(\beta_n^2 - \alpha_r^2)} \cos \alpha_r x\\
&- ABl\omega_n \beta_n \sum_{m=0}^{\infty} \left(\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}}\right) (\cos(\beta_n - \alpha_m)x - \cos(\beta_n + \alpha_m)x) + l^2 \omega_n^2 B^2\\
&+ 2l^2 \omega_n^2 B^2 \sum_{m=0}^{\infty} \left(\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}}\right) \left[\frac{2}{L\alpha_m} - \sum_{r=1}^{\infty} \frac{4\alpha_m}{L(\beta_r^2 - \alpha_m^2)} \cos \beta_r x\right] + \frac{1}{2} B^2 l^2 \omega_n^2 \sum_{m=0}^{\infty} \left(\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}}\right)^2 (1 - \cos 2\alpha_m x),\\
&+ B^2 l^2 \omega_n^2 \sum_{m=0}^{\infty} \sum_{q=m+1}^{\infty} \left(\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}}\right) \left(\frac{4\omega_n^2}{L\alpha_q \sigma_{n,q}}\right) (\cos(\alpha_q - \alpha_m)x - \cos(\alpha_q + \alpha_m)x),
\end{aligned}$$

$$\begin{aligned}
\hat{\phi}_{1y}^2 &= \frac{1}{2} A^2 \beta_n^2 \tanh^2 \beta_n h_0 (1 + \cos 2\beta_n x)\\
&+ ABl\omega_n \beta_n \tanh \beta_n h_0 \sum_{m=0}^{\infty} \left(-\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}} \tanh \alpha_m h_0\right) (\cos(\beta_n - \alpha_m)x + \cos(\beta_n + \alpha_m)x),\\
&+ \frac{1}{2} B^2 l^2 \omega_n^2 \sum_{m=0}^{\infty} \left(-\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}} \tanh \alpha_m h_0\right)^2 (1 + \cos 2\alpha_m x),\\
&+ B^2 l^2 \omega_n^2 \sum_{m=0}^{\infty} \sum_{q=m+1}^{\infty} \left(-\frac{4\omega_n^2}{L\alpha_m \sigma_{m,n}} \tanh \alpha_m h_0\right) \left(-\frac{4\omega_n^2}{L\alpha_q \sigma_{n,q}} \tanh \alpha_q h_0\right) (\cos(\alpha_q - \alpha_m)x + \cos(\alpha_q + \alpha_m)x),
\end{aligned}$$

and

$$\hat{\phi}_{1yy}\hat{\phi}_1 = \frac{1}{2}A^2\beta_n^2\left(1+\cos 2\beta_n x\right) + \frac{1}{2}ABl\omega_n\sum_{m=0}^{\infty}\left(-\frac{4\omega_n^2}{L\sigma_{m,n}}\right)\left(\cos(\beta_n-\alpha_m)x+\cos(\beta_n+\alpha_m)x\right),$$
$$+\frac{1}{2}ABl\omega_n\beta_n^2\sum_{m=0}^{\infty}\left(p_m-\frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\right)\left(\cos(\beta_n-\alpha_m)x+\cos(\beta_n+\alpha_m)x\right),$$
$$+\frac{1}{2}B^2l^2\omega_n^2\sum_{m=0}^{\infty}\left(p_m-\frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\right)\left(-\frac{4\omega_n^2}{L\sigma_{m,n}}\right)\left(1+\cos 2\alpha_m x\right),$$
$$+\frac{1}{2}B^2l^2\omega_n^2\sum_{m=0}^{\infty}\sum_{q=m+1}^{\infty}\left(p_m-\frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\right)\left(-\frac{4\omega_n^2}{L\sigma_{n,q}}\right)\left(\cos(\alpha_m-\alpha_q)x+\cos(\alpha_m+\alpha_q)x\right)$$
$$+\frac{1}{2}B^2l^2\omega_n^2\sum_{m=0}^{\infty}\sum_{q=m+1}^{\infty}\left(-\frac{4\omega_n^2}{L\sigma_{m,n}}\right)\left(p_q-\frac{4\omega_n^2}{L\alpha_q^2\sigma_{n,q}}\right)\left(\cos(\alpha_m-\alpha_q)x+\cos(\alpha_m+\alpha_q)x\right).$$

Also on $y = h_0$

$$\hat{\phi}_{1x} = -A\beta_n\sin\beta_n x + Bl\omega_n\left(1+\sum_{m=0}^{\infty}\left(\frac{4\omega_n^2}{L\alpha_m\sigma_{m,n}}\right)\sin\alpha_m x\right).$$

We note that the first 3 terms on the RHS of the free surface condition are written as infinite cosine expansions (or as constants) while the final term is expressed as infinite sine expansions. Therefore, in order to simplify our overall solution for ϕ_2 we convert the infinite sine expansions into cosine expansions by using the expressions

$$\sin\beta_n x = \sum_{r=0}^{\infty}\frac{4\beta_n}{L(\beta_n^2-\alpha_r^2)}\cos\alpha_r x,$$
$$\sin\alpha_m x = \frac{2}{L\alpha_m} - \sum_{r=1}^{\infty}\frac{4\alpha_m}{L(\beta_r^2-\alpha_m^2)}\cos\beta_r x.$$

Thus, we can see from the above expressions that the RHS of the free surface boundary condition can be written as $F + f(x)$ where F is a constant, and $f(x)$ is an infinite cosine expansion. We can again remove the constant term from this boundary condition by introducing the new velocity potential

$$\bar{\phi}_2 = \tilde{\phi}_2 + \frac{1}{4\omega_n^2}F,$$

for which the system of equations become

$$\begin{array}{rcll}
\bar{\phi}_{2xx}+\bar{\phi}_{2yy}=0 & \text{in} & 0<y<h_0 \quad 0<x<L, & \\
-4\omega_n^2\bar{\phi}_2+g\bar{\phi}_{2y}=f(x) & \text{on} & y=h_0, & \\
\bar{\phi}_{2y}=0 & \text{on} & y=0, & \\
\bar{\phi}_{2x}=-2\omega_n l\hat{\theta}_2 & \text{on} & x=0,L, & \\
-4\omega_n^2 m_v\hat{\theta}_2+\frac{g}{l}(m_v+m_f)\hat{\theta}_2 & = & -\frac{2\omega_n W}{l}\int_0^L\int_0^{h_0}\rho\bar{\phi}_{2x}\,dydx-\frac{\omega_n^2 W}{gl}\int_0^L\rho\hat{\phi}_1\hat{\phi}_{1x}(x,y=h_0)\,dx, &
\end{array} \tag{2.2}$$

where

$$F = \frac{1}{4}\omega_n A^2\beta_n^2(1+3\tanh^2\beta_n h_0) + B^2l^2\omega_n^5\sum_{m=0}^{\infty}\frac{1}{L\sigma_{m,n}}\left[\frac{8\omega_n^2}{L\alpha_m^2\sigma_{m,n}}+\frac{12\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\tanh^2\alpha_m h_0\right.$$
$$\left.+p_m-\frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\right]-\frac{1}{2}l^2\omega_n^3B^2-8l^2\omega_n^5B^2\sum_{m=0}^{\infty}\frac{1}{L^2\alpha_m^2\sigma_{m,n}},$$

and

$$
\begin{aligned}
f(x) = {} & \frac{1}{2}\omega_n \sum_{r=1}^{\infty} \Bigg[\frac{3}{2}A^2\beta_n^2\delta_{r,2n}(\tanh^2\beta_n h_0 - 1) + \frac{2\omega_n^4}{L\sigma_{r,n}} l^2 B^2 \delta_r \Bigg\{ -\frac{32\omega_n^2}{L\beta_r^2\sigma_{r,n}} \\
& + \frac{48\omega_n^2}{L\beta_r^2\sigma_{r,n}} \tanh^2 \frac{1}{2}\beta_r h_0 + p_r - \frac{16\omega_n^2}{L\beta_r^2\sigma_{r,n}} \Bigg\} - 16 l^2\omega_n^4 B^2 \sum_{m=0}^{\infty} \frac{1}{L^2\sigma_{m,n}(\beta_r^2 - \alpha_m^2)} \\
& + \frac{1}{2} l^2\omega_n^2 B^2 \sum_{\substack{m=0 \\ q=m+r}}^{\infty} \Bigg\{ \frac{64\omega_n^4}{L^2\alpha_m\alpha_q\sigma_{m,n}\sigma_{q,n}} + \frac{96\omega_n^4}{L^2\alpha_m\alpha_q\sigma_{m,n}\sigma_{q,n}} \tanh\alpha_m h_0 \tanh\alpha_q h_0 \\
& + \left(p_m - \frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\right)\left(\frac{4\omega_n^2}{L\sigma_{q,n}}\right) + \left(\frac{4\omega_n^2}{L\sigma_{m,n}}\right)\left(p_q - \frac{4\omega_n^2}{L\alpha_q^2\sigma_{q,n}}\right)\Bigg\} \\
& + \frac{1}{2} l^2\omega_n^2 B^2 \sum_{\substack{m=0 \\ q=r-1-m \\ q>m}}^{\infty} \Bigg\{ -\frac{64\omega_n^4}{L^2\alpha_m\alpha_q\sigma_{m,n}\sigma_{q,n}} + \frac{96\omega_n^4}{L^2\alpha_m\alpha_q\sigma_{m,n}\sigma_{q,n}} \tanh\alpha_m h_0 \tanh\alpha_q h_0 \\
& + \left(p_m - \frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\right)\left(\frac{4\omega_n^2}{L\sigma_{q,n}}\right) + \left(\frac{4\omega_n^2}{L\sigma_{m,n}}\right)\left(p_q - \frac{4\omega_n^2}{L\alpha_q^2\sigma_{q,n}}\right)\Bigg\} \Bigg] \cos\beta_r x \\
& + \frac{1}{4} l\omega_n^2 AB \sum_{r=0}^{\infty} \Bigg[\sum_{\substack{q=n+r \\ q=n-r-1 \\ q>0}} \Bigg\{ -\frac{16\beta_n\omega_n^2}{L\alpha_q\sigma_{q,n}} - \frac{24\beta_n\omega_n^2}{L\alpha_q\sigma_{q,n}} \tanh\beta_n h_0 \tanh\alpha_q h_0 \\
& + \frac{4\omega_n^2}{L\sigma_{q,n}} - \beta_n^2 \left(p_q - \frac{4\omega_n^2}{L\alpha_q^2\sigma_{q,n}}\right)\Bigg\} + \sum_{\substack{q=r-n \\ q>0}} \Bigg\{ \frac{16\beta_n\omega_n^2}{L\alpha_q\sigma_{q,n}} - \frac{24\beta_n\omega_n^2}{L\alpha_q\sigma_{q,n}} \tanh\beta_n h_0 \tanh\alpha_q h_0 \\
& + \frac{4\omega_n^2}{L\sigma_{q,n}} - \beta_n^2 \left(p_q - \frac{4\omega_n^2}{L\alpha_q^2\sigma_{q,n}}\right)\Bigg\} - \left(\frac{8\beta_n^2}{L(\beta_n^2 - \alpha_r^2)}\right) \Bigg] \cos\alpha_r x, \\
= {} & \sum_{r=1}^{\infty} \left(A^2 d_r^A + B^2 d_r^B\right)\cos\beta_r x + \sum_{r=0}^{\infty} AB d_r^{AB} \cos\alpha_r x.
\end{aligned}
$$

The parameters $\delta_{r,2n}$ and δ_r appearing in $f(x)$ are defined by

$$
\delta_{r,2n} = \begin{cases} 1 & r = 2n \\ 0 & \text{otherwise} \end{cases}, \qquad \delta_r = \begin{cases} 1 & \text{for } r \text{ odd} \\ 0 & \text{otherwise} \end{cases}.
$$

In order to solve the system (2.2) we assume that $\bar{\phi}_2(x, y)$ can be split into two components, one proportional to $\hat{\theta}_2$ and one not, so we write

$$
\bar{\phi}_2(x, y) = H(x, y) + \hat{\theta}_2 G(x, y),
$$

where H satisfies the homogeneous x boundary conditions and inhomogeneous y boundary conditions, while G satisfies the homogeneous y boundary conditions and inhomogeneous x boundary conditions. Substituting this expression into the governing equations gives the following equations for G

$$
\begin{aligned}
G_{xx} + G_{yy} &= 0 \quad \text{in} \quad 0 < y < h_0 \quad 0 < x < L, \\
-4\omega_n^2 G + g G_y &= 0 \quad \text{on} \quad y = h_0, \\
G_y &= 0 \quad \text{on} \quad y = 0, \\
G_x &= -2\omega_n l \quad \text{on} \quad x = 0, L.
\end{aligned}
$$

In order to take into account the x boundary conditions, we shift them into the surface boundary condition by introducing $\hat{G}(x, y)$ such that

$$
G = -2\omega_n l \left(x - \frac{L}{2}\right) + \hat{G},
$$

and $\hat{G}$ satisfies

$$\begin{aligned} \hat{G}_{xx} + \hat{G}_{yy} &= 0 \quad \text{in} \quad 0 < y < h_0 \quad 0 < x < L, \\ -4\omega_n^2 \hat{G} + g\hat{G}_y &= -8\omega_n^3 l \left(x - \frac{L}{2} \right) \quad \text{on} \quad y = h_0, \\ \hat{G}_y &= 0 \quad \text{on} \quad y = 0, \\ \hat{G}_x &= 0 \quad \text{on} \quad x = 0, L. \end{aligned}$$

The solution for $\hat{G}$ satisfying all the equations except the surface boundary condition is

$$\hat{G} = \sum_{r=0}^{\infty} a_r \frac{\cosh(\alpha_r y)}{\cosh(\alpha_r h_0)} \cos(\alpha_r x).$$

The expansion of $x - L/2$ in terms of cosines is given by (1.5), and so to find the unknown constants a_r, the solution for $\hat{G}$ is substituted into the surface boundary condition giving

$$a_r = \frac{8\omega_n^3 l p_r}{4\omega_n^2 - g\alpha_r \tanh \alpha_r h_0}.$$

The problem for the function $H(x, y)$ is very similar, and the solution procedure follows similarly. The function H satisfies

$$\begin{aligned} H_{xx} + H_{yy} &= 0 \quad \text{in} \quad 0 < y < h_0 \quad 0 < x < L, \\ -4\omega_n^2 H + gH_y &= f(x) \quad \text{on} \quad y = h_0, \\ H_y &= 0 \quad \text{on} \quad y = 0, \\ H_x &= 0 \quad \text{on} \quad x = 0, L. \end{aligned}$$

Therefore the solution for H satisfying all the equations except the surface boundary condition is

$$H = \sum_{r=1}^{\infty} b_r \frac{\cosh(\beta_r y)}{\cosh(\beta_r h_0)} \cos(\beta_r x) + \sum_{r=0}^{\infty} c_r \frac{\cosh(\alpha_r y)}{\cosh(\alpha_r h_0)} \cos(\alpha_r x).$$

This can then be substituted into the surface boundary condition and the $b_r's$ and $c_r's$ can then be read off by comparing coefficients of $\cos(\beta_r x)$ and $\cos(\alpha_r x)$. Thus,

$$b_r = -\frac{A^2 d_{Ar} + B^2 d_{Br}}{4\omega_n^2 - g\beta_r \tanh \beta_r h_0} \quad \text{and} \quad c_r = -\frac{ABd_{ABr}}{4\omega_n^2 - g\alpha_r \tanh \alpha_r h_0}.$$

Therefore

$$\begin{aligned} \hat{\phi}_2 = \frac{1}{2} l\omega_n B^2 y - \frac{l\omega_n h_0 B^2}{2} - \frac{1}{4\omega_n^2} F - \sum_{r=1}^{\infty} \left(\frac{A^2 d_{Ar} + B^2 d_{Br}}{4\omega_n^2 - g\beta_r \tanh \beta_r h_0} \right) \frac{\cosh \beta_r y}{\cosh \beta_r h_0} \cos \beta_r x \\ + \sum_{r=0}^{\infty} \left(\frac{-ABd_{ABr} + \hat{\theta}_2 8\omega_n^3 l p_r}{4\omega_n^2 - g\alpha_r \tanh \alpha_r h_0} \right) \frac{\cosh \alpha_r y}{\cosh \alpha_r h_0} \cos \alpha_r x - 2\omega_n l \left(x - \frac{L}{2} \right) \hat{\theta}_2. \end{aligned}$$

The coupling equation for $\hat{\theta}_2$ can be expressed as

$$-4\omega_n^2 m_v \hat{\theta}_2 + \frac{g}{l}(m_v + m_f)\hat{\theta}_2 = -\frac{2\omega_n \rho W}{l} \int_0^L \int_0^{h_0} \left(H_x + \hat{\theta}_2 G_x \right) dy dx - \frac{\omega_n^2 \rho W}{gl} \left[\frac{1}{2} \hat{\phi}_1^2 \right]_0^L,$$

which rearranges to

$$\hat{\theta}_2 = -\frac{2\omega_n \rho W}{l\Lambda} \int_0^L \int_0^{h_0} H_x \, dy dx - \frac{2AB\omega_n^3 \rho W}{g\Lambda} \left(\frac{1}{2} L + \sum_{r=0}^{\infty} \frac{4\omega_n^2}{L\alpha_r^2 \sigma_{r,n}} \right),$$

where

$$\Lambda = -4\omega_n^2 m_v + \frac{g}{l}(m_v + m_f) + \frac{2\omega_n \rho W}{l} \int_0^L \int_0^{h_0} G_x \, dy dx.$$

Evaluating the double integrals gives

$$\begin{aligned}\int_0^L\int_0^{h_0} G_x\,dydx &= -2\omega_n l\int_0^L\int_0^{h_0} dydx - \int_0^L\int_0^{h_0}\sum_{r=0}^{\infty} a_r\alpha_r\frac{\cosh\alpha_r y}{\cosh\alpha_r h_0}\sin\alpha_r x\,dydx,\\ &= -2\omega_n l h_0 L - \int_0^L\sum_{r=0}^{\infty} a_r\tanh\alpha_r h_0\sin\alpha_r x\,dx,\\ &= -2\omega_n l h_0 L - 2\sum_{r=0}^{\infty}\frac{a_r}{\alpha_r}\tanh\alpha_r h_0,\end{aligned}$$

and similarly

$$\int_0^L\int_0^{h_0} H_x\,dydx = -2\sum_{r=0}^{\infty}\frac{c_r}{\alpha_r}\tanh\alpha_r h_0.$$

Thus, $\hat{\theta}_2$ is proportional to AB, i.e. it is made up solely of antisymmetric modes, while $\hat{\phi}_2$ contains both symmetric and antisymmetric modes.

Therefore the final form of $\phi_2(x,y)$ and $\theta_2(t)$ are

$$\begin{aligned}\phi_2(x,y,t) &= \big(A^2\zeta_0(x,y) + AB\zeta_1(x,y) + B^2\zeta_2(x,y) + B^2 Z_1 y + B^2 Z_2 + A^2 Z_3\big)\sin(2t),\\ \theta_2(x,y,t) &= AB\Theta_1\cos(2t),\end{aligned}$$

where

$$\zeta_0 = \sum_{r=1}^{\infty}\frac{d_r^A}{g\beta_r\tanh\beta_r h_0 - 4\omega_n^2}\frac{\cosh(\beta_r y)}{\cosh(\beta_r h_0)}\cos(\beta_r x), \tag{2.3}$$

$$\zeta_1 = \sum_{r=0}^{\infty}\frac{d_r^{AB}}{g\alpha_r\tanh\alpha_r h_0 - 4\omega_n^2}\frac{\cosh(\alpha_r y)}{\cosh(\alpha_r h_0)}\cos(\alpha_r x), \tag{2.4}$$

$$\zeta_2 = \sum_{r=1}^{\infty}\frac{d_r^B}{g\beta_r\tanh\beta_r h_0 - 4\omega_n^2}\frac{\cosh(\beta_r y)}{\cosh(\beta_r h_0)}\cos(\beta_r x), \tag{2.5}$$

$$Z_1 = \frac{1}{2}l\omega_n, \tag{2.6}$$

$$\begin{aligned}Z_2 = -\frac{1}{2}l\omega_n h_0 - \frac{1}{4}l^2\omega_n^3\sum_{m=0}^{\infty}\frac{1}{L\sigma_{m,n}}\bigg[&\frac{8\omega_n^2}{L\alpha_m^2\sigma_{m,n}} + \frac{12\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\tanh^2\alpha_m h_0\\ &+p_m - \frac{4\omega_n^2}{L\alpha_m^2\sigma_{m,n}}\bigg] + \frac{1}{8}l^2\omega_n + 2l^2\omega_n^3 B^2\sum_{m=0}^{\infty}\frac{1}{L^2\alpha_m^2\sigma_{m,n}},\end{aligned} \tag{2.7}$$

$$Z_3 = -\frac{1}{16\omega_n}\beta_n^2(1 + 3\tanh^2\beta_n h_0), \tag{2.8}$$

$$\Theta_1 = \frac{4\omega_n\rho W}{l\Lambda}\sum_{r=0}^{\infty}\frac{d_r^{AB}}{(g\alpha_r\tanh\alpha_r h_0 - 4\omega_n^2)\alpha_r}\tanh\alpha_r h_0 - \frac{2\omega_n^3\rho W}{g\Lambda}\left(\frac{1}{2}L + \sum_{r=0}^{\infty}\frac{4\omega_n^2}{L\alpha_r^2\sigma_{r,n}}\right). \tag{2.9}$$

3. The Lagrangian formulation

In order to formulate the normal form equations about the 1 : 1 resonance, we assume that the arbitrary eigenmode constants A and B now depend upon a slow time variable $\tau = \epsilon^2 t$. Also, the governing equations are variational, i.e. they can be determined by taking the first variation of a Lagrangian functional. It is useful to use a Lagrangian formulation when calculating the normal form to reduce the amount of algebra required. Alemi Ardakani *et al.* (2012*a*) defined a reduced Lagrangian for this system in order to derive the vessel equation, but did not include the constraint terms needed to correctly describe the fluid motion. The complete Lagrangian for this system is

$$\mathscr{L} = \int_{\tau_1}^{\tau_2}\int_{t_1}^{t_2}\mathcal{L}\,dt, \tag{3.1}$$

where $[t_1, t_2]$ and $[\tau_1, \tau_2]$ are arbitrary time intervals of the fast and slow time, and

$$\mathcal{L} = \rho W \int_0^L \int_0^{h(x,t)} \left(\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g(y - h_0) - Be \right) dydx \tag{3.2}$$
$$-\frac{1}{2} m_v l^2 \dot{\theta}^2 - (m_v + m_f) gl \cos\theta - l\dot{\theta}\rho W \int_0^L \int_0^{h(x,t)} (\phi_x \cos\theta + \phi_y \sin\theta)\, dydx\,,$$

where h_0 is the mean fluid height. Therefore, we insert our perturbation expansion up to second order into the Lagrangian, and then take variations with respect to A^* and B^* (where $*$ denotes the complex conjugate) to determine the normal form equations. In order to determine the form of the Lagrangian, we first require the form of the free surface $h(x, t, \tau)$ up to second order in ϵ. So that phase effects are not missed when calculating the form of h, we write the expansions for ϕ, θ and h in exponential form, as A and B are now complex. Thus, the forms of ϕ, h and θ which are substituted into the Lagrangian are

$$\begin{aligned}
\phi(x, y, t, \tau) = {} & \Phi_0 + \frac{1}{2}\epsilon\Big[(AS_0(x, y) + BS_1(x, y))e^{\mathrm{i}t} + (A^* S_0(x, y) + B^* S_1(x, y))e^{-\mathrm{i}t}\Big] \\
& -\frac{1}{2}\epsilon^2 \mathrm{i}\Big[(A^2\zeta_0(x, y) + AB\zeta_1(x, y) + B^2\zeta_2(x, y) + B^2 Z_1 y + B^2 Z_2 + A^2 Z_3)e^{2\mathrm{i}t} \\
& -(A^{*2}\zeta_0(x, y) + A^* B^*\zeta_1(x, y) + B^{*2}\zeta_2(x, y) + B^{*2} Z_1 y + B^{*2} Z_2 + A^{*2} Z_3)e^{-2\mathrm{i}t}\Big] \\
& +O(\epsilon^3),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
h(x, t, \tau) = {} & h_0 - \frac{1}{2}\epsilon\mathrm{i}\Big[(AH_0(x) + BH_1(x))e^{\mathrm{i}t} - (A^* H_0(x) + B^* H_1(x))e^{-\mathrm{i}t}\Big] \\
& +\frac{1}{2}\epsilon^2\Big[(A^2\lambda_0(x) + AB\lambda_1(x) + B^2\lambda_2(x))e^{2\mathrm{i}t} + (A^{*2}\lambda_0(x) + A^* B^*\lambda_1(x) + B^{*2}\lambda_2(x))e^{-2\mathrm{i}t} \\
& +(\Lambda_0(x) - C_0)|A|^2 + (\Lambda_1(x) - C_1)AB^* + (\Lambda_2(x) - C_2)A^* B + (\Lambda_3(x) - C_3)|B|^2\Big] \\
& +O(\epsilon^3),
\end{aligned} \tag{3.4}$$

$$\theta(t, \tau) = -\frac{1}{2}\epsilon\mathrm{i}\Big[Be^{\mathrm{i}t} - B^* e^{-\mathrm{i}t}\Big] + \frac{1}{2}\epsilon^2\Big[AB\Theta_1 e^{2\mathrm{i}t} + A^* B^*\Theta_1 e^{-2\mathrm{i}t}\Big] + O(\epsilon^3), \tag{3.5}$$

$$Be = -\frac{1}{2}\epsilon^2 g\left(C_0|A|^2 + C_1 AB^* + C_2 A^* B + C_3|B|^2\right), \tag{3.6}$$

where the ϵ^2 part of Be is determined from (1.6). Here the constants C_i are chosen such that

$$\frac{1}{L}\int_0^L h(x, t, \tau)\, dx = h_0,$$

i.e. that the mean free surface height remains at $y = h_0$. This gives the definition for the C_i's as

$$C_i = \frac{1}{L}\int_0^L \Lambda_i(x)\, dx.$$

The unknown functions $H_0, H_1, \lambda_0, \lambda_1, \lambda_2, \Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3$ are determined by substituting (3.3)-(3.5) into the nonlinear free surface boundary condition

$$\omega\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) - l\omega\dot{\theta}\cos\theta\phi_x - l\omega\dot{\theta}\sin\theta\phi_y + g(y - h_0) = Be \quad \text{on} \quad y = h,$$

and equating the combination of terms proportional to A, B, A^2, AB, AB^*, A^*B, B^2, $|A|^2$ and $|B|^2$ equal to be zero. The convention in the work which follows is that the functions S_0, S_1, ζ_0, ζ_1 and ζ_2 will be assumed to be evaluated at $y = h_0$ unless they lie under an integral with respect to y. The nonlinear free surface condition leads to

$$
\begin{aligned}
H_0(x) &= \frac{\omega_n}{g} S_0, \\
H_1(x) &= \frac{\omega_n}{g} S_1, \\
\lambda_0(x) &= -\frac{1}{4g}\left(S_{0x}^2 + S_{0y}^2 + 2\omega_n H_0 S_{0y} + 8\omega_n \zeta_0 + 8\omega_n Z_3\right), \\
\lambda_1(x) &= -\frac{1}{2g}\left(S_{0x}S_{1x} + S_{0y}S_{1y} + \omega_n H_0 S_{1y} + \omega_n H_1 S_{0y} - l\omega_n S_{0x} + 4\omega_n \zeta_1\right), \\
\lambda_2(x) &= -\frac{1}{4g}\left(S_{1x}^2 + S_{1y}^2 + 2\omega_n H_1 S_{1y} - 2l\omega_n S_{1x} + 8\omega_n \zeta_2 + 8\omega_n Z_1 h_0 + 8\omega_n Z_2\right), \\
\Lambda_0(x) &= -\frac{1}{2g}\left(S_{0x}^2 + S_{0y}^2 - 2\omega_n H_0 S_{0y}\right), \\
\Lambda_1(x) = \Lambda_2(x) &= -\frac{1}{2g}\left(S_{0x}S_{1x} + S_{0y}S_{1y} - \omega_n H_0 S_{1y} - \omega_n H_1 S_{0y} - l\omega_n S_{0x}\right), \\
\Lambda_3(x) &= -\frac{1}{2g}\left(S_{1x}^2 + S_{1y}^2 - 2\omega_n H_1 S_{1y} - 2l\omega_n S_{1x}\right).
\end{aligned}
$$

The consequence of $\Lambda_1(x) = \Lambda_2(x)$ is that $C_1 = C_2$.

Substituting the above expressions for ϕ, θ and h into $\hat{L}$ and integrating with respect to t between 0 and 2π we obtain an expression of the Lagrangian in the following form

$$
\begin{aligned}
L = \int_{\tau_1}^{\tau_2} \Big[& \epsilon^{-3} A_0 + \frac{1}{2} a_0 \mathrm{i}\left(A^* A_\tau - A A_\tau^*\right) + \frac{1}{2} b_0 \mathrm{i}\left(B^* B_\tau - B B_\tau^*\right) + c_0 \mathrm{i}\left(B^* A_\tau - B A_\tau^*\right) + c_1 \mathrm{i}\left(A^* B_\tau - A B_\tau^*\right) \\
& - a_1 \omega_{n,2} |A|^2 - b_1 \omega_{n,2} |B|^2 - c_2 \omega_{n,2}\left(AB^* + A^* B\right) - \frac{1}{2} a_2 |A|^4 - \frac{1}{2} b_2 |B|^4 - a_3 |A|^2 |B|^2 \\
& - \frac{1}{2} a_4\left(B^2 A^{*2} + A^2 B^{*2}\right) - c_3\left(AB|A|^2 + AB^*|A|^2\right) - c_4\left(AB|B|^2 + A^* B|B|^2\right) \Big] \, d\tau.
\end{aligned}
$$

The coefficients of the terms containing τ derivatives are

$$
\begin{aligned}
a_0 &= \frac{1}{2}\frac{\omega_n^2}{g}\rho W \int_0^L S_0^2 \, dx, \\
b_0 &= \frac{1}{2} l\omega_n \rho W \int_0^L \int_0^{h_0} S_{1x} \, dx + \frac{1}{2}\frac{\omega_n^2}{g}\rho W \int_0^L S_1^2 \, dx + \frac{1}{2} l^2 \omega_n^2 m_v, \\
c_0 &= \frac{1}{4}\frac{\omega_n^2}{g}\rho W \int_0^L S_0 S_1 \, dx, \\
c_1 &= \frac{1}{4} l\omega_n \rho W \int_0^L \int_0^{h_0} S_{0x} + \frac{1}{4}\frac{\omega_n^2}{g}\rho W \int_0^L S_0 S_1 \, dx,
\end{aligned}
$$

the linear terms are

$$
\begin{aligned}
a_1 \omega_{n,2} &= -\rho\frac{1}{4} W \int_0^L \int_0^{h_0} \left(S_{0x}^2 + S_{0y}^2\right) dy dx + \frac{1}{4g}\omega_n(\omega_n + 2\omega_{n,2})\rho W \int_0^L S_0^2 \, dx, \\
b_1 \omega_{n,2} &= -\frac{1}{4}\rho W \int_0^L \int_0^{h_0} \left(S_{1x}^2 + S_{1y}^2 - 2l(\omega_n + \omega_{n,2}) S_{1x}\right) dy dx + \frac{1}{4g}\omega_n(\omega_n + 2\omega_{n,2})\rho W \int_0^L S_1^2 \, dx \\
&\quad - \frac{1}{4} l\left(g(m_v + m_f) - m_v l(\omega_n + \omega_{n,2})^2\right), \\
c_2 \omega_{n,2} &= -\frac{1}{4}\rho W \int_0^L \int_0^{h_0} \left(S_{0x}S_{1x} + S_{0y}S_{1y} - l(\omega_n + \omega_{n,2}) S_{0x}\right) dy dx + \frac{1}{4g}\omega_n(\omega_n + 2\omega_{n,2})\rho W \int_0^L S_0 S_1 \, dx,
\end{aligned}
$$

and the nonlinear terms are

$$
\begin{aligned}
a_2 = & -\frac{1}{2}\rho W \int_0^L \int_0^{h_0} \left(\zeta_{0x}^2 + \zeta_{0y}^2\right) \, dydx \\
& + \frac{1}{32g^3}\rho W \int_0^L \big(-4g\omega_n^2 S_0 S_{0y} S_{0x}^2 - 4g\omega_n^2 S_0^2 S_{0y} S_{0yy} - 4g\omega_n^2 S_0^2 S_{0x} S_{0xy} + 6g^2 S_{0x}^2 S_{0y}^2 \\
& + 3g^2 S_{0x}^4 + 3g^2 S_{0y}^4 - 4g\omega_n^2 S_0 S_{0y}^3 + 4\omega_n^4 S_0^3 S_{0yy} + 12\omega_n^4 S_0^2 S_{0y}^2 + 32g\omega_n^3 S_0 S_{0y} Z_3 \\
& + 64g^2\omega_n^2 Z_3^2 + 16g^2\omega_n S_{0x}^2 Z_3 + 16g^2\omega_n S_{0y}^2 Z_3 + 128g^2\omega_n^2 Z_3\zeta_0 + 32g\omega_n^3 S_0 S_{0y}\zeta_0 \\
& - 16g^2\omega_n S_0 S_{0y}\zeta_{0y} - 16g^2\omega_n S_0 S_{0x}\zeta_{0x} + 64g^2\omega_n^2\zeta_0^2 + 16g\omega_n^3 S_0^2\zeta_{0y} + 16g^2\omega_n S_{0x}^2\zeta_0 \\
& + 16g^2\omega_n S_{0y}^2\zeta_0 + 8g^4 C_0^2 - 16\omega_n^2 g^2 S_0 S_{0y} C_0 + 8g^3 S_{0x}^2 C_0 + 8g^3 S_{0y}^2 C_0 \ \big) \, dx, \\
b_2 = & -\frac{1}{8}\rho W \int_0^L \int_0^{h_0} \left(4\zeta_{2x}^2 + 4\zeta_{2y}^2 - 4l\omega_n\zeta_{2y} + 8Z_1\zeta_{2y} + l\omega_n S_{1x} + 4Z_1^2 - 4l\omega_n Z_1\right) \, dydx \\
& + \frac{1}{32g^3}\rho W \int_0^L \big(\ 6g^2 S_{1x}^2 S_{1y}^2 - 4g\omega_n^2 S_1^2 S_{1y} S_{1yy} - 4g\omega_n^2 S_1^2 S_{1x} S_{1xy} + 4\omega_n^4 S_1^3 S_{1yy} \\
& - 4g\omega_n^2 S_1 S_{1y}^3 + 3g^2 S_{1x}^4 + 3g^2 S_{1y}^4 - 4g\omega_n^2 S_1 S_{1y} S_{1x}^2 + 12\omega_n^4 S_1^2 S_{1y}^2 + 4g\omega_n^3 l S_1^2 S_{1xy} \\
& - 12g^2\omega_n l S_{1x}^3 + 8g\omega_n^3 l S_1 S_{1x} S_{1y} - 12g^2\omega_n l S_{1x} S_{1y}^2 + 12g^2\omega_n^2 l^2 S_{1x}^2 + 8g^2\omega_n^2 l S_1 S_{1y} \\
& + 64g^2\omega_n^2\zeta_2^2 + 16g\omega_n^3 S_1^2\zeta_{2y} + 16g^2\omega_n S_{1x}^2\zeta_2 + 16g^2\omega_n S_{1y}^2\zeta_2 - 16g^2\omega_n S_1 S_{1y}\zeta_{2y} \\
& - 16g^2\omega_n S_1 S_{1x}\zeta_{2x} + 32g\omega_n^3 S_1 S_{1y}\zeta_2 + 16g^2\omega_n^2 l S_1\zeta_{2x} - 32g^2\omega_n^2 l S_{1x}\zeta_2 + 128g^2\omega_n^2 Z_1 h_0\zeta_2 \\
& + 64g^2\omega_n^2 Z_2^2 - 64g^2\omega_n^2 Z_1^2 h_0^2 + 16g\omega_n^3 S_1^2 Z_1 + 16g^2\omega_n S_{1x}^2 Z_2 + 16g^2\omega_n S_{1y}^2 Z_2 \\
& + 128g^2\omega_n^2 Z_2\zeta_2 + 32g\omega_n^3 S_1 S_{1y} Z_1 h_0 - 32g^2\omega_n^2 l S_{1x} Z_1 h_0 + 32g\omega_n^3 S_1 S_{1y} Z_2 \\
& - 32g^2\omega_n^2 l S_{1x} Z_2 + 16g^2\omega_n Z_1 h_0 S_{1x}^2 + 16g^2\omega_n Z_1 h_0 S_{1y}^2 - 16g^2\omega_n S_1 S_{1y} Z_1 \\
& + 128g^2\omega_n^2 Z_1^2 h_0^2 + 128g^2\omega_n^2 Z_1 h_0 Z_2 - 16\omega_n^2 g^2 S_1 S_{1y} C_3 - 16g^3\omega_n l S_{1x} C_3 \\
& + 8g^3 S_{1x}^2 C_3 + 8g^3 S_{1y}^2 C_3 + 8g^4 C_3^2 \ \big) \, dx + \frac{1}{32} lg(m_v + m_f), \\
a_3 = & -\frac{1}{4}\rho W \int_0^L \int_0^{h_0} \left(\zeta_{1x}^2 + \zeta_{1y}^2 + 4l\omega_n\Theta_1\zeta_{1x} + l\omega_n\Theta_1 S_{0y}\right) \, dydx \\
& + \frac{1}{16g^3}\rho W \int_0^L \big(\ 3g^2 S_{1x}^2 S_{0x}^2 + g^2 S_{1x}^2 S_{0y}^2 + g^2 S_{1y}^2 S_{0x}^2 + 3g^2 S_{1y}^2 S_{0y}^2 + 4g^2 S_{0x} S_{1x} S_{0y} S_{1y} \\
& - 2g\omega_n^2 S_{0x}^2 S_1 S_{1y} - 2g\omega_n^2 S_{0y}^2 S_1 S_{1y} - 2g\omega_n^2 S_0 S_{1y}^2 S_{0y} - 2g\omega_n^2 S_0 S_{0y} S_{1x}^2 + 2\omega_n^4 S_0 S_1^2 S_{0yy} \\
& + 2\omega_n^4 S_1 S_0^2 S_{1yy} - 2g\omega_n^2 S_1^2 S_{0y} S_{0yy} - 2g\omega_n^2 S_0^2 S_{1y} S_{1yy} - 2g\omega_n^2 S_1^2 S_{0x} S_{0xy} + 8\omega_n^4 S_0 S_{0y} S_1 S_{1y} \\
& + 2\omega_n^4 S_0^2 S_{1y}^2 + 2\omega_n^4 S_{0y}^2 S_1^2 - 2g\omega_n^2 S_0^2 S_{1x} S_{1xy} - 6g^2\omega_n l S_{1x} S_{0x}^2 - 2g^2\omega_n l S_{1x} S_{0y}^2 + 4g\omega_n^3 l S_0 S_{0y} S_{1x} \\
& - 4g^2\omega_n l S_{0x} S_{1y} S_{0y} + 2g\omega_n^3 l S_0^2 S_{1xy} + 2g^2\omega_n^2 l^2 S_{0x}^2 + 8g\omega_n^3 S_0 S_{1y}\zeta_1 + 8g\omega_n^3 S_0 S_1\zeta_{1y} \\
& + 8g^2\omega_n S_{0x} S_{1x}\zeta_1 + 8g^2\omega_n S_{0y} S_{1y}\zeta_1 + 8g\omega_n^3 S_1 S_{0y}\zeta_1 - 8g^2\omega_n^2 l S_{0x}\zeta_1 - 4g^2\omega_n S_1 S_{0y}\zeta_{1y} \\
& - 4g^2\omega_n S_1 S_{0x}\zeta_{1x} - 4g^2\omega_n S_0 S_{1y}\zeta_{1y} - 4g^2\omega_n S_0 S_{1x}\zeta_{1x} + 4g^2\omega_n^2 l S_0\zeta_{1x} + 16g^2\omega_n^2\zeta_1^2 \\
& - 8g^2\omega_n^2 l\Theta_1 S_1 S_{0x} - 8g^2\omega_n^2 l\Theta_1 S_0 S_{1x} + 2g^3 S_{0x} S_{1x} C_1 + 2g^3 S_{0y} S_{1y} C_1 + 4g^4 C_0 C_3 + 4g^4 C_1 C_2 \\
& + 2g^3 S_{0x}^2 C_3 + 2g^3 S_{0y}^2 C_3 + 2g^3 S_{1x}^2 C_0 + 2g^3 S_{1y}^2 C_0 + 2g^3 S_{0x} S_{1x} C_2 + 2g^3 S_{0y} S_{1y} C_2 \\
& - 2g^2\omega_n^2 S_{0y} S_1 C_2 - 2g^2\omega_n^2 S_0 S_{1y} C_2 - 2g^3\omega_n l S_{0x} C_2 - 4g^2\omega_n^2 S_1 S_{1y} C_0 - 4g^2\omega_n^2 S_0 S_{0y} C_3 \\
& - 4g^3\omega_n l S_{1x} C_0 - 2g^3\omega_n l S_{0x} C_1 - 2g^2\omega_n^2 S_0 S_{1y} C_1 - 2g^2\omega_n^2 S_{0y} S_1 C_1 \ \big) \, dx \\
& - \frac{1}{4} l\Theta_1^2 \left(g(m_v + m_f) - 4\omega_n^2 m_v l\right),
\end{aligned}
$$

$$
\begin{aligned}
a_4 = &-\frac{1}{4}\rho W \int_0^L \int_0^{h_0} \left(2\zeta_{0y}\zeta_{2y} + 2\zeta_{0x}\zeta_{2x} - l\omega_n\zeta_{0y} + 2Z_1\zeta_{0y}\right)\, dydx \\
&+\frac{1}{32g^3}\rho W \int_0^L \Big(\; 3g^2S_{1x}^2S_{0x}^2 + g^2S_{1x}^2S_{0y}^2 + g^2S_{1y}^2S_{0x}^2 + 3g^2S_{1y}^2S_{0y}^2 + 4g^2S_{0x}S_{0y}S_{1x}S_{1y} \\
&+2g\omega_n^2S_1S_{1y}S_{0x}^2 - 2g\omega_n^2S_{0y}^2S_1S_{1y} - 2g\omega_n^2S_0S_{0y}S_{1y}^2 + 2g\omega_n^2S_0S_{0y}S_{1x}^2 - 4g\omega_n^2S_0S_{0yy}S_1S_{1y} \\
&-4g\omega_n^2S_0S_{0y}S_1S_{1yy} - 4g\omega_n^2S_0S_{0x}S_1S_{1xy} - 4g\omega_n^2S_0S_{0xy}S_1S_{1x} - 4g\omega_n^2S_0S_{0x}S_{1x}S_{1y} \\
&-4g\omega_n^2S_{0x}S_{0y}S_1S_{1x} + 2\omega_n^4S_0S_{0yy}S_1^2 + 2\omega_n^4S_0^2S_1S_{1yy} + 2g\omega_n^2S_{0y}S_{0yy}S_1^2 + 2g\omega_n^2S_0^2S_{1y}S_{1yy} \\
&+2g\omega_n^2S_{0x}S_{0xy}S_1^2 + 2g\omega_n^2S_0^2S_{1x}S_{1xy} + 8\omega_n^4S_0S_{0y}S_1S_{1y} + 2\omega_n^4S_0^2S_{1y}^2 + 2\omega_n^4S_{0y}^2S_1^2 \\
&+2g^2\omega_n^2l^2S_{0x}^2 + 4g^2\omega_n^2lS_0S_{0y} - 6g^2\omega_nlS_{0x}^2S_{1x} - 2g^2\omega_nlS_{0y}^2S_{1x} + 4g\omega_n^3lS_0S_{0xy}S_1 \\
&+4g\omega_n^3lS_0S_{0x}S_{1y} - 4g\omega_n^3lS_0S_{0y}S_{1x} + 4g\omega_n^3lS_{0x}S_{0y}S_1 - 4g^2\omega_nlS_{0x}S_{0y}S_{1y} - 2g\omega_n^3lS_0^2S_{1xy} \\
&+8g^2\omega_nS_{1x}^2\zeta_0 + 8g^2\omega_nS_{1y}^2\zeta_0 + 8g^2\omega_nS_{0x}^2\zeta_2 + 8g^2\omega_nS_{0y}^2\zeta_2 + 64g^2\omega_n^2\zeta_0\zeta_2 + 8g\omega_n^3S_1^2\zeta_{0y} \\
&+8g\omega_n^3S_0^2\zeta_{2y} + 16g\omega_n^3S_1S_{1y}\zeta_0 + 16g\omega_n^3S_0S_{0y}\zeta_2 - 8g^2\omega_nS_1S_{1x}\zeta_{0x} - 8g^2\omega_nS_0S_{0x}\zeta_{2x} \\
&-8g^2\omega_nS_0S_{0y}\zeta_{2y} - 8g^2\omega_nS_1S_{1y}\zeta_{0y} + 8g^2\omega_n^2lS_1\zeta_{0x} - 16g^2\omega_n^2lS_{1x}\zeta_0 + 8g^2\omega_nS_{1x}^2Z_3 \\
&+8g^2\omega_nS_{1y}^2Z_3 + 64g^2\omega_n^2Z_3\zeta_2 + 8g^2\omega_nZ_2S_{0x}^2 + 8g^2\omega_nZ_2S_{0y}^2 + 64g^2\omega_n^2Z_2\zeta_0 + 8g\omega_n^3Z_1S_0^2 \\
&+64g^2\omega_n^2Z_1Z_3h_0 + 16g\omega_n^3S_1S_{1y}Z_3 + 16g\omega_n^3Z_2S_0S_{0y} - 8g^2\omega_nZ_1S_0S_{0y} - 16g^2\omega_n^2lZ_3S_{1x} \\
&+8g^2\omega_nZ_1h_0S_{0x}^2 + 8g^2\omega_nZ_1h_0S_{0y}^2 + 64g^2\omega_n^2Z_1h_0\zeta_0 + 64g^2\omega_n^2Z_2Z_3 + 16g\omega_n^3Z_1h_0S_0S_{0y} \\
&+8g^3S_{0x}S_{1x}C_1 + 8g^3S_{0y}S_{1y}C_1 + 8g^4C_1^2 - 8g^3\omega_nlS_{0x}C_1 - 8g^2\omega_n^2S_0S_{1y}C_1 - 8g^2\omega_n^2S_{0y}S_1C_1 \;\Big)\, dx, \\
c_3 = &-\frac{1}{4}\rho W \int_0^L \int_0^{h_0} \left(\zeta_{0x}\zeta_{1x} + \zeta_{0y}\zeta_{1y} + 2l\omega_n\Theta_1\zeta_{0x}\right)\, dydx \\
&+\frac{1}{32g^3}\rho W \int_0^L \Big(\; -g\omega_n^2S_{0x}^2S_{0y}S_1 - g\omega_n^2S_0^2S_{0xy}S_{1x} - g\omega_n^2S_0^2S_{0x}S_{1xy} - g\omega_n^2S_0^2S_{0y}S_{1yy} \\
&-g\omega_n^2S_0^2S_{0yy}S_{1y} + 3\omega_n^4S_0^2S_{0yy}S_1 - 2g\omega_n^2S_0S_{0y}S_{0yy}S_1 - 2g\omega_n^2S_0S_{0x}S_{0xy}S_1 \\
&+\omega_n^4S_0^3S_{1yy} - 3g\omega_n^2S_0S_{0y}^2S_{1y} - g\omega_n^2S_0S_{0x}^2S_{1y} + 3g^2S_{0x}^2S_{0y}S_{1y} + 6\omega_n^4S_0^2S_{0y}S_{1y} + 6\omega_n^4S_0S_{0y}^2S_1 \\
&+3g^2S_{0y}^2S_{0x}S_{1x} - g\omega_n^2S_{0y}^3S_1 - 2g\omega_n^2S_0S_{0y}S_{0x}S_{1x} + 3g^2S_{0x}^3S_{1x} + 3g^2S_{0y}^3S_{1y} \\
&-3g^2\omega_nlS_{0x}S_{0y}^2 + g\omega_n^3lS_0^2S_{0xy} - 3g^2\omega_nlS_{0x}^3 + 2g\omega_n^3lS_0S_{0y}S_{0x} - 8g^2\omega_n^2l\Theta_1S_0S_{0x} + 8g\omega_n^3S_0S_{0y}\zeta_1 \\
&+8g\omega_n^3S_0S_{1y}\zeta_0 + 8g\omega_n^3S_1S_{0y}\zeta_0 - 8g^2\omega_n^2lS_{0x}\zeta_0 - 4g^2\omega_nS_0S_{0x}\zeta_{1x} - 4g^2\omega_nS_{0y}S_1\zeta_{0y} \\
&-4g^2\omega_nS_0S_{1y}\zeta_{0y} - 4g^2\omega_nS_{0x}S_1\zeta_{0x} - 4g^2\omega_nS_0S_{0y}\zeta_{1y} + 4g^2\omega_n^2lS_0\zeta_{0x} - 4g^2\omega_nS_0S_{1x}\zeta_{0x} \\
&+8g\omega_n^3S_0S_1\zeta_{0y} + 8g^2\omega_nS_{0x}S_{1x}\zeta_0 + 8g^2\omega_nS_{0y}S_{1y}\zeta_0 + 4g\omega_n^3S_0^2\zeta_{1y} + 4g^2\omega_nS_{0x}^2\zeta_1 \\
&+4g^2\omega_nS_{0y}^2\zeta_1 + 32g^2\omega_n^2\zeta_0\zeta_1 + 8gZ_3\omega_n^3S_0S_{1y} + 8g\omega_n^3Z_3S_{0y}S_1 + 8g^2\omega_nZ_3S_{0x}S_{1x} + 8g^2\omega_nZ_3S_{0y}S_{1y} \\
&-8g^2\omega_n^2lZ_3S_{0x} + 32g^2\omega_n^2Z_3\zeta_1 - 4g^3\omega_nlS_{0x}C_0 - 4g^2\omega_n^2S_0S_{1y}C_0 - 4g^2\omega_n^2S_{0y}S_1C_0 + 4g^3S_{0y}^2C_1 \\
&+8g^4C_0C_1 + 4g^3S_{0x}^2C_1 + 4g^3S_{0x}S_{1x}C_0 + 4g^3S_{0y}S_{1x}C_0 - 8g^2\omega_n^2S_0S_{0y}C_1 \;\Big)\, dx, \\
c_4 = &-\frac{1}{32}\rho W \int_0^L \int_0^{h_0} \left(8\zeta_{1x}\zeta_{2x} + 8\zeta_{1y}\zeta_{2y} - 4l\omega_n\zeta_{1y} + 16l\omega_n\Theta_1\zeta_{2x} + l\omega_nS_{0x} + 4l\omega_n\Theta_1S_{1y} + 8Z_1\zeta_{1y}\right)\, dydx \\
&+\frac{1}{32g^3}\rho W \int_0^L \Big(\; -3g\omega_n^2S_{0y}S_1S_{1y}^2 - g\omega_n^2S_0S_{1y}S_{1x}^2 - g\omega_n^2S_{0y}S_1S_{1x}^2 + 3g^2S_{0x}S_{1x}^3 + 3g^2S_{0y}S_{1y}^3 \\
&-g\omega_n^2S_{0x}S_1^2S_{1xy} - g\omega_n^2S_{0xy}S_1^2S_{1x} - g\omega_n^2S_{0y}S_1^2S_{1yy} + 6\omega_n^4S_0S_{1y}^2S_1 + 6\omega_n^4S_{0y}S_{1y}S_1^2 \\
&-g\omega_n^2S_{0yy}S_1^2S_{1y} + 3g^2S_{0y}S_{1y}S_{1x}^2 + 3g^2S_{0x}S_{1x}S_{1y}^2 - g\omega_n^2S_0S_{1y}^3 - 2g\omega_n^2S_{0x}S_1S_{1y}S_{1x} \\
&+3\omega_n^4S_0S_1^2S_{1yy} - 2g\omega_n^2S_0S_1S_{1y}S_{1yy} - 2g\omega_n^2S_0S_1S_{1x}S_{1xy} + \omega_n^4S_{0yy}S_1^3 - 9g^2\omega_nlS_{0x}S_{1x}^2 \\
&-3g^2\omega_nlS_{0x}S_{1y}^2 + g\omega_n^3lS_{0xy}S_1^2 + 6g^2\omega_n^2l^2S_{0x}S_{1x} + 2g\omega_n^3lS_{0x}S_1S_{1y} + 2g\omega_n^3lS_0S_{1x}S_{1y} \\
&+2g\omega_n^3lS_{0y}S_1S_{1x} - 6g^2\omega_nlS_{0y}S_{1x}S_{1y} - 8g^2\omega_n^2l\Theta_1S_1S_{1x} + 2g^2\omega_n^2lS_0S_{1y} + 2g^2\omega_n^2lS_{0y}S_1 \\
&+2g\omega_n^3lS_0S_1S_{1xy} + 8g\omega_n^3S_1S_{1y}\zeta_1 + 8g\omega_n^3S_0S_{1y}\zeta_2 + 8g\omega_n^3S_{0y}S_1\zeta_2 + 8g\omega_n^3S_0S_1\zeta_{2y} + 8g^2\omega_nS_{0x}S_{1x}\zeta_2 \\
&+8g^2\omega_nS_{0y}S_{1y}\zeta_2 + 4g\omega_n^3S_1^2\zeta_{1y} + 4g^2\omega_nS_{1x}^2\zeta_1 + 4g^2\omega_nS_{1y}^2\zeta_1 + 32g^2\omega_n^2\zeta_1\zeta_2 - 8g^2\omega_n^2lS_{1x}\zeta_1 \\
&-8g^2\omega_n^2lS_{0x}\zeta_2 - 4g^2\omega_nS_1S_{1y}\zeta_{1y} - 4g^2\omega_nS_0S_{1x}\zeta_{2x} - 4g^2\omega_nS_0S_{1y}\zeta_{2y} + 4g^2\omega_n^2lS_1\zeta_{1x} \\
&+4g^2\omega_n^2lS_0\zeta_{2x} - 4g^2\omega_nS_{0y}S_1\zeta_{2y} - 4g^2\omega_nS_{0x}S_1\zeta_{2x} - 4g^2\omega_nS_1S_{1x}\zeta_{1x} + 8g\omega_n^3Z_2S_0S_{1y} \\
&+8g\omega_n^3Z_2S_{0y}S_1 + 8g\omega_n^3Z_1S_0S_1 + 8g^2\omega_nZ_2S_{0x}S_{1x} + 8g^2\omega_nZ_2S_{0y}S_{1y} + 8g\omega_n^3Z_1h_0S_0S_{1y} \\
&+8g\omega_n^3Z_1h_0S_{0y}S_1 - 8g^2\omega_n^2lZ_1h_0S_{0x} + 8g^2\omega_nZ_1h_0S_{0x}S_{1x} + 8g^2\omega_nZ_1h_0S_{0y}S_{1y} \\
&-8g^2\omega_n^2lZ_2S_{0x} + 32g^2\omega_n^2Z_1h_0\zeta_1 + 32g^2\omega_n^2Z_2\zeta_1 - 4g^2\omega_nZ_1S_{0y}S_1 - 4g^2\omega_nZ_1S_0S_{1y} - 4g^2\omega_n^2S_0S_{1y}C_3 \\
&-4g^2\omega_n^2S_{0y}S_1C_3 - 4g^3\omega_nlS_{0x}C_3 - 8g^3\omega_nlS_{1x}C_2 - 8g^2\omega_n^2S_1S_{1y}C_2 + 4g^3S_{0x}S_{1x}C_3 \\
&+4g^3S_{0y}S_{1y}C_3 + 8g^4C_2C_3 + 4g^3S_{1x}^2C_2 + 4g^3S_{1y}^2C_2 \;\Big)\, dx.
\end{aligned}
$$

The above coefficients can be simplified by noting that

$$\rho W \int_0^L \int_0^{h_0} S_{0x} \, dydx = 0, \tag{3.7}$$

$$\rho W \int_0^L \int_0^{h_0} S_{1x} \, dydx = \frac{g(m_v + m_f)}{\omega_n} - m_v l \omega_n, \tag{3.8}$$

$$\rho W \int_0^L \int_0^{h_0} \left(S_{0x}^2 + S_{0y}^2\right) \, dydx - \frac{\omega_n^2 \rho W}{g} \int_0^L S_0^2 \, dx = 0, \tag{3.9}$$

$$\rho W \int_0^L \int_0^{h_0} \left(S_{1x}^2 + S_{1y}^2\right) \, dydx - \frac{\omega_n^2 \rho W}{g} \int_0^L S_1^2 \, dx = l\omega_n \left(\frac{g(m_v + m_f)}{\omega_n} - m_v l \omega_n \right), \tag{3.10}$$

and also that $S_{0y} = \omega_n^2/gS_0$ and $S_{1y} = \omega_n^2/gS_1$ on $y = h_0$. The forms of the integrands of $c_0,\ c_1,\ c_2,\ c_3$ and c_4 are all anti-symmetric about the centre-line of the vessel at $x = L/2$ and therefore the integral of these terms is 0. The constants C_1 and C_2 are also both 0. Thus the coefficients of the Lagrangian simplify to

$$\begin{aligned}
a_0 &= \frac{1}{2} \frac{\omega_n^2}{g} \rho W \int_0^L S_0^2 \, dx, \\
b_0 &= \frac{1}{2} \frac{\omega_n^2}{g} \rho W \int_0^L S_1^2 \, dx + \frac{1}{4} lg(m_v + m_f), \\
a_1 &= \frac{1}{2g} \omega_n \rho W \int_0^L S_0^2 \, dx, \\
b_1 &= \frac{1}{2g} \omega_n \rho W \int_0^L S_1^2 \, dx + \frac{lg}{2\omega_n}(m_v + m_f),
\end{aligned}$$

$$\begin{aligned}
a_2 = {} & -\frac{\rho W}{2} \int_0^L \int_0^{h_0} \left(\zeta_{0x}^2 + \zeta_{0y}^2\right) \, dydx + \frac{\rho W}{32 g^3} \int_0^L \Big(-2\omega_n^4 S_0^2 S_{0x}^2 + 3g^2 S_{0x}^4 + 11 \frac{\omega_n^8}{g^2} S_0^4 \\
& + 48\omega_n^5 S_0^2 Z_3 + 64 g^2 \omega_n^2 Z_3^2 + 16 g^2 \omega_n S_{0x}^2 Z_3 + 128 g^2 \omega_n^2 Z_3 \zeta_0 + 48 \omega_n^5 S_0^2 \zeta_0 \\
& - 16 g^2 \omega_n S_0 S_{0x} \zeta_{0x} + 64 g^2 \omega_n^2 \zeta_0^2 + 16 g^2 \omega_n S_{0x}^2 \zeta_0 - 8 g^2 C_0^2 \ \Big) \, dx,
\end{aligned}$$

$$\begin{aligned}
b_2 = {} & -\frac{\rho W}{8} \int_0^L \int_0^{h_0} \left(4\zeta_{2x}^2 + 4\zeta_{2y}^2 - 4l\omega_n \zeta_{2y} + 8 Z_1 \zeta_{2y} + l\omega_n S_{1x} + 4 Z_1^2 - 4 l \omega_n Z_1\right) \, dydx \\
& + \frac{\rho W}{32 g^3} \int_0^L \Big(-2\omega_n^4 S_1^2 S_{1x}^2 + 11 \frac{\omega_n^8}{g^2} S_1^4 + 3 g^2 S_{1x}^4 - 12 g^2 \omega_n l S_{1x}^3 + 12 g^2 \omega_n^2 l^2 S_{1x}^2 + 8 g \omega_n^4 l S_1^2 \\
& + 64 g^2 \omega_n^2 \zeta_2^2 + 16 g^2 \omega_n S_{1x}^2 \zeta_2 + 48 \omega_n^5 S_1^2 \zeta_2 - 16 g^2 \omega_n S_1 S_{1x} \zeta_{2x} + 16 g^2 \omega_n^2 l S_1 \zeta_{2x} \\
& - 32 g^2 \omega_n^2 l S_{1x} \zeta_2 + 128 g^2 \omega_n^2 Z_1 h_0 \zeta_2 + 64 g^2 \omega_n^2 Z_2^2 - 64 g^2 \omega_n^2 Z_1^2 h_0^2 + 16 g^2 \omega_n S_{1x}^2 Z_2 \\
& + 48 \omega_n^5 S_1^2 Z_2 + 128 g^2 \omega_n^2 Z_2 \zeta_2 + 48 \omega_n^5 Z_1 h_0 S_1^2 - 32 g^2 \omega_n^2 l S_{1x} Z_1 h_0 - 32 g^2 \omega_n^2 l S_{1x} Z_2 \\
& + 16 g^2 \omega_n Z_1 h_0 S_{1x}^2 + 128 g^2 \omega_n^2 Z_1^2 h_0^2 + 128 g^2 \omega_n^2 Z_1 h_0 Z_2 - 8 g^4 C_3^2 \ \Big) \, dx + \frac{lg}{64}(m_v + m_f),
\end{aligned}$$

h_0	a_0	a_1	a_2	a_3	a_4	b_0	b_1	b_2
0.01	24.323	6.504	−346965.582	−7607.023	85845.591	15.400	4.118	−20410.023
0.02	47.965	9.134	−84160.573	−10554.333	27934.742	26.249	4.999	−7799.324
0.03	70.321	11.059	−35408.536	−13199.384	17437.132	42.719	6.718	−5590.563
0.04	90.915	12.575	−18248.991	−14766.293	14314.677	65.599	9.073	−4835.622
0.05	109.435	13.796	−10208.916	−15441.721	13846.522	96.411	12.154	−4417.360
0.06	125.732	14.788	−5751.113	−14884.675	14908.399	136.935	16.106	−3885.173
0.07	139.802	15.593	−2986.405	−24117.712	17151.009	188.837	21.063	−2817.218
0.08	151.750	16.246	−1132.049	−21628.028	20415.617	253.051	27.091	−76.818
0.09	161.755	16.773	181.996	−23028.441	24522.598	328.864	34.102	2951.772
0.1	170.035	17.197	1149.094	−24730.839	29157.071	412.908	41.761	8777.781
0.125	184.686	17.923	2670.280	−34910.791	39465.003	609.095	59.109	29915.277
0.15	193.239	18.333	3479.054	−29949.511	41170.085	676.320	64.164	41503.246
0.175	198.103	18.562	3920.644	−23030.135	33998.371	592.523	55.520	32728.922
0.2	200.827	18.690	4163.392	−15867.775	24599.383	454.975	42.341	19134.032
0.225	202.340	18.760	4296.987	−10608.343	17086.217	335.303	31.087	10187.484
0.25	203.177	18.799	4370.497	−7218.494	11999.003	249.554	23.089	5558.420
0.3	203.893	18.832	4433.167	−3758.248	6563.936	152.517	14.086	2173.503
0.35	204.110	18.842	4452.114	−2293.214	4153.070	107.002	9.877	1389.215
0.4	204.175	18.845	4457.840	−1584.454	2947.428	83.631	7.719	1488.541

TABLE 2. A selection of coefficient values for the normal form equations.

$$
\begin{aligned}
a_3 = & -\frac{\rho W}{4}\int_0^L\int_0^{h_0}\left(\zeta_{1x}^2+\zeta_{1y}^2+4l\omega_n\Theta_1\zeta_{1x}\right)\,dydx \\
& +\frac{\rho W}{16g^3}\int_0^L \Big(\, 3g^2S_{1x}^2S_{0x}^2-3\omega_n^4S_0^2S_{1x}^2-3\omega_n^4S_{0x}^2S_1^2+11\frac{\omega_n^8}{g^2}S_0^2S_1^2+4\omega_n^4S_0S_{0x}S_1S_{1x} \\
& -6g^2\omega_n lS_{0x}^2S_{1x}+4\omega_n^5lS_0^2S_{1x}-4\omega_n^5lS_0S_{0x}S_1+2g^2\omega_n^2l^2S_{0x}^2+24\omega_n^5S_0S_1\zeta_1+8g^2\omega_nS_{0x}S_{1x}\zeta_1 \\
& -8g^2\omega_n^2lS_{0x}\zeta_1-4g^2\omega_nS_{0x}S_1\zeta_{1x}-4g^2\omega_nS_0S_{1x}\zeta_{1x}+4g^2\omega_n^2lS_0\zeta_{1x}+16g^2\omega_n^2\zeta_1^2 \\
& -8g^2\omega_n^2l\Theta_1S_1S_{0x}-8g^2\omega_n^2l\Theta_1S_0S_{1x}-4g^4C_0C_3\,\Big)\,dx+\frac{l\Theta_1^2}{4}\left(g(m_v+m_f)-4\omega_n^2m_vl\right),
\end{aligned}
$$

$$
\begin{aligned}
a_4 = & -\frac{\rho W}{2}\int_0^L\int_0^{h_0}\left(\zeta_{0y}\zeta_{2y}+\zeta_{0x}\zeta_{2x}\right)\,dydx \\
& +\frac{\rho W}{32g^3}\int_0^L \Big(\, 3g^2S_{0x}^2S_{1x}^2+5\omega_n^4S_0^2S_{1x}^2+5\omega_n^4S_{0x}^2S_1^2+11\frac{\omega_n^8}{g^2}S_0^2S_1^2-12\omega_n^4S_0S_{0x}S_1S_{1x} \\
& +2g^2\omega_n^2l^2S_{0x}^2+4g\omega_n^4lS_0^2-6g^2\omega_nlS_{0x}^2S_{1x}-8\omega_n^5lS_0^2S_{1x}+8\omega_n^5lS_0S_{0x}S_1+8g^2\omega_nS_{1x}^2\zeta_0 \\
& +48\omega_n^5S_1^2\zeta_0+8g^2\omega_nS_{0x}^2\zeta_2+48\omega_n^5S_0^2\zeta_2+64g^2\omega_n^2\zeta_0\zeta_2-8g^2\omega_nS_1S_{1x}\zeta_{0x}-8g^2\omega_nS_0S_{0x}\zeta_{2x} \\
& +8g^2\omega_n^2lS_1\zeta_{0x}-16g^2\omega_n^2lS_{1x}\zeta_0+8g^2\omega_nS_{1x}^2Z_3+48\omega_n^5S_1^2Z_3+64g^2\omega_n^2Z_3\zeta_2+8g^2\omega_nZ_2S_{0x}^2 \\
& +48\omega_n^5Z_2S_0^2+64g^2\omega_n^2Z_2\zeta_0+64g^2\omega_n^2Z_1Z_3h_0-16g^2\omega_n^2lZ_3S_{1x}+8g^2\omega_nZ_1h_0S_{0x}^2+48\omega_n^5Z_1h_0S_0^2 \\
& +64g^2\omega_n^2Z_1h_0\zeta_0+64g^2\omega_n^2Z_2Z_3\,\Big)\,dx.
\end{aligned}
$$

By taking variations of the Lagrangian with respect to A^* and B^* respectively gives the normal form equations

$$
\begin{aligned}
\mathrm{i}a_0A_\tau &= a_1\omega_{n,2}A+a_2|A|^2A+a_3|B|^2A+a_4B^2A^*, \\
\mathrm{i}b_0B_\tau &= b_1\omega_{n,2}B+b_2|B|^2B+a_3|A|^2B+a_4A^2B^*.
\end{aligned}
$$

These equations are analysed fully in the main paper, but a selection of these parameter values are given as a function of h_0 in table 2.

Appendix A. Solvability condition at $O(\epsilon^2)$

The system of equations (1.6) have terms proportional to both $\cos(2t)$, $\sin(2t)$ and $\cos(t)$, $\sin(t)$. The terms proportional to the first harmonic should not appear in the problem, and we can choose $\omega_{n,1}$ in such a way as to remove these terms. This is achieved via a solvability condition at $O(\epsilon^2)$. Assume that the solutions for $\phi_2(x,t)$ and $\theta_2(t)$ can be expressed as

$$\begin{aligned}\phi_2(x,y,t) &= \hat{\phi}_2(x,y)\sin(2t) + \hat{\phi}_{2,1}(x,y)\cos(t),\\ \theta_2(t) &= \hat{\theta}_2\cos(2t) + \hat{\theta}_{2,1}\sin(t),\end{aligned}$$

then $\hat{\phi}_{2,1}$ and $\hat{\theta}_{2,1}$ satisfy the system of equations

$$\begin{aligned}\hat{\phi}_{2,1xx} + \hat{\phi}_{2,1yy} &= A(x,y) \quad \text{in} \quad 0<y<h_0 \quad 0<x<L,\\ \hat{\phi}_{2,1y} - \frac{\omega_n^2}{g}\hat{\phi}_{2,1} &= b(x) \quad \text{on} \quad y=h_0,\\ \hat{\phi}_{2,1y} &= a(x) \quad \text{on} \quad y=0,\\ \hat{\phi}^{(1)}_{2,1x} - l\omega_n\hat{\theta}_{2,1} &= c(y) \quad \text{on} \quad x=0,L,\\ -\frac{1}{\rho}\left(\frac{g(m_v+m_f)}{\omega_n} - lm_v\omega_n\right)\hat{\theta}_{2,1} + W\int_0^L\int_0^{h_0}\hat{\phi}^{(1)}_{2,1x}\,dydx &= d,\end{aligned}$$

where A, a, b, c, d are the terms on the RHS of the system (1.6) which are proportional to the first harmonic solution. Therefore, in this case

$$\begin{aligned}A(x,y) &= 0,\\ a(x) &= 0,\\ b(x) &= 2\omega_n\omega_{n,1}\left(AS_0(x,h_0) + BS_1(x,h_0)\right),\\ c(y) &= l\omega_{n,1}B,\\ d &= -\frac{\omega_{n,1}\rho W}{\omega_n}\int_0^L\int_0^{h_0}\left(AS_{0x} + BS_{1x}\right), = -\frac{\omega_{n,1}B}{\omega_n}\left(\frac{g(m_f+m_v)}{\omega_n} - m_vl\omega_n\right).\end{aligned}$$

These expressions are then substituted into the solvability condition

$$\begin{aligned}\rho W\int_0^L\int_0^{h_0} S(x,y)A(x,y)\,dydx + l\omega_n\theta d = \rho W\int_0^{h_0}(S(L,y)-S(0,y))c(y)\,dy\\ +\rho W\int_0^L S(x,h_0)b(x)\,dx - \rho W\int_0^L S(x,0)a(x)\,dx,\end{aligned}$$

where the two sets of (S,θ) come from the homogeneous solutions and are $(S_0(x,y),0)$ and $(S_1(x,y),1)$.

Therefore, the two resulting equations after applying this solvability condition are

$$\begin{aligned}0 &= 2\omega_n\omega_{n,1}A\int_0^L S_0^2\,dx = \omega_n\omega_{n,1}A,\\ 0 &= 2\omega_{n,1}B\left[l\left(\frac{g(m_f+m_v)}{\omega_n} - m_vl\omega_n\right) + \omega_n\int_0^L S_1^2\,dx\right],\end{aligned}$$

with the only nontrivial solution being $\omega_{n,1} = 0$.

Appendix B. Equivalence with study of Tadjbakhsh & Keller (1960)

The forms of $\phi_2(x,y)$ and $h_2(x,t)$ in (3.3) and (3.4) can be shown to be equivalent to equations (30) and (31) of Tadjbakhsh & Keller (1960) for the first symmetric sloshing mode, i.e. when $B=0$ and $n=1$.

In Tadjbakhsh & Keller (1960) all quantities are non-dimensionalised by the length scale β_1^{-1} and by the timescale $\sqrt{\beta_1 g}$. The linear solutions of Tadjbakhsh & Keller (1960) agree with those in the current paper (albeit by moving our free surface from $y=h(x,t)$ to $y=h(x,t)-h_0$) if we choose

$$A = \frac{g}{\beta_1\omega_1}.$$

This choice of A is important, as now both definitions of ϵ are equivalent. Also, the non-dimensionalisation of the frequencies gives the expressions

$$\omega_0^{\rm TK} = \frac{\omega_1}{\sqrt{\beta_1 g}} \quad \text{and} \quad \omega_2^{\rm TK} = \frac{2\omega_{1,2}}{\sqrt{\beta_1 g}},$$

where the superscript TK denotes the non-dimensional variable in Tadjbakhsh & Keller (1960).

Now,

$$\phi_2 = \frac{gC_0}{2\omega_1}|A|^2 t + A^2 Z_3 \sin 2t + A^2 \zeta_0(x,y)\sin 2t,$$

by absorbing the Bernoulli constant into the velocity potential. Thus,

$$\phi_2 = -\frac{g^2}{8\omega_1^3}(1-T_1^2)t - \frac{g^2}{16\omega_1^3}(1+3T_1^2)\sin 2t - \frac{3g(T_1 - T_1^{-3})}{16\beta_1\omega_1}\frac{\cosh 2\beta_1 y}{\cosh 2\beta_1 h_0}\cos 2\beta_1 x \sin 2t,$$

using (5.4) of the main paper, (2.8) and by noting that (2.3) reduces to

$$\zeta_0 = -\frac{3g(T_1 - T_1^{-3})}{16\beta_1\omega_1}\frac{\cosh 2\beta_1 y}{\cosh 2\beta_1 h_0}\cos 2\beta_1 x.$$

Here $T_1 = \tanh \beta_1 h_0 = \omega_0^{\rm TK}$, which when substituted into the above expression gives

$$\begin{aligned}\phi_2 = \frac{1}{\beta_1}\sqrt{\frac{g}{\beta_1}}\Bigg[&\frac{1}{8}\left(\omega_0^{\rm TK} - (\omega_0^{\rm TK})^{-3}\right)t - \frac{1}{16}\left((\omega_0^{\rm TK})^{-3} + 3\omega_0^{\rm TK}\right)\sin 2t \\ &\frac{3}{16}\left(\omega_0^{\rm TK} - (\omega_0^{\rm TK})^{-7}\right)\frac{\cosh 2\beta_1 y}{\cosh 2\beta_1 h_0}\cos 2\beta_1 x \sin 2t\Bigg].\end{aligned}$$

When this expression is non-dimensionalised it then agrees with (31) of Tadjbakhsh & Keller (1960).

Similarly it can be shown by simplifying $\lambda_0(x)$ and $\Lambda_0(x)$ in §3 that

$$h_2(x,t) = \frac{1}{8\beta_1}\left[(\omega_0^{\rm TK})^2 - (\omega_0^{\rm TK})^{-2} + \left((\omega_0^{\rm TK})^{-2} - 3(\omega_0^{\rm TK})^{-6}\right)\cos 2t\right]\cos 2\beta_1 x,$$

which is the dimensional form of (30).

From (1.2) of the main paper with $B = 0$ and $n = 1$ we observe that

$$\omega_{1,2} = -\frac{a_2}{a_1}|A|^2.$$

Using the exact forms of a_1 and a_2 given in appendix B of the main paper, this becomes

$$\begin{aligned}\omega_{1,2} &= \frac{\beta_1^4}{64\omega_1}|A|^2\left[\frac{9}{T_1^2} + 12 + 3T_1^2 + T_1^4\right], \\ &= \frac{\sqrt{\beta_1 g}}{64}\left[9(\omega_0^{\rm TK})^{-7} + 12(\omega_0^{\rm TK})^{-3} - 3\omega_0^{\rm TK} - 2(\omega_0^{\rm TK})^5\right], \\ &= \frac{\sqrt{\beta_1 g}}{2}\omega_2^{\rm TK},\end{aligned}$$

and thus agrees with the form of $\omega_2^{\rm TK}$ from (35) of Tadjbakhsh & Keller (1960).

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