Supplementary Materials for: Global modes, receptivity, and sensitivity analysis of diffusion flames coupled with duct acoustics

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1 Galerkin-discretized equations

The discretized linear system, $\boldsymbol{\chi}$, evolves according to

$$\mathcal{M}\dot{\boldsymbol{\chi}} = B\boldsymbol{\chi}. \tag{1}$$

where the overdot is the temporal derivative. Matrices M and B are

$$\mathcal{M} = I - \frac{1}{2T_{av}}T, \tag{2}$$

$$B = -D - A - E - F + \frac{1}{2T_{av}}U + \frac{1}{2T_{av}}J, \tag{3}$$

and I is the identity matrix. All matrices in (2), (3) have dimension $(NM + 2K) \times (NM + 2K)$, where N is the number of Galerkin modes with which the flame is discretized along η (including the zero-th mode), M is the number of Galerkin modes with which the flame is discretized along ξ, and K is the number of Galerkin modes for the acoustics. In addition, we define $n = 0, 1, 2, \ldots, (N - 1)$ and $m, l = 1, 2, \ldots, M$;

First, we define a row vector $\mathbf{c}$, whose dimension is $1 \times (NM + 2K)$:

$$c_{1j} = \begin{cases} \cos(\omega_j \sqrt{\rho_1} x_f), & \text{if } j = (NM + 1), \ldots, (NM + K), \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

and a column vector $\mathbf{s}$, whose dimension is $(NM + 2K) \times 1$:

$$s_{j1} = \begin{cases} \sin(\omega_j \sqrt{\rho_1} x_f)/E_j, & \text{if } j = (NM + K + 1), \ldots, (NM + 2K), \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where E_j is the acoustic pressure energy given, similarly to Zhao (2012), by

$$E_j = \frac{1}{2}\left[x_f + (1 - x_f)\left(\frac{\sin\gamma_j}{\sin\beta_j}\right)^2\right]. \tag{6}$$

1.1 Diffusion matrix

The following matrix represents the discretized diffusion term in the linearized z-equation. This matrix couples $G_{n,m}$ to $\dot{G}_{n,m}$:

$$D_{pq} = \begin{cases} \frac{1}{\mathrm{Pe}}\left[\frac{\left(m-\frac{1}{2}\right)^2\pi^2}{L_c^2} + n^2\pi^2\right], & \text{if } p,q = nM+m, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

1.2 Advection matrices

The following matrices represent the advective terms in the linearized z-equation. They couple $G_{n,m}$ to $\dot{G}_{n,m}$:

$$C_{nm} = \frac{2}{L_c}\int_0^{L_c}\int_{-1}^{1}\frac{\partial \bar{Z}}{\partial \xi}\sin\left[\left(m-\frac{1}{2}\right)\frac{\pi\xi}{L_c}\right]\cos(n\pi\eta)\,\mathrm{d}\xi\mathrm{d}\eta, \tag{8}$$

$$W_{ml} = \frac{2}{L_c}\left(l-\frac{1}{2}\right)\frac{\pi}{L_c}\int_0^{L_c}\sin\left[\left(m-\frac{1}{2}\right)\frac{\pi\xi}{L_c}\right]\cos\left[\left(l-\frac{1}{2}\right)\frac{\pi\xi}{L_c}\right]\mathrm{d}\xi. \tag{9}$$

Integration of (9) yields:

$$W_{ml} = \begin{cases} \frac{2}{L_c}\left(l-\frac{1}{2}\right)\left[\frac{1-\cos[(m+l-1)\pi]}{2(m+l-1)} + \frac{1-\cos[(m-l)\pi]}{2(m-l)}\right], & \text{if } l \neq m, \\ \frac{1}{2}\frac{2}{L_c}, & \text{if } l = m. \end{cases} \tag{10}$$

Matrix C_{nm} is arranged as a column vector $\mathbf{y}$ whose dimension is $(NM+2K)\times 1$. This vector consists of the concatenated rows of C:

$$y_{j1} = \begin{cases} [C_{01}, C_{02}, \ldots, C_{0M}, \ldots, C_{21}, C_{22}, \ldots, C_{2M}, \ldots, C_{NM}], & \text{if } j = 1, \ldots, NM, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

where C_{0m} coefficients are zero. Matrix E is defined as:

$$E = \mathbf{y} \otimes \mathbf{c}. \tag{12}$$

Finally, matrix A is given by

$$A_{pq} = \begin{cases} W_{ml}, & \text{if } p = nM+m \text{ and } q = nM+l, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

1.3 Heat-release matrices

The following term represents the heat release integrated over the flame domain:

$$R_{nm} = \begin{cases} \frac{-1}{1-Z_{sto}}\int_0^{L_c}\int_{-\eta_{sto}}^{\eta_{sto}}\sin\left[\left(m-\frac{1}{2}\right)\frac{\pi\xi}{L_c}\right]\cos(n\pi\eta)\,\mathrm{d}\xi\mathrm{d}\eta & \text{if } n \neq 0, \\ \frac{2L_c}{\left(m-\frac{1}{2}\right)}(1-c), & \text{if } n = 0, \end{cases} \tag{14}$$

where η_{sto} is the vertical coordinate of the stoichiometric curve $\bar{Z} = Z_{sto}$; $c = 0$ for closed flames, $c = -1/\left(1 - Z_{sto}\right)\cos\left[\left(m - \frac{1}{2}\right)\frac{\pi}{2}\frac{L_f}{L_c}\right]$ for open flames, where L_f is the length of the flame. Matrix R_{nm} is arranged as a row vector $\mathbf{r}$ whose dimension is $1 \times (NM + 2K)$. This vector consists of the concatenated rows of R:

$$r_{1j} = \begin{cases} [R_{01}, R_{02}, \dots, R_{0M}, \dots, R_{21}, R_{22}, \dots, R_{2M}, \dots, R_{NM}], & \text{if } j = 1, \dots, NM, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Matrix T, which couples $\dot{G}_{n,m}$ with $\dot{\alpha}_j$, is then given by:

$$T = \mathbf{s} \otimes \mathbf{r}. \tag{16}$$

Matrix U, which couples η_j with $\dot{\alpha}_j$, is

$$U = \bar{Q}\mathbf{s} \otimes \mathbf{c}, \tag{17}$$

where $\bar{Q}$ is the steady heat release. Matrix J is

$$J = \mathbf{s} \otimes \mathbf{j}, \tag{18}$$

where row vector $\mathbf{j}$ has non-trivial components associated only with mode $n = 0$, which are

$$j_{0m} = \begin{cases} 2\left(-1\right)^{(m-1)}, & \text{closed flame,} \\ 2\frac{-Z_{sto}}{1-Z_{sto}}\left(-1\right)^{(m-1)}, & \text{open flame.} \end{cases} \tag{19}$$

1.4 Acoustics matrices

$$S_{ij} = \omega_j \delta_{ij}, \tag{20}$$

$$Z_{ij} = \zeta_j \delta_{ij}, \tag{21}$$

where $i, j = 1, 2, \dots, K$ and δ_{ij} is the Kronecker delta. Finally, matrix F is

$$F = \begin{bmatrix} [0]_{NM\times NM} & [0]_{NM\times 2K} \\ [0]_{K\times K} & S \\ -S & -Z \end{bmatrix}.$$

2 Wave approach

Figure 1 shows the acoustic geometry with the nomenclature required to solve the thermo-acoustic problem via the wave approach (see e.g. Dowling, 1995). Steady heat release gives rise to a change in temperature across the flame (and therefore a change in the mean speed of sound from $\bar{c}_1$ before combustion to $\bar{c}_2$

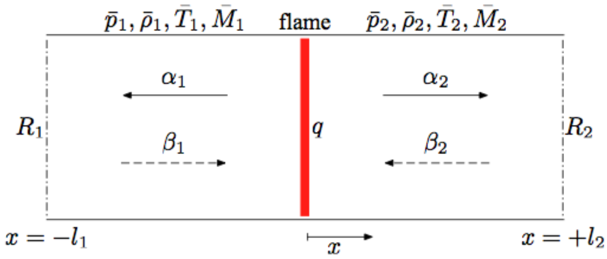

Figure 1: Upstream- and downstream-travelling waves in a duct geometry with reflection coefficients R_1 and R_2 under the compact-flame assumption.

after combustion). A perturbation in the heat release, $q'(t)$, generates outward-travelling waves that propagate both upstream α_1 and downstream α_2. These waves are reflected when they reach the upstream/downstream ends of the duct. This reflection is characterized by the reflection coefficients, R_1 and R_2, and gives rise to reflected waves β_1 and β_2. Notice that we have defined the spatial coordinate system, x, such that $x = 0$ at the flame. Then $x \in [-l_1, 0]$ upstream of the flame and $x \in [0, l_2]$ downstream of the flame. This makes the analysis easier. We solve the wave equation in each of the two regions upstream and downstream of the flame shown in figure 1. Considering the region upstream of the flame, the acoustic pressure and velocity perturbations can be written in terms of the upstream- and downstream-travelling waves, $\alpha_1(t, x)$ and $\beta_1(t, x)$:

$$p_1' = \alpha_1 \left(t + \frac{x}{\bar{c}_1 - \bar{u}_1} \right) + \beta_1 \left(t - \frac{x}{\bar{c}_1 + \bar{u}_1} \right), \tag{22}$$

$$u_1' = \frac{1}{\bar{\rho}_1 \bar{c}_1} \left(-\alpha_1 \left(t + \frac{x}{\bar{c}_1 - \bar{u}_1} \right) + \beta_1 \left(t - \frac{x}{\bar{c}_1 + \bar{u}_1} \right) \right), \tag{23}$$

$$\rho_1' = \frac{1}{\bar{c}_1^2} p_1'. \tag{24}$$

We can write similar expressions for the downstream region:

$$p_2' = \alpha_2 \left(t - \frac{x}{\bar{c}_2 + \bar{u}_2} \right) + \beta_2 \left(t + \frac{x}{\bar{c}_2 - \bar{u}_2} \right), \tag{25}$$

$$u_2' = \frac{1}{\bar{\rho}_2 \bar{c}_2} \left(\alpha_2 \left(t - \frac{x}{\bar{c}_2 + \bar{u}_2} \right) - \beta_2 \left(t + \frac{x}{\bar{c}_2 - \bar{u}_2} \right) \right), \tag{26}$$

$$\rho_2' = \frac{1}{\bar{c}_2^2} p_2'. \tag{27}$$

Boundary conditions are required at the ends of the duct and across the flame. The boundary condition at the upstream end, $x = -l_1$, is:

$$\beta_1\left(t + \frac{l_1}{\bar{c}_1 + \bar{u}_1}\right) = R_1\alpha_1\left(t - \frac{l_1}{\bar{c}_1 - \bar{u}_1}\right), \tag{28}$$

$$\implies \beta_1(t) = R_1\alpha_1\left(t - \tau_1\right), \tag{29}$$

where $\tau_1 = 2l_1\bar{c}_1/\left(\bar{c}_1^2 - \bar{u}_1^2\right)$. Similarly, at the downstream end $x = l_2$, we have:

$$\beta_2\left(t + \frac{l_2}{\bar{c}_2 - \bar{u}_2}\right) = R_2\alpha_2\left(t - \frac{l_2}{\bar{c}_2 + \bar{u}_2}\right), \tag{30}$$

$$\implies \beta_2(t) = R_2\alpha_2\left(t - \tau_2\right), \tag{31}$$

where $\tau_2 = 2l_2\bar{c}_2/\left(\bar{c}_2^2 - \bar{u}_2^2\right)$.

At the flame $x = 0$, the integral momentum and energy equations are, respectively (see e.g. Dowling, 1997)

$$p_2 - p_1 + \rho_1 u_1\left(u_2 - u_1\right) = 0 \tag{32}$$

$$\frac{\gamma}{\gamma - 1}\left(p_2u_2 - p_1u_1\right) + \frac{1}{2}\rho_1u_1\left(u_2^2 - u_1^2\right) = Q/A, \tag{33}$$

Linearizing (32),(33); substituting the wave solutions (22)–(31); taking the Laplace transform in the σ-domain, gives

$$\begin{bmatrix} X_{11} - R_1Y_{11}\exp(-\sigma\tau_1) & X_{12} - R_2Y_{12}\exp(-\sigma\tau_2) \\ X_{21} - R_1Y_{21}\exp(-\sigma\tau_1) & X_{22} - R_2Y_{22}\exp(-\sigma\tau_2) \end{bmatrix} \begin{pmatrix} \alpha_1(\sigma) \\ \alpha_2(\sigma) \end{pmatrix} = \begin{pmatrix} 0 \\ q(\sigma)/\bar{c}_1 \end{pmatrix} \tag{34}$$

where

$$\begin{aligned}
X_{11} &= -1 + \bar{M}_1\left(2 - \frac{\bar{u}_2}{\bar{u}_1}\right) - \bar{M}_1^2\left(1 - \frac{\bar{u}_2}{\bar{u}_1}\right), \\
X_{12} &= 1 + \bar{M}_2, \\
X_{21} &= \frac{1 - \gamma\bar{M}_1}{\gamma - 1} + \bar{M}_1^2 - \frac{1}{2}\bar{M}_1^2\left(1 - \bar{M}_1\right)\left(\frac{\bar{u}_2^2}{\bar{u}_1^2} - 1\right), \\
X_{22} &= \frac{\bar{c}_2}{\bar{c}_1}\left(\frac{1 + \gamma\bar{M}_2}{\gamma - 1} + \bar{M}_2^2\right), \\
Y_{11} &= 1 + \bar{M}_1\left(2 - \frac{\bar{u}_2}{\bar{u}_1}\right) + \bar{M}_1^2\left(1 - \frac{\bar{u}_2}{\bar{u}_1}\right), \\
Y_{12} &= -1 + \bar{M}_2, \\
Y_{21} &= \frac{1 + \gamma\bar{M}_1}{\gamma - 1} + \bar{M}_1^2 - \frac{1}{2}\bar{M}_1^2\left(1 + \bar{M}_1\right)\left(\frac{\bar{u}_2^2}{\bar{u}_1^2} - 1\right), \\
Y_{22} &= \frac{\bar{c}_2}{\bar{c}_1}\left(\frac{1 - \gamma\bar{M}_2}{\gamma - 1} + \bar{M}_2^2\right),
\end{aligned} \tag{35}$$

The acoustic angular frequencies are the imaginary parts of the eigenvalues of (34), by setting no damping, i.e. $R_1 = R_2 = -1$.

(Note that in paper DOI 10.1017/jfm.2014.328, which this supplementary material is associated with, we used a little different notation, i.e. $p' \rightarrow p$, $u' \rightarrow u$, $\rho' \rightarrow \rho$, $\bar{M} \rightarrow M$.)

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