# Supplementary Materials for: Nonlinear and Dispersive Free Surface Waves Propagating over Fluids with Weak Vertical and Horizontal Density Variation

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**Abstract**

In this auxillary material, we provide the detailed model equation derivation as well as the numerical solution scheme.

## 1 Derivation of Boussinesq-type Equations for Dispersive Waves over a Variable-Density Fluid

Following Kim and Lynett (2011), we obtain the dimensionless form of the spatially filtered continuity and Navier–Stokes equations for incompressible flow:

$$\nabla \cdot \boldsymbol{u} + w_z = 0, \tag{1}$$

$$\rho \left( \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + w \boldsymbol{u}_z \right) + \nabla p = \alpha \mu \nabla \cdot \left( \rho \nu_t^h \nabla \boldsymbol{u} \right) + \frac{\beta}{\mu} \left( \rho \nu_t^v \boldsymbol{u}_z \right)_z , \tag{2}$$

$$\mu^2 \rho \left( w_t + \boldsymbol{u} \cdot \nabla w + w w_z \right) + p_z + \rho = \alpha \mu^3 \nabla \cdot \left( \rho \nu_t^h \nabla w \right) + \beta \mu \left( \rho \nu_t^v w_z \right)_z , \tag{3}$$

where $\nabla$ is the horizontal derivative operator, subscripts $z$ and $t$ function as vertical and time differentiations, respectively, and $\boldsymbol{u} = (u, v)$ is the horizontal velocity vector. The conditions applied at the free surface and at the bottom boundary are expressed in dimensionless form as well:

$$w = \zeta_t + \boldsymbol{u} \cdot \nabla \zeta \quad \text{at} \quad z = \zeta, \tag{4}$$

$$w + \boldsymbol{u} \cdot \nabla h = 0 \quad \text{at} \quad z = -h. \tag{5}$$

The physical parameters $p$, $u$, $v$, and $w$ are expanded as a power series of $\mu^2$:

$$f = \sum_{n=1}^{\infty} \mu^{2n} f_n . \tag{6}$$

The leading order terms of Equation 3 yield the hydrostatic condition,

$$\left( p_0 \right)_z + \rho_0 = 0, \tag{7}$$

which accordingly guarantees

$$\boldsymbol{u}(x,y,z,t) = \boldsymbol{u}_0(x,y,t) + O(\mu^2). \tag{8}$$

By integrating the continuity equation over depth and applying the free-surface and bottom-boundary conditions, Equations 4 and 5 will give the relationship between horizontal and vertical velocities as

$$w_0 = -z\left(\nabla\cdot\boldsymbol{u}_0\right) - \nabla\cdot\left(h\boldsymbol{u}_0\right) = -zS - T. \tag{9}$$

To determine the horizontal velocity profile, the horizontal vorticity ($\boldsymbol{\omega}$) is examined:

$$\boldsymbol{\omega}' = \boldsymbol{u}'_z - \nabla w' = \mu^2\sqrt{\frac{g}{h_0}}\boldsymbol{\omega}_1 + O(\mu^4); \tag{10}$$

this equation indicates that the horizontal vorticity can at most first appear at $O(\mu^2)$ within our scaling. Assuming that $\boldsymbol{\omega}_1$ is not zero permits rotational effects induced by bottom stress to be directly included in the velocity profile (Kim et al., 2009).

From the horizontal vorticity expression, the vertical profile of the horizontal velocity can be approximated by vertically integrating $\boldsymbol{\omega}_1$ from $-h$ to $z$,

$$\begin{aligned}\boldsymbol{u}_1 \quad = \quad & \boldsymbol{u}_1|_{z=-h} - \left\{\left(\frac{1}{2}z^2\nabla S + z\nabla T\right) - \left(\frac{1}{2}h^2\nabla S - h\nabla T\right)\right\} \\ & + \int_{-h}^{z}\boldsymbol{\omega}_1 dz + O(\mu^2).\end{aligned} \tag{11}$$

The integral of the vorticity term appearing above remains as a “residual” velocity component, and it can be specified depending on the particular physics of the configuration at hand (e.g., bottom or free-surface stress or stratification effects).

The horizontal velocity up to $O(\mu^2)$ can then be expressed as

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_0 \quad & + \quad \mu^2\left\{\boldsymbol{u}_1|_{z=-h} - \frac{1}{2}z^2\nabla S - z\nabla T + \frac{1}{2}h^2\nabla S - h\nabla T\right. \\ & + \quad \left.\int_{-h}^{z}\boldsymbol{\omega}_1 dz\right\} + O(\mu^4)\end{aligned} \tag{12}$$

and, following Nwogu's (1993) approach, we can define $\boldsymbol{u}_\alpha$ evaluated at $z = z_\alpha$ as

$$\begin{aligned}\boldsymbol{u}_\alpha = \boldsymbol{u}_0 \quad & + \quad \mu^2\left\{\boldsymbol{u}_1|_{z=-h} - \frac{1}{2}z_\alpha^2\nabla S - z_\alpha\nabla T + \frac{1}{2}h^2\nabla S - h\nabla T\right. \\ & + \quad \left.\int_{-h}^{z_\alpha}\boldsymbol{\omega}_1 dz\right\} + O(\mu^4).\end{aligned} \tag{13}$$

Subtracting Equation 13 from Equation 12 finalizes the expression of $\boldsymbol{u}$ in terms of $\boldsymbol{u}_\alpha$ as

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_\alpha \quad &+ \quad \mu^2 \left\{ \frac{1}{2} \left( z_\alpha^2 - z^2 \right) \nabla S + \left( z_\alpha - z \right) \nabla T + \int_{z_\alpha}^{z} \boldsymbol{\omega}_1 dz \right\} \\ &+ \quad O(\mu^4).\end{aligned} \tag{14}$$

Because a main purpose of this study is to extend the derivation to include internal motion resulting from stratification, the horizontal velocity of the background internal motion, $\boldsymbol{u}^i(z)$, should be included. Here, this component is interpreted as part of the residual vorticity acting on the barotropic wave, i.e.,

$$\int_{z_\alpha}^{z} \boldsymbol{\omega}_1 dz = \boldsymbol{u}^i(z) + \int_{z_\alpha}^{z} \boldsymbol{\omega}_1^s dz, \tag{15}$$

where $\boldsymbol{\omega}_1^s$ represents the horizontal vorticity resulting from a bottom stress only. This decomposition yields

$$\begin{aligned}\boldsymbol{u} \quad &= \quad \boldsymbol{u}_\alpha + \mu^2 \boldsymbol{u}^i(z) \\ &+ \quad \mu^2 \left\{ \frac{1}{2} \left( z_\alpha^2 - z^2 \right) \nabla S + \left( z_\alpha - z \right) \nabla T + \int_{z_\alpha}^{z} \boldsymbol{\omega}_1^s dz \right\} \\ &+ \quad O(\mu^4).\end{aligned} \tag{16}$$

Note that, in the above, it is implicitly admitted that $O(\gamma) = O(\mu^2)$, because $\boldsymbol{u}^i(z)$ is usually scaled by reduced gravity (e.g., Lynett and Liu, 2002b). As a result, the direct inclusion of density-driven internal kinematics into the velocity structure allows for consideration of internal motion effects on free-surface waves. Although the above expression is flexible in terms of the $\boldsymbol{u}^i(z)$ that can be accommodated, the theory is only valid if the magnitude of this component is small.

For a quantitative description of $\boldsymbol{u}^i(z)$, specific cases will be chosen later for verification. In any case, the horizontal velocity with rotational terms ($\int_{z_\alpha}^{z} \boldsymbol{\omega}_1^s dz'$) remains undetermined; these rotational terms are assumed to be related only to the bottom stress ($\boldsymbol{\tau}$) (i.e., Kim et al., 2009). Imposing a linear stress profile from zero at the free surface to $\boldsymbol{\tau}_b$ at the bed,

$$\boldsymbol{\omega}_1^s \quad = \quad \frac{\boldsymbol{\tau}_b}{\rho \nu_t^v} \left( \frac{\zeta - z}{\zeta + h} \right), \tag{17}$$

is all the information that is required. Integration of $\boldsymbol{\omega}_1^s$ produces a rotational velocity component of $O(\mu^2)$ as follows:

$$\int_{z_\alpha}^{z} \boldsymbol{\omega}_1^s dz' = \boldsymbol{\Psi} \left\{ \frac{1}{2} \left( z_\alpha^2 - z^2 \right) + \zeta \left( z - z_\alpha \right) \right\}, \tag{18}$$

where $\boldsymbol{\Psi} = \boldsymbol{\tau}_b / \left\{ \rho \nu_t^v \left( \zeta + h \right) \right\}$.

With this, the horizontal velocity up to $O(\mu^2)$ can be finally expressed as

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_\alpha &+ \mu^2 \boldsymbol{u}^i + \mu^2 \left\{ \frac{1}{2} \left( z_\alpha^2 - z^2 \right) \nabla S + \left( z_\alpha - z \right) \nabla T \right\} \\ &+ \mu^2 \boldsymbol{\Psi} \left\{ \frac{1}{2} \left( z_\alpha^2 - z^2 \right) + \zeta \left( z - z_\alpha \right) \right\} \\ &+ O(\mu^4).\end{aligned} \tag{19}$$

By utilizing the horizontal velocity profile above, the exact continuity equation can be given in a depth-integrated format. Integrating Equation 1 from $-h$ to $\zeta$ and applying boundary conditions (Equations 4 and 5) gives

$$\nabla \cdot \int_{-h}^{\zeta} \boldsymbol{u} dz + \zeta_t = 0. \tag{20}$$

To express this equation in terms of $\boldsymbol{u}_\alpha$, Equation 19 is substituted, and after manipulation this yields

$$\zeta_t + \nabla \cdot \left\{ \left( \zeta + h \right) \boldsymbol{u}_\alpha \right\} + \mu^2 \left( \mathcal{N}_D + \mathcal{N}_B + \mathcal{N}_I \right) = O(\mu^4), \tag{21}$$

where the second-order terms are

$$\mathcal{N}_D = -\nabla \cdot \left[ \left( \zeta + h \right) \left\{ \left( \frac{\zeta^2 - \zeta h + h^2}{6} - \frac{z_\alpha^2}{2} \right) \nabla S + \left( \frac{\zeta - h}{2} - z_\alpha \right) \nabla T \right\} \right], \tag{22}$$

$$\mathcal{N}_B = \nabla \cdot \left[ \boldsymbol{\psi} \left( \zeta + h \right) \left\{ \frac{z_\alpha^2}{2} - z_\alpha \zeta + \frac{\left( 2\zeta^2 - 2\zeta h - h^2 \right)}{6} \right\} \right], \tag{23}$$

$$\mathcal{N}_I = \nabla \cdot \left\{ \left( \zeta + h \right) \overline{\boldsymbol{u}^i} \right\}, \tag{24}$$

and where $\overline{\boldsymbol{u}^i}$ is the depth-averaged horizontal velocity vector resulting from internal motion.

Although integration of the continuity equation was straightforward, derivation of a depth-integrated momentum equation requires a fairly complex procedure owing to nonlinearity in the equation and the desired removal (through substitution) of the hydrodynamic pressure term.

From the vertical momentum equation and density profile assumed, the pressure field can be extracted. Substituting Equation 19 into Equation 3 yields

$$\begin{aligned}\mu^2 \rho_0 \left\{ (w_0)_t + \boldsymbol{u}_\alpha \cdot \nabla w_0 + w_0 \left( w_0 \right)_z \right\} + p_z + \rho_0 \left\{ 1 - \gamma \tanh \left( \frac{z - z_0}{\delta} \right) \right\} \\ = O \left( \mu^4, \alpha\mu^3, \beta\mu^3, \gamma\mu^2 \right).\end{aligned} \tag{25}$$

Integrating this equation with respect to $z$ gives an expression for pressure,

$$
\begin{aligned}
p &= \rho_0\left(\zeta - z\right) - \gamma\delta\rho_0\left[\ln\left\{\cosh\left(\frac{\zeta - z_0}{\delta}\right)\right\} - \ln\left\{\cosh\left(\frac{z - z_0}{\delta}\right)\right\}\right] \\
&\quad + \mu^2\rho_0\left\{\frac{1}{2}\left(z^2 - \zeta^2\right)S_t + \left(z - \zeta\right)T_t + \frac{1}{2}\left(z^2 - \zeta^2\right)\boldsymbol{u}_\alpha\cdot\nabla S\right. \\
&\qquad \left. + \left(z - \zeta\right)\boldsymbol{u}_\alpha\cdot\nabla T - \frac{1}{2}\left(z^2 - \zeta^2\right)S^2 - \left(z - \zeta\right)TS\right\} \\
&\qquad\qquad + O\left(\mu^4, \alpha\mu^3, \beta\mu^3, \gamma\mu^2\right),
\end{aligned}
\tag{26}
$$

where we have applied Equation 9.

To derive a depth-integrated momentum equation for $\boldsymbol{u}_\alpha$, Equations 19 and 26 can be applied to Equation 2, where each term is written as,

$$
\begin{aligned}
\rho\boldsymbol{u}_t &= \rho_0\left(\boldsymbol{u}_\alpha\right)_t - \gamma\tanh\left(\frac{z - z_0}{\delta}\right)\rho_0\left(\boldsymbol{u}_\alpha\right)_t + \mu^2\rho_0\left(\boldsymbol{u}^i\right)_t \\
&\quad + \mu^2\rho_0\left\{\frac{1}{2}\left(z_\alpha^2 - z^2\right)\nabla S + \left(z_\alpha - z\right)\nabla T\right\}_t \\
&\quad + \mu^2\rho_0\left[\boldsymbol{\Psi}\left\{\frac{1}{2}\left(z_\alpha^2 - z^2\right) + \zeta\left(z - z_\alpha\right)\right\}\right]_t \\
&\quad + O(\mu^4, \gamma\mu^2),
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} &= \rho_0\boldsymbol{u}_\alpha\cdot\nabla\boldsymbol{u}_\alpha - \gamma\tanh\left(\frac{z - z_0}{\delta}\right)\rho_0\boldsymbol{u}_\alpha\cdot\nabla\boldsymbol{u}_\alpha + \mu^2\rho_0\nabla\left(\boldsymbol{u}_\alpha\cdot\boldsymbol{u}^i\right) \\
&\quad + \mu^2\rho_0\nabla\left[\boldsymbol{u}_\alpha\cdot\left\{\frac{1}{2}\left(z_\alpha^2 - z^2\right)\nabla S + \left(z_\alpha - z\right)\nabla T\right\}\right] \\
&\quad + \mu^2\rho_0\nabla\left[\boldsymbol{u}_\alpha\cdot\boldsymbol{\Psi}\left\{\frac{1}{2}\left(z_\alpha^2 - z^2\right) + \zeta\left(z - z_\alpha\right)\right\}\right] \\
&\quad + \mu^2\rho_0\boldsymbol{\xi} + O(\mu^4, \gamma\mu^2),
\end{aligned}
\tag{28}
$$

$$
\rho w\left(\boldsymbol{u}\right)_z = \mu^2\rho_0\left(z^2 S\nabla S + zT\nabla S + zS\nabla T + T\nabla T + w_0\boldsymbol{\omega}_1\right) + O(\mu^4, \gamma\mu^2), \tag{29}
$$

$$
\begin{aligned}
\nabla p &= \nabla\left\{\rho_0\left(\zeta - z\right)\right\} \\
&- \gamma\nabla\left(\rho_0\delta\left[\ln\left\{\cosh\left(\frac{\zeta - z_0}{\delta}\right)\right\} - \ln\left\{\cosh\left(\frac{z - z_0}{\delta}\right)\right\}\right]\right) \\
&+ \mu^2\rho_0\left\{\frac{1}{2}z^2\nabla S_t - \frac{1}{2}\nabla\left(\zeta^2 S_t\right) + z\nabla T_t - \nabla\left(\zeta T_t\right) + \frac{1}{2}z^2\nabla\left(\boldsymbol{u}_\alpha\cdot\nabla S\right)\right. \\
&\quad - \frac{1}{2}\nabla\left(\zeta^2\boldsymbol{u}_\alpha\cdot\nabla S\right) + z\nabla\left(\boldsymbol{u}_\alpha\cdot\nabla T\right) - \nabla\left(\zeta\boldsymbol{u}_\alpha\cdot\nabla T\right) - \frac{1}{2}z^2\nabla S^2
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{1}{2}\nabla\left(\zeta^2 S^2\right) - z\nabla\left(TS\right) + \nabla\left(\zeta TS\right)\Big\} \\
+ \quad & \mu^2\left(\nabla\rho_0\right)\left\{\frac{1}{2}\left(z^2-\zeta^2\right)S_t + \left(z-\zeta\right)T_t + \frac{1}{2}\left(z^2-\zeta^2\right)\boldsymbol{u}_\alpha\cdot\nabla S\right. \\
& \left. + \left(z-\zeta\right)\boldsymbol{u}_\alpha\cdot\nabla T - \frac{1}{2}\left(z^2-\zeta^2\right)S^2 - \left(z-\zeta\right)TS\right\} \\
+ \quad & O(\mu^4, \alpha\mu^3, \beta\mu^3, \gamma\mu^2),
\end{aligned} \tag{30}
$$

$$
\alpha\mu\nabla\cdot\left(\rho\nu_t^h\nabla\boldsymbol{u}\right) = \alpha\mu\nabla\cdot\left(\rho_0\nu_t^h\nabla\boldsymbol{u}_\alpha\right) + O(\alpha\mu^3, \alpha\gamma\mu), \tag{31}
$$

$$
\frac{\beta}{\mu}\left(\rho\nu_t^v\boldsymbol{u}_z\right)_z = -\beta\mu\rho_0\nu_t^v\nabla S - \beta\mu\frac{\boldsymbol{\tau}_b}{\zeta+h} + \beta\mu\rho_0\nu_t^v\left(\boldsymbol{u}^i\right)_{zz} + O(\beta\mu^3, \beta\gamma\mu). \tag{32}
$$

In Equation 28, $\boldsymbol{\xi}$ is defined as

$$
\begin{aligned}
\boldsymbol{\xi} = \quad & - \quad \boldsymbol{u}_\alpha\times\left[\nabla\times\left\{\frac{1}{2}\left(z_\alpha^2-z^2\right)\nabla S + \left(z_\alpha-z\right)\nabla T\right\}\right] \\
& - \quad \left\{\frac{1}{2}\left(z_\alpha^2-z^2\right)\nabla S + \left(z_\alpha-z\right)\nabla T\right\}\times\left(\nabla\times\boldsymbol{u}_\alpha\right) \\
& - \quad \boldsymbol{u}_\alpha\times\left(\nabla\times\boldsymbol{u}^i\right) - \boldsymbol{u}^i\times\left(\nabla\times\boldsymbol{u}_\alpha\right) \\
& - \quad \boldsymbol{u}_\alpha\times\left(\nabla\times\boldsymbol{\Psi}\right)\left\{\frac{1}{2}\left(z_\alpha^2-z^2\right) + \zeta\left(z-z_\alpha\right)\right\} \\
& - \quad \boldsymbol{\Psi}\times\left(\nabla\times\boldsymbol{u}_\alpha\right)\left\{\frac{1}{2}\left(z_\alpha^2-z^2\right) + \zeta\left(z-z_\alpha\right)\right\}.
\end{aligned} \tag{33}
$$

Equations 27 to 32 are substituted into Equation 2 to produce the horizontal momentum equation written in terms of $\boldsymbol{u}_\alpha$. Thus,

$$
\begin{aligned}
\left(\boldsymbol{u}_\alpha\right)_t \quad + \quad & \boldsymbol{u}_\alpha\cdot\nabla\boldsymbol{u}_\alpha + \nabla\zeta + \frac{\nabla\rho_0}{\rho_0}\left(\zeta-z\right) \\
+ \quad & \mu^2\left\{-\frac{1}{2}\nabla\left(\zeta^2 S_t\right) - \nabla\left(\zeta T_t\right) + \frac{1}{2}z_\alpha^2\nabla S_t + z_\alpha\nabla T_t\right. \\
& -\frac{1}{2}\nabla\left(\zeta^2\boldsymbol{u}_\alpha\cdot\nabla S\right) - \nabla\left(\zeta\boldsymbol{u}_\alpha\cdot\nabla T\right) + \frac{1}{2}\nabla\left(\zeta^2 S^2\right) \\
& \left. +\frac{1}{2}\nabla\left(z_\alpha^2\boldsymbol{u}_\alpha\cdot\nabla S\right) + \nabla\left(z_\alpha\boldsymbol{u}\cdot\nabla T\right) + T\nabla T + \nabla\left(\zeta TS\right)\right\} \\
+ \quad & \mu^2\left[\boldsymbol{\Psi}\left\{\frac{1}{2}\left(z_\alpha^2-z^2\right) + \zeta\left(z-z_\alpha\right)\right\}\right]_t \\
+ \quad & \mu^2\nabla\left[\boldsymbol{u}_\alpha\cdot\boldsymbol{\Psi}\left\{\frac{1}{2}\left(z_\alpha^2-z^2\right) + \zeta\left(z-z_\alpha\right)\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \quad \mu^2 \boldsymbol{\Psi} (\zeta - z)(zS + T) + \mu^2 \boldsymbol{\xi} + \mu^2 \left( \boldsymbol{u}_t^i + \nabla \boldsymbol{u}_\alpha \cdot \boldsymbol{u}^i \right) \\
&+ \quad \mu^2 \frac{\nabla \rho_0}{\rho_0} \left\{ \frac{1}{2} \left( z^2 - \zeta^2 \right) S_t + (z - \zeta) T_t + \frac{1}{2} \left( z^2 - \zeta^2 \right) \boldsymbol{u}_\alpha \cdot \nabla S \right. \\
&\qquad \left. + (z - \zeta) \boldsymbol{u}_\alpha \cdot \nabla T - \frac{1}{2} \left( z^2 - \zeta^2 \right) S^2 - (z - \zeta) TS \right\} \\
&- \quad \gamma \left\{ \tanh \left( \frac{z - z_0}{\delta} \right) (\boldsymbol{u}_\alpha)_t + \tanh \left( \frac{z - z_0}{\delta} \right) \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha \right. \\
&+ \quad \left. \frac{1}{\rho_0} \nabla \left( \rho_0 \delta \left[ \ln \left\{ \cosh \left( \frac{\zeta - z_0}{\delta} \right) \right\} - \ln \left\{ \cosh \left( \frac{z - z_0}{\delta} \right) \right\} \right] \right) \right\} \\
&= \quad \alpha \mu \frac{1}{\rho_0} \nabla \cdot \left( \rho_0 \nu_t^h \nabla \boldsymbol{u}_\alpha \right) \\
&\qquad -\beta \mu \nu_t^v \nabla S - \beta \mu \frac{1}{\rho_0} \frac{\boldsymbol{\tau}_b}{\zeta + h} + \beta \mu \nu_t^v \left( \boldsymbol{u}^i \right)_{zz}. \qquad (34)
\end{aligned}
$$

Now, the remaining procedure is to eliminate the $z$ dependency in the above equation; depth-averaging is employed (e.g., see Chen, 2006) over the entire equation. The resultant form of the depth-integrated momentum equation appears as

$$
\begin{aligned}
(\boldsymbol{u}_\alpha)_t \quad &+ \quad \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha + \nabla \zeta + \frac{\nabla \rho_0}{\rho_0} \frac{(\zeta + h)}{2} + \gamma \mathcal{R}_P^v \\
&+ \quad \mu^2 \left( \mathcal{R}_D + \mathcal{R}_B + \mathcal{R}_I + \mathcal{R}_P^h + \overline{\boldsymbol{\xi}} \right) \\
&- \quad \alpha \mu \frac{1}{\rho_0} \nabla \cdot \left( \rho_0 \nu_t^h \nabla \boldsymbol{u}_\alpha \right) + \beta \mu \nu_t^v \nabla S \\
&+ \quad \beta \mu \frac{1}{\rho_0} \frac{\boldsymbol{\tau}_b}{\zeta + h} - \beta \mu \nu_t^v \left\{ \left( \boldsymbol{u}^i \right)_z |_{z=\zeta} - \left( \boldsymbol{u}^i \right)_z |_{z=-h} \right\} \\
&= \quad O(\mu^4, \alpha \mu^3, \beta \mu^3), \qquad (35)
\end{aligned}
$$

in which the higher-order terms are defined as

$$
\begin{aligned}
\mathcal{R}_D \quad &= \quad \frac{1}{2} z_\alpha^2 \nabla S_t + z_\alpha \nabla T_t - \frac{1}{2} \nabla \left( \zeta^2 S_t \right) - \nabla (\zeta T_t) + T \nabla T \\
&+ \quad \frac{1}{2} \nabla \left( z_\alpha^2 \boldsymbol{u}_\alpha \cdot \nabla S \right) + \nabla (z_\alpha \boldsymbol{u}_\alpha \cdot \nabla T) + \frac{1}{2} \nabla \left( \zeta^2 S^2 \right) \\
&- \quad \frac{1}{2} \nabla \left( \zeta^2 \boldsymbol{u}_\alpha \cdot \nabla S \right) - \nabla (\zeta \boldsymbol{u}_\alpha \cdot \nabla T) + \nabla (\zeta T S), \qquad (36)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_B \quad &= \quad \frac{(\zeta - h)}{2} (\boldsymbol{\Psi} \zeta)_t - \frac{\left( \zeta^2 - \zeta h + h^2 \right)}{6} \boldsymbol{\Psi}_t + \left[ \boldsymbol{\Psi} \left( \frac{z_\alpha^2}{2} - \zeta z_\alpha \right) \right]_t \qquad (37) \\
&+ \quad \frac{(\zeta - h)}{2} \nabla \{ \boldsymbol{u}_\alpha \cdot (\boldsymbol{\Psi} \zeta) \} - \frac{\left( \zeta^2 - \zeta h + h^2 \right)}{6} \nabla (\boldsymbol{u}_\alpha \cdot \boldsymbol{\Psi}) \\
&+ \quad \nabla \left[ \boldsymbol{u}_\alpha \cdot \left\{ \boldsymbol{\Psi} \left( \frac{1}{2} z_\alpha^2 - \zeta z_\alpha \right) \right\} \right] - \boldsymbol{\Psi} \left\{ \frac{\left( \zeta^2 - \zeta h - 2h^2 \right) S}{6} + \frac{(\zeta + h) T}{2} \right\},
\end{aligned}
$$

$$\mathcal{R}_I = \overline{\boldsymbol{u}_t^i} + \nabla\left(\boldsymbol{u}_\alpha \cdot \overline{\boldsymbol{u}^i}\right), \tag{38}$$

$$\begin{aligned}\mathcal{R}_P^h &= \frac{\nabla\rho_0}{\rho_0}(\zeta+h)\left\{\frac{-2\zeta+h}{6}\left(S_t+\boldsymbol{u}_\alpha\cdot\nabla S-S^2\right)\right.\\ &\quad\left.-\frac{1}{2}\left(T_t+\boldsymbol{u}_\alpha\cdot\nabla T-ST\right)\right\},\end{aligned} \tag{39}$$

$$\begin{aligned}\mathcal{R}_P^v &= \frac{\delta}{\zeta+h}\left\{(\boldsymbol{u}_\alpha)_t+\boldsymbol{u}_\alpha\cdot\nabla\boldsymbol{u}_\alpha\right\}\\ &\quad\times\left[\ln\left\{\cosh\left(\frac{-h-z_0}{\delta}\right)\right\}-\ln\left\{\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\right]\\ &+ \frac{1}{\rho_0}\nabla\left\{\rho_0\delta\ln\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\\ &- \frac{1}{\rho_0}\frac{1}{\zeta+h}\int_{-h}^{\zeta}\nabla\left\{\rho_0\delta\ln\cosh\left(\frac{z-z_0}{\delta}\right)\right\}dz.\end{aligned} \tag{40}$$

$\overline{\boldsymbol{\xi}} = \left(\overline{\xi^x}, \overline{\xi^y}\right)$ are written as

$$\begin{aligned}\overline{\xi^x} &= -v_\alpha\left\{(z_\alpha)_x(z_\alpha S_y+T_y)-(z_\alpha)_y(z_\alpha S_x+T_x)\right\}\\ &- \left\{(v_\alpha)_x-(u_\alpha)_y\right\}\left[\left\{\frac{z_\alpha^2}{2}-\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}\right\}S_y+\left\{z_\alpha-\frac{(\zeta-h)}{2}\right\}T_y\right]\\ &- v_\alpha\left[\left\{\psi^y\left(\frac{1}{2}z_\alpha^2-z_\alpha\zeta\right)\right\}_x-\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}(\psi^y)_x+\frac{(\zeta-h)}{2}(\psi^y\zeta)_x\right.\\ &- \left.\left\{\psi^x\left(\frac{1}{2}z_\alpha^2-z_\alpha\zeta\right)\right\}_y+\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}(\psi^x)_y-\frac{(\zeta-h)}{2}(\psi^x\zeta)_y\right]\\ &- \left\{(v_\alpha)_x-(u_\alpha)_y\right\}\psi^y\left\{\frac{z_\alpha^2}{2}-z_\alpha\zeta+\frac{\left(2\zeta^2-2\zeta h-h^2\right)}{6}\right\}\\ &+ v_\alpha^i\left\{(v_\alpha)_x-(u_\alpha)_y\right\},\end{aligned} \tag{41}$$

$$\begin{aligned}\overline{\xi^y} &= u_\alpha\left\{(z_\alpha)_x(z_\alpha S_y+T_y)-(z_\alpha)_y(z_\alpha S_x+T_x)\right\}\\ &+ \left\{(v_\alpha)_x-(u_\alpha)_y\right\}\left[\left\{\frac{z_\alpha^2}{2}-\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}\right\}S_x+\left\{z_\alpha-\frac{(\zeta-h)}{2}\right\}T_x\right]\\ &+ u_\alpha\left[\left\{\psi^y\left(\frac{1}{2}z_\alpha^2-z_\alpha\zeta\right)\right\}_x-\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}(\psi^y)_x+\frac{(\zeta-h)}{2}(\psi^y\zeta)_x\right.\\ &- \left.\left\{\psi^x\left(\frac{1}{2}z_\alpha^2-z_\alpha\zeta\right)\right\}_y+\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}(\psi^x)_y-\frac{(\zeta-h)}{2}(\psi^x\zeta)_y\right]\end{aligned} \tag{42}$$

$$
\begin{aligned}
&+ \quad \Big\{(v_\alpha)_x - (u_\alpha)_y\Big\}\, \psi^x \left\{ \frac{z_\alpha^2}{2} - z_\alpha \zeta + \frac{\left(2\zeta^2 - 2\zeta h - h^2\right)}{6} \right\} \\
&+ \quad u_\alpha^i \Big\{(u_\alpha)_y - (v_\alpha)_x\Big\},
\end{aligned}
$$

where $\boldsymbol{u}_\alpha = (u_\alpha, v_\alpha)$ and $\boldsymbol{\Psi} = (\psi^x, \psi^y)$.

# 2 Numerical Formulation

The derived Equations above are discretized to determine numerical solutions. In the present work, a conservative-form finite-volume method is adopted for spatial derivatives while third-order Adams–Bashforth predictor and fourth-order Adams–Moulton corrector schemes are used for time integration.

Prior to discretization of the governing system, the continuity and momentum equations are converted to the conservative form before applying the finite-volume method. In this section, all dimensions are recovered with primes ($'$) omitted for convenience. By utilizing a fixed bottom assumption ($h_t = 0$), the conservative form of the continuity and momentum equations can be obtained as

$$
H_t + (Hu_\alpha)_x + (Hv_\alpha)_y + (\mathcal{N}_D + \mathcal{N}_B + \mathcal{N}_I) = 0, \tag{43}
$$

$$
\begin{aligned}
(Hu_\alpha)_t \quad &+ \quad \left(Hu_\alpha^2 + \frac{1}{2} gH^2\right)_x + (Hu_\alpha v_\alpha)_y - gHh_x + \frac{1}{2}\frac{(\rho_0)_x}{\rho_0} gH^2 \\
&+ \quad H\mathcal{M}^x + u_\alpha (\mathcal{N}_D + \mathcal{N}_B + \mathcal{N}_I) = 0,
\end{aligned} \tag{44}
$$

$$
\begin{aligned}
(Hv_\alpha)_t \quad &+ \quad (Hu_\alpha v_\alpha)_x + \left(Hv_\alpha^2 + \frac{1}{2} gH^2\right)_y - gHh_y + \frac{1}{2}\frac{(\rho_0)_y}{\rho_0} gH^2 \\
&+ \quad H\mathcal{M}^y + v_\alpha (\mathcal{N}_D + \mathcal{N}_B + \mathcal{N}_I) = 0,
\end{aligned} \tag{45}
$$

where $H = \zeta + h$ and terms of $O(\mu^2, \gamma, \alpha\mu, \beta\mu)$ are given by

$$
\begin{aligned}
(\mathcal{M}^x, \mathcal{M}^y) \quad &= \quad \mathcal{R}_D + \mathcal{R}_B + \mathcal{R}_I + \mathcal{R}_P^h + \mathcal{R}_P^v + \overline{\boldsymbol{\xi}} \\
&- \quad \frac{1}{\rho_0} \nabla \cdot \left(\rho_0 \nu_t^h \nabla \boldsymbol{u}_\alpha\right) + \nu_t^v \nabla S + \frac{\boldsymbol{\tau}_b}{H\rho_0} \\
&- \quad \nu_t^v \left(\boldsymbol{u}_z^i|_{z=\zeta} - \boldsymbol{u}_z^i|_{z=-h}\right).
\end{aligned} \tag{46}
$$

## 2.1 Time Integration

The time derivative terms in the above equations are solved by using a third-order Adams–Bashforth predictor and a fourth-order Adams–Moulton corrector scheme (Wei et al., 1995; Lynett and Liu, 2002a) to minimize truncation error to $O(\Delta t^3)$ (Liu and Wang, 2012). Through an iterative predictor–corrector time-marching scheme, the solution at the next time step, $(n + 1)$, can be found.

The explicit predictor step is given by

$$\zeta^{n+1} = \zeta^n + \frac{\Delta t}{12}\left(23E^n - 16E^{n-1} + 5E^{n-2}\right), \tag{47}$$

$$\begin{aligned} P^{n+1} &= P^n + \frac{\Delta t}{12}\left(23F^n - 16F^{n-1} + 5F^{n-2}\right) \\ &\quad + 2F_3^n - 3F_3^{n-1} + F_3^{n-2} + F_4^p, \end{aligned} \tag{48}$$

$$\begin{aligned} Q^{n+1} &= Q^n + \frac{\Delta t}{12}\left(23G^n - 16G^{n-1} + 5G^{n-2}\right) \\ &\quad + 2G_3^n - 3G_3^{n-1} + G_3^{n-2} + G_4^p, \end{aligned} \tag{49}$$

and the implicit corrector step is written as

$$\zeta^{n+1} = \zeta^n + \frac{\Delta t}{24}\left(9E^{n+1} + 19E^n - 5E^{n-1} + E^{n-2}\right), \tag{50}$$

$$\begin{aligned} P^{n+1} &= P^n + \frac{\Delta t}{24}\left(9F^{n+1} + 19F^n - 5F^{n-1} + F^{n-2}\right) \\ &\quad + F_3^{n+1} - F_3^n + F_4^c, \end{aligned} \tag{51}$$

$$\begin{aligned} Q^{n+1} &= Q^n + \frac{\Delta t}{24}\left(9G^{n+1} + 19G^n - 5G^{n-1} + G^{n-2}\right) \\ &\quad + G_3^{n+1} - G_3^n + G_4^c, \end{aligned} \tag{52}$$

where the superscript $n$ denotes the time step and $P$ and $Q$ are defined numerically as (Kim et al., 2009)

$$\begin{aligned} P &= (u_\alpha)_{i-1,j} H_{i,j} \left\{ \frac{z_\alpha^2 - \zeta^2}{2\Delta x^2} + \frac{(z_\alpha - \zeta) h_{i-1,j}}{\Delta x^2} + \frac{\zeta_x \zeta}{2\Delta x} + \frac{\zeta_x h_{i-1,j}}{2\Delta x} \right\} \\ &+ (u_\alpha)_{i,j} H_{i,j} \left\{ 1 - \frac{z_\alpha^2 - \zeta^2}{\Delta x^2} - \frac{2(z_\alpha - \zeta) h_{i,j}}{\Delta x^2} \right\} \\ &+ (u_\alpha)_{i+1,j} H_{i,j} \left\{ \frac{z_\alpha^2 - \zeta^2}{2\Delta x^2} + \frac{(z_\alpha - \zeta) h_{i+1,j}}{\Delta x^2} - \frac{\zeta_x \zeta}{2\Delta x} - \frac{\zeta_x h_{i+1,j}}{2\Delta x} \right\}, \end{aligned} \tag{53}$$

$$\begin{aligned} Q &= (v_\alpha)_{i,j-1} H_{i,j} \left\{ \frac{z_\alpha^2 - \zeta^2}{2\Delta y^2} + \frac{(z_\alpha - \zeta) h_{i,j-1}}{\Delta y^2} + \frac{\zeta_y \zeta}{2\Delta y} + \frac{\zeta_y h_{i,j-1}}{2\Delta y} \right\} \\ &+ (v_\alpha)_{i,j} H_{i,j} \left\{ 1 - \frac{z_\alpha^2 - \zeta^2}{\Delta y^2} - \frac{2(z_\alpha - \zeta) h_{i,j}}{\Delta y^2} \right\} \\ &+ (v_\alpha)_{i,j+1} H_{i,j} \left\{ \frac{z_\alpha^2 - \zeta^2}{2\Delta y^2} + \frac{(z_\alpha - \zeta) h_{i,j+1}}{\Delta y^2} - \frac{\zeta_y \zeta}{2\Delta y} - \frac{\zeta_y h_{i,j+1}}{2\Delta y} \right\}. \end{aligned} \tag{54}$$

Subscripts $i$ and $j$ in $P$ and $Q$ identify the cell location. The other terms included in Equations 47–52 are

$$E = E_1 + E_2, \tag{55}$$

$$F = F_1 + F_2 + u_\alpha E_2, \tag{56}$$

$$G = G_1 + G_1 + v_\alpha E_2, \tag{57}$$

where $E_1$, $F_1$, and $G_1$ can rewritten as

$$E_1 = -\left\{Hu_\alpha\right\}_x - \left\{Hv_\alpha\right\}_y, \tag{58}$$

$$F_1 = -\left\{Hu_\alpha^2 + \frac{1}{2}gH^2\right\}_x - \left\{Hu_\alpha v_\alpha\right\}_y + gHh_x - \frac{1}{2}\frac{(\rho_0)_x}{\rho_0}gH^2, \tag{59}$$

$$G_1 = -\left\{Hu_\alpha v_\alpha\right\}_x - \left\{Hv_\alpha^2 + \frac{1}{2}gH^2\right\}_y + gHh_y - \frac{1}{2}\frac{(\rho_0)_y}{\rho_0}gH^2, \tag{60}$$

and $E_2$, $F_2$, $G_2$, $F_3$, and $G_3$ are expressed as

$$\begin{aligned} E_2 &= \left[H\left\{\left(\frac{\zeta^2 - \zeta h + h^2}{6} - \frac{1}{2}z_\alpha^2\right)\nabla S + \left(\frac{\zeta - h}{2} - z_\alpha\right)\nabla T\right\}\right]_x \\ &+ \left[H\left\{\left(\frac{\zeta^2 - \zeta h + h^2}{6} - \frac{1}{2}z_\alpha^2\right)\nabla S + \left(\frac{\zeta - h}{2} - z_\alpha\right)\nabla T\right\}\right]_y \\ &- \left[H\psi^x\left\{\frac{z_\alpha^2}{2} - z_\alpha\zeta + \frac{\left(2\zeta^2 - 2\zeta h - h^2\right)}{6}\right\}\right]_x \\ &- \left[H\psi^y\left\{\frac{z_\alpha^2}{2} - z_\alpha\zeta + \frac{\left(2\zeta^2 - 2\zeta h - h^2\right)}{6}\right\}\right]_y \\ &- \left[H\overline{u^i}\right]_x - \left[H\overline{v^i}\right]_y, \end{aligned} \tag{61}$$

$$\begin{aligned} (F_2, G_2) &= H\left[\frac{1}{2}\nabla\left(\zeta^2\boldsymbol{u}_\alpha\cdot\nabla S\right) + \nabla\left(\zeta\boldsymbol{u}_\alpha\cdot\nabla T\right) - \frac{1}{2}\nabla\left(\zeta^2 S^2\right)\right. \\ &- \frac{1}{2}\nabla\left(z_\alpha^2\boldsymbol{u}_\alpha\cdot\nabla S\right) - \nabla\left(z_\alpha\boldsymbol{u}_\alpha\cdot\nabla T\right) - \nabla\left(\zeta TS\right) \\ &- \left.(T\nabla T) - \nabla\left\{E\left(\zeta S + T\right)\right\}\right] - H\overline{\boldsymbol{\xi}} \\ &- E\left(\zeta S + T\right)\nabla\zeta - \frac{1}{2}\left(\zeta^2 - z_\alpha^2\right)E\nabla S - \left(\zeta - z_\alpha\right)E\nabla T \end{aligned}$$

$$
\begin{aligned}
&+ \quad H\left[\frac{\left(\zeta^2-\zeta h+h^2\right)}{6}\nabla\left(\boldsymbol{u}_\alpha\cdot\boldsymbol{\Psi}\right)-\frac{(\zeta-h)}{2}\nabla\left\{\boldsymbol{u}_\alpha\cdot(\boldsymbol{\Psi}\zeta)\right\}\right.\\
&+ \quad \boldsymbol{\Psi}\left\{\frac{\left(\zeta^2+\zeta h-2h^2\right)S}{6}+\frac{HT}{2}\right\}\\
&- \quad \left.\nabla\left\{\boldsymbol{u}_\alpha\cdot\left(\boldsymbol{\Psi}\left(\frac{z_\alpha^2}{2}-\zeta z_\alpha\right)\right)\right\}\right]-H\nabla\left(\boldsymbol{u}_\alpha\cdot\overline{\boldsymbol{u}^i}\right)\\
&- \quad \frac{\nabla\rho_0}{\rho_0}H\left\{\frac{H(-2\zeta+h)}{6}(\boldsymbol{u}_\alpha\cdot\nabla S-S^2)-\frac{H}{2}(\boldsymbol{u}_\alpha\cdot\nabla T-ST)\right\}\\
&- \quad \delta(\boldsymbol{u}_\alpha\cdot\nabla\boldsymbol{u}_\alpha)\left[\ln\left\{\cosh\left(\frac{-h-z_0}{\delta}\right)\right\}-\ln\left\{\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\right]\\
&- \quad \frac{H}{\rho_0}\nabla\left\{\rho_0\delta\ln\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\\
&+ \quad \frac{1}{\rho_0}\int_{-h}^{\zeta}\nabla\left\{\rho_0\delta\ln\cosh\left(\frac{z-z_0}{\delta}\right)\right\}dz\\
&+ \quad H\left[\frac{1}{\rho_0}\nabla\cdot\left(\rho_0\nu_t^h\nabla\boldsymbol{u}_\alpha\right)-\nu_t^v\nabla S-\frac{\boldsymbol{\tau}_b}{H\rho_0}\right.\\
&+ \quad \left.\nu_t^v\left\{\boldsymbol{u}_z^i|_{z=\zeta}-\boldsymbol{u}_z^i|_{z=-h}\right\}\right], \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
F_3 &= \frac{1}{2}H\left(\zeta^2-z_\alpha^2\right)(v_\alpha)_{xy}-H\left(z_\alpha-\zeta\right)(hv_\alpha)_{xy}\\
&+ H\zeta_x\left\{\zeta\left(v_\alpha\right)_y+\left(hv_\alpha\right)_y\right\}, \qquad (63)
\end{aligned}
$$

$$
\begin{aligned}
G_3 &= \frac{1}{2}H\left(\zeta^2-z_\alpha^2\right)(u_\alpha)_{xy}-H\left(z_\alpha-\zeta\right)(hu_\alpha)_{xy}\\
&+ H\zeta_y\left\{\zeta\left(u_\alpha\right)_x+\left(hu_\alpha\right)_x\right\}. \qquad (64)
\end{aligned}
$$

$F_4^p$, $G_4^p$, $F_4^c$, and $G_4^c$ can be rewritten as

$$
\begin{aligned}
F_4^p &= \frac{H^n\left(\zeta^2-\zeta h+h^2-3z_\alpha^2\right)^n}{6}\Sigma^p(\psi^x)-\frac{H^n\left(\zeta-h-2z_\alpha\right)^n}{2}\Sigma^p(\psi^x\zeta)\\
&+ H^n\Sigma^p(\overline{u^i})-\frac{H^n\left(h-2\zeta\right)^n}{6}\frac{(\rho_0)_x}{\rho_0}\Sigma^p(S)+\frac{H^n}{2}\frac{(\rho_0)_x}{\rho_0}\Sigma^p(T)\\
&- \delta\left[\ln\left\{\cosh\left(\frac{-h-z_0}{\delta}\right)\right\}-\ln\left\{\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\right]^n\Sigma^p(u_\alpha), \qquad (65)
\end{aligned}
$$

$$
\begin{aligned}
G_4^p &= \frac{H^n\left(\zeta^2-\zeta h+h^2-3z_\alpha^2\right)^n}{6}\Sigma^p(\psi^y)-\frac{H^n\left(\zeta-h-2z_\alpha\right)^n}{2}\Sigma^p(\psi^y\zeta)\\
&+ H^n\Sigma^p\left(\overline{v^i}\right)-\frac{H^n\left(h-2\zeta\right)^n}{6}\frac{(\rho_0)_y}{\rho_0}\Sigma^p(S)+\frac{H^n}{2}\frac{(\rho_0)_y}{\rho_0}\Sigma^p(T)\\
&- \delta\left[\ln\left\{\cosh\left(\frac{-h-z_0}{\delta}\right)\right\}-\ln\left\{\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\right]^n\Sigma^p(v_\alpha), \qquad (66)
\end{aligned}
$$

$$
\begin{aligned}
F_4^c &= \frac{H^{n+1}\left(\zeta^2-\zeta h+h^2-3z_\alpha^2\right)^{n+1}}{6}\Sigma^c(\psi^x)-\frac{H^{n+1}\left(\zeta-h-2z_\alpha\right)^{n+1}}{2}\Sigma^c(\psi^x\zeta) \\
&+ H^{n+1}\Sigma^c\left(\overline{u^i}\right)-\frac{H^{n+1}\left(h-2\zeta\right)^{n+1}}{6}\frac{\left(\rho_0\right)_x}{\rho_0}\Sigma^c(S)+\frac{H^{n+1}}{2}\frac{\left(\rho_0\right)_x}{\rho_0}\Sigma^c(T) \\
&- \delta\left[\ln\left\{\cosh\left(\frac{-h-z_0}{\delta}\right)\right\}-\ln\left\{\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\right]^{n+1}\Sigma^c\left(u_\alpha\right), \qquad (67)
\end{aligned}
$$

$$
\begin{aligned}
G_4^c &= \frac{H^{n+1}\left(\zeta^2-\zeta h+h^2-3z_\alpha^2\right)^{n+1}}{6}\Sigma^c(\psi^y)-\frac{H^{n+1}\left(\zeta-h-2z_\alpha\right)^{n+1}}{2}\Sigma^c(\psi^y\zeta) \\
&+ H^{n+1}\Sigma^c\left(\overline{v^i}\right)-\frac{H^{n+1}\left(h-2\zeta\right)^{n+1}}{6}\frac{\left(\rho_0\right)_y}{\rho_0}\Sigma^c(S)+\frac{H^{n+1}}{2}\frac{\left(\rho_0\right)_y}{\rho_0}\Sigma^c(T) \\
&- \delta\left[\ln\left\{\cosh\left(\frac{-h-z_0}{\delta}\right)\right\}-\ln\left\{\cosh\left(\frac{\zeta-z_0}{\delta}\right)\right\}\right]^{n+1}\Sigma^c\left(v_\alpha\right), \qquad (68)
\end{aligned}
$$

where $\Sigma^p(\phi)=2\phi^n-3\phi^{n-1}+\phi^{n-2}$ and $\Sigma^c(\phi)=\phi^{n+1}-\phi^n$.

## 2.2 Spatial Discretization: Finite-Volume Method

Recently, finite-volume schemes coupled with Riemann solvers have been successfully applied to shallow-water (Erduran et al., 2005; Dutykh et al., 2011) and Boussinesq-type (Tonelli and Petti, 2009; Kim et al., 2009; Shi et al., 2012) equations and have exhibited relatively robust performance. For the shallow-water terms embedded in Equations 43, 44, and 45, a fourth-order compact MUSCL-TVD scheme has been applied and combined with the HLL Riemann solver (see Kim et al., 2009). The remaining terms, including higher order spatial derivatives, are differenced by the cell-averaged finite-volume method proposed by Lacor et al. (2003).

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