Supplementary information: full mathematical details to

The Graetz-Nusselt problem extended to continuum flows with finite slip

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1 GOVERNING EQUATION

The Graetz-Nusselt problem considers transport of heat between a fluid and a tube, beginning at some location in the hydrodynamically developed flow region. The tube entrance length is unheated, and the wall temperature T_1 of the heated or cooled section is constant but different from that of the entering fluid temperature T_0. The fluid has constant physical properties, and viscous dissipation and axial heat conduction are neglected. In this study we extend this classical problem to fluid flows with a finite slip velocity u_s at the wall at $r = R$. Wall slip can be quantified by a slip length b, which is defined according to Navier's slip condition,

$$u_s = -b \left.\frac{\partial u}{\partial r}\right|_{r=R}. \tag{1.1}$$

The governing equation describing stationary heat transport in such an axisymmetric cylindrical system, under the assumption of constant density ρ and thermal conductivity k, can be written as

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{1.2}$$

where $u(r,b)$ decribes the velocity profile of the laminar fluid flowing in the x-direction, and $\alpha = k/(\rho C_p)$ is the thermal diffusivity. The initial and boundary condition are $T(0,r) = T_0$ and $T(x,R) = T_1$.

Solving the Navier-Stokes equation for stationary slip flow in the axial direction, again assuming constant ρ and k, yields the following expression for the velocity profile,

$$\tilde{u} = \frac{2\left(1-\tilde{r}^2\right)+4\tilde{b}}{1+4\tilde{b}}, \tag{1.3}$$

where $\tilde{u} = u/u_{av}$, $\tilde{r} = r/R$, and $\tilde{b} = b/R$. $\tilde{u}$ can written as the sum of the variable velocity $\tilde{u}_v(\tilde{r},\tilde{b})$ and the slip velocity $\tilde{u}_s(\tilde{b})$ at the wall:

$$\tilde{u} = \tilde{u}_v + \tilde{u}_s = \frac{2\left(1-\tilde{r}^2\right)}{1+4\tilde{b}} + \frac{4\tilde{b}}{1+4\tilde{b}}. \tag{1.4}$$

The governing heat equation can now be non-dimensionalised using $\Theta = (T_1 - T)/(T_1 - T_0)$ and $\tilde{x} = x/L$ (L being the length of the heated or cooled section of the pipe),

$$\frac{\left(1-\tilde{r}^2\right)+2\tilde{b}}{2(1+4\tilde{b})}\frac{\partial\Theta}{\partial\left(\tilde{x}/Gz\right)} = \frac{1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\frac{\partial\Theta}{\partial\tilde{r}}\right), \tag{1.5}$$

with $\Theta(0,\tilde{r}) = 1$ and $\Theta(\tilde{x},1) = 0$. The Graetz number Gz is defined as

$$Gz = RePr\frac{D}{L} = 4\frac{u_{av}R^2}{\alpha L}, \tag{1.6}$$

where $Re = u_{av}D/\nu$ is the Reynolds number, $Pr = \nu/\alpha$ is the Prandtl number, and D is the diameter of the tube. For $\tilde{x}/Gz < 0.01$ the fluid flow is thermally developing, while for $\tilde{x}/Gz > 0.1$ the fluid flow is said to be thermally developed.

2 NUSSELT NUMBER

The local Nusselt number Nu_x is defined, by definition, as

$$Nu_x = \frac{h_x D}{k}, \tag{2.1}$$

where h_x is the local heat transfer coefficient, being a function of the dimensionless position $\tilde{x}/Gz$. Following Newton's law of cooling and using Fourier's Law of thermal conduction, the heat transfer coefficient h_x can be rewritten as

$$h_x = \frac{q"}{\Delta T} = -\frac{k}{\langle T\rangle - T_1} \left.\frac{\partial T}{\partial r}\right|_{r=R}, \tag{2.2}$$

where q" represents the local heat flux between the wall and the fluid. Rewriting the temperature gradient in dimensionless form gives

$$h_x = -\frac{2k}{D}\frac{1}{\langle\Theta\rangle} \left.\frac{\partial \Theta}{\partial \tilde{r}}\right|_{\tilde{r}=1}, \tag{2.3}$$

where $\langle\Theta\rangle = (T_1 - \langle T\rangle)/(T_1 - T_0)$. Here, it is always true that $\partial_{\tilde{r}}\Theta \leq 0$. Now, Nu_x can be written as

$$Nu_x = -\frac{2}{\langle\Theta\rangle} \left.\frac{\partial \Theta}{\partial \tilde{r}}\right|_{\tilde{r}=1}, \tag{2.4}$$

where $\langle\Theta\rangle\,(\tilde{x}/Gz)$. The dimensionless flow-averaged or mixing-cup temperature $\langle\Theta\rangle$ can be calculated according to

$$\langle\Theta\rangle = \frac{\int_0^1 \Theta(\tilde{x}/Gz, \tilde{r})\tilde{u}(\tilde{r})\tilde{r}d\tilde{r}}{\int_0^1 \tilde{u}(\tilde{r})\tilde{r}d\tilde{r}}. \tag{2.5}$$

When $\tilde{x}/Gz > 0.1$, $Nu_x \to Nu_\infty$.

3 Lévêque approximation

3.1 Governing equation and velocity profile

To solve the temperature field near the very entrance of the pipe the Lévêque approximation is followed. This involves the following assumptions:

- curvature effects are neglected;
- infinite bulk is assumed;
- the velocity profile is regarded as linear, with the slope given by the slope of the velocity profile at the wall.

Then governing equation can be solved in Cartesian coordinates:

$$u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}. \tag{3.1}$$

The wall at $r = R$ is now located at $y = 0$, *i.e.* the directions of the r- and y-axis are reversed.

Under the assumptions listed above, the velocity profile can be writted as

$$\tilde{u} = \frac{4}{1+4\tilde{b}}(\tilde{y}+\tilde{b}), \tag{3.2}$$

where $\tilde{y} = y/R$.

Three different flow regimes can be distinguished:

1. no-slip, *i.e.* $\tilde{b} = 0$ and hence $\tilde{u} = 4\tilde{y}$;
2. finite slip, *i.e.* $0 < \tilde{b} < \infty$;
3. no-shear, *i.e.* $\tilde{b} = \infty$ and thus $\tilde{u} = 1$.

In the next sections an analytical expression for the local Nusselt number Nu_x is derived for each flow regime. In general,

$$Nu_x \propto \left(\frac{\tilde{x}}{Gz}\right)^{-\beta}. \tag{3.3}$$

The exponent β is constant for the two limiting cases and equals $1/3$ for no-slip flow, and $1/2$ for no-shear flow. In case of slip flow, $\beta = f(\tilde{x}/Gz, \tilde{b})$.

3.2 No-slip

Making the governing energy equation dimensionless results in the following differential equation for no-slip flow,

$$Gz\tilde{y}\frac{\partial \Theta}{\partial \tilde{x}} = \frac{\partial^2 \Theta}{\partial \tilde{y}^2}, \tag{3.4}$$

with $\Theta(0,\tilde{y}) = 1$, $\Theta(\tilde{x},0) = 0$, and $\Theta(\tilde{x},\infty) = 1$.

By introducing a similarity variable η, where

$$\eta = \left(\frac{Gz\tilde{y}^3}{9\tilde{x}}\right)^{\frac{1}{3}}, \tag{3.5}$$

and subsequent rewriting of the energy balance, we obtain the following ODE:

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\eta^2} + 3\eta^2 \frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = 0, \tag{3.6}$$

with boundary conditions $\Theta(0) = 0$ and $\Theta(\infty) = 1$. This ordinary differential equation has a known solution, and with the boundary conditions given the final solution for Θ becomes

$$\Theta = \frac{1}{\Gamma(\frac{4}{3})} \int_0^\eta \exp(-\bar{\eta}^3)\, \mathrm{d}\bar{\eta}. \tag{3.7}$$

Previously, an expression for the Nusselt number was found using the dimensionless temperature gradient $\partial_{\tilde{r}}\Theta$. Rewriting this in terms of $\partial_{\tilde{y}}\Theta$, the expression for Nu becomes

$$Nu_x = \frac{2}{\langle\Theta\rangle} \left.\frac{\partial\Theta}{\partial\tilde{y}}\right|_{\tilde{y}=0}. \tag{3.8}$$

However, in the Lévêque approximation $\langle\Theta\rangle = 1$. Furthermore, the dimensionless temperature gradient is rewritten in terms of $\mathrm{d}_\eta\Theta$. Then

$$Nu_x = \frac{2\eta}{\tilde{y}} \left.\frac{\mathrm{d}\Theta}{\mathrm{d}\eta}\right|_{\eta=0}. \tag{3.9}$$

Always the temperature gradient $\mathrm{d}_\eta\Theta > 0$. Now, using the Leibniz formula for differentiation of integrals, an expression for the temperature gradient can be found:

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = \frac{\exp(-\eta^3)}{\Gamma(\frac{4}{3})}. \tag{3.10}$$

Evaluating this at $\eta = 0$ and substituting this in the expression for Nu_x we find

$$Nu_x = \frac{2}{\Gamma(\frac{4}{3})} \frac{\eta}{\tilde{y}}, \tag{3.11}$$

or,

$$Nu_x = \frac{2}{9^{\frac{1}{3}}\Gamma(\frac{4}{3})} \left(\frac{\tilde{x}}{Gz}\right)^{-\frac{1}{3}}. \tag{3.12}$$

Thus, we find that for the no-slip regime the exponent is $\beta = 1/3$.

3.3 No-shear

For the no-shear case, which implies that we have plug flow, the velocity profile is uniform. Rewriting the governing equation yields the following dimensionless PDE:

$$\frac{Gz}{4}\frac{\partial \Theta}{\partial \tilde{x}} = \frac{\partial^2 \Theta}{\partial \tilde{y}^2}, \tag{3.13}$$

with $\Theta(0,\tilde{y}) = 1$, $\Theta(\tilde{x},0) = 0$, and $\Theta(\tilde{x},\infty) = 1$. Using the similarity variable

$$\eta = \left(\frac{Gz\tilde{y}^2}{16\tilde{x}}\right)^{\frac{1}{2}}, \tag{3.14}$$

this PDE turns into the following ODE having a known solution,

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\eta^2} + 2\eta\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = 0, \tag{3.15}$$

with $\Theta(0) = 0$ and $\Theta(\infty) = 1$. Solving this ODE gives

$$\Theta = \frac{2}{\sqrt{\pi}}\int_0^\eta \exp(-\bar{\eta}^2)\,\mathrm{d}\bar{\eta} = \mathrm{erf}(\eta). \tag{3.16}$$

In order to find an expression for Nu_x, we need an equation for the temperature gradient. This is

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} = \frac{2}{\sqrt{\pi}}\exp(-\eta^2). \tag{3.17}$$

Ultimately this results in the following expression for Nu_x:

$$Nu_x = \frac{1}{\sqrt{\pi}}\left(\frac{\tilde{x}}{Gz}\right)^{-\frac{1}{2}}. \tag{3.18}$$

Here, $\beta = 1/2$.

3.4 Finite slip

To find an analytical expression for Nu_x for fluid flow with finite slip, we start by rewriting the expression for the velocity profile:

$$u = 4u_{av}\frac{y+b}{R+4b}. \tag{3.19}$$

Plugging this into the governing equation for heat transport, and using the dimensionless variables

$$\Theta = \frac{T_1 - T}{T_1 - T_0}, \tag{3.20}$$

$$Y = \frac{y}{b}, \tag{3.21}$$

$$X = x\frac{\alpha(R+4b)}{4u_{av}b^3}, \tag{3.22}$$

we can non-dimensionalize the governing equation. This yields

$$(1+Y)\frac{\partial\Theta}{\partial X} = \frac{\partial^2\Theta}{\partial Y^2}, \tag{3.23}$$

with $\Theta(0,Y) = 1$, $\Theta(X,0) = 0$, and $\Theta(X,\infty) = 1$. To reduce the number of variables, we perform a Laplace transformation of $\partial_X\Theta$ in X,

$$\mathscr{L}_X\left[\frac{\partial\Theta}{\partial X}\right] = \int_0^\infty \frac{\partial\Theta}{\partial X}e^{-pX}\ \mathrm{d}X = p\bar{\Theta}(p,Y) - \Theta(0,Y) = p\bar{\Theta}(p,Y) - 1, \tag{3.24}$$

where $\bar{\Theta}$ is the Laplace transform of Θ,

$$\bar{\Theta}(p,Y) = \mathscr{L}_X[\Theta] = \int_0^\infty \Theta(X,Y)e^{-pX}\ \mathrm{d}X. \tag{3.25}$$

Furthermore,

$$\mathscr{L}_X\left[\frac{\partial^2\Theta}{\partial Y^2}\right] = \frac{\partial^2\bar{\Theta}}{\partial Y^2}. \tag{3.26}$$

The governing equation now becomes

$$(1+Y)(p\bar{\Theta} - 1) = \frac{\partial^2\bar{\Theta}}{\partial Y^2}, \tag{3.27}$$

with $\bar{\Theta}(p,0) = 0$ and $\bar{\Theta}(p,\infty) = 1/p$. To convert this into an ODE with a known solution, we change variable by introducing

$$\hat{\Theta} = \bar{\Theta} - \frac{1}{p}. \tag{3.28}$$

Now we obtain

$$p(1+Y)\hat{\Theta} = \frac{\partial^2\hat{\Theta}}{\partial Y^2} \tag{3.29}$$

with $\hat{\Theta}(p,0) = -1/p$ and $\hat{\Theta}(p,\infty) = 0$. Now we change variable a second time by defining

$$\eta = p^{1/3}(1+Y). \tag{3.30}$$

Following Faà di Bruno's formula for second order derivatives saying

$$\frac{\partial^2\hat{\Theta}}{\partial Y^2} = \frac{\partial^2\hat{\Theta}}{\partial\eta^2}\left(\frac{\partial\eta}{\partial Y}\right)^2 + \frac{\partial\hat{\Theta}}{\partial\eta}\frac{\partial^2\eta}{\partial Y^2}, \tag{3.31}$$

we find

$$\frac{\mathrm{d}^2\hat{\Theta}}{\mathrm{d}\eta^2} - \eta\hat{\Theta} = 0 \tag{3.32}$$

with $\hat{\Theta}(p^{1/3}) = -1/p$ and $\hat{\Theta}(\infty) = 0$. This ODE is known as the Airy equation, whose general solution of the first kind is the Airy function $\mathrm{Ai}(\eta)$,

$$\mathrm{Ai}(\eta) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \exp\left(\eta z - \frac{z^3}{3}\right) \, \mathrm{d}z, \tag{3.33}$$

or,

$$\mathrm{Ai}(\eta) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\eta z + \frac{z^3}{3}\right) \, \mathrm{d}z, \tag{3.34}$$

with $\lim_{\eta\to\infty} \mathrm{Ai}(\eta) = 0$. Then the solution for $\hat{\Theta}$ becomes

$$\hat{\Theta} = -\frac{\mathrm{Ai}(p^{1/3}(1+Y))}{p\mathrm{Ai}(p^{1/3})}, \tag{3.35}$$

and

$$\bar{\Theta} = \frac{\mathrm{Ai}(p^{1/3}) - \mathrm{Ai}(p^{1/3}(1+Y))}{p\mathrm{Ai}(p^{1/3})}. \tag{3.36}$$

This expression for $\bar{\Theta}$ recovers both $\bar{\Theta}(p,0) = 0$ and $\bar{\Theta}(p,\infty) = 1/p$. To obtain Θ, we take the inverse Laplace transform in X, giving

$$\Theta(X,Y) = \mathscr{L}_X^{-1}\left[\bar{\Theta}\right] = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{Ai}(p^{1/3}) - \mathrm{Ai}(p^{1/3}(1+Y))}{p\mathrm{Ai}(p^{1/3})} e^{pX} \, \mathrm{d}p. \tag{3.37}$$

For Nu_x we can derive that

$$Nu_x = \frac{2}{\tilde{b}} \left.\frac{\partial\Theta}{\partial Y}\right|_{Y=0}, \tag{3.38}$$

where, following Leibniz' integral rule,

$$\frac{\partial\Theta}{\partial Y} = -\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{Ai}'(p^{1/3}(1+Y))e^{pX}}{p^{2/3}\mathrm{Ai}(p^{1/3})} \, \mathrm{d}p. \tag{3.39}$$

Then, finally, we obtain

$$Nu_x = -\frac{1}{\tilde{b}\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{Ai}'(p^{1/3})e^{pX}}{p^{2/3}\mathrm{Ai}(p^{1/3})} \, \mathrm{d}p, \tag{3.40}$$

or

$$Nu_x = \frac{2}{\tilde{b}} g(X), \tag{3.41}$$

with

$$g(X) = -\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{Ai}'(p^{1/3})e^{pX}}{p^{2/3}\mathrm{Ai}(p^{1/3})} \, \mathrm{d}p. \tag{3.42}$$

X turns out to be a function of $\tilde{x}/Gz$ and $\tilde{b}$, as

$$X = \frac{x}{L}\frac{\alpha L}{4u_{av}R^2}\frac{R^2(R+4b)}{b^3} = \left(\frac{\tilde{x}}{Gz}\right)\frac{1+4\tilde{b}}{\tilde{b}^3}. \quad (3.43)$$

The function $g(X)$ is universal, as it does not depend on the slip length b. The slip length affects the scaling between X and $\tilde{x}/Gz$, so it determines with part of the function $g(X)$ is relevant. When $\tilde{b} \to 0$, $X \to \infty$, while for $\tilde{b} \to \infty$, $X \to 0$. As a check, when $\tilde{b} \to 0$ we expect that

$$g(X) \propto X^{-1/3} \quad (3.44)$$

for large X. Then

$$Nu_x \propto \frac{1}{\tilde{b}}\left(\frac{\tilde{x}}{Gz}\frac{1}{\tilde{b}^3}\right)^{-1/3} = \left(\frac{\tilde{x}}{Gz}\right)^{-1/3}. \quad (3.45)$$

When $\tilde{b} \to \infty$ we expect that

$$g(X) \propto X^{-1/2} \quad (3.46)$$

for small X. Then

$$Nu_x \propto \frac{1}{\tilde{b}}\left(\frac{\tilde{x}}{Gz}\frac{4\tilde{b}}{\tilde{b}^3}\right)^{-1/2} = \left(\frac{\tilde{x}}{Gz}\right)^{-1/2}. \quad (3.47)$$

To evaluate $g(X)$, let $p = s^3$. Then

$$g(X) = -\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{Ai}'(s)e^{s^3X}}{s^2\mathrm{Ai}(s)}3s^2\,\mathrm{d}s = -\frac{3}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathrm{Ai}'(s)e^{s^3X}}{\mathrm{Ai}(s)}\,\mathrm{d}s. \quad (3.48)$$

The integral should start in the sector $-\pi/2 < \arg(s) < -\pi/6$ and finish in the sector $\pi/6 < \arg(s) < \pi/2$. The convergence properties of the integrand are best if we integrate along the rays $\arg(s) = \pm\pi/3$. On the ray $\infty e^{-\mathrm{i}\pi/3}$ to the origin we let $s = ke^{-\mathrm{i}\pi/3}$ with k going from ∞ to 0. On the ray from the origin to $\infty e^{\mathrm{i}\pi/3}$ we let $s = ke^{\mathrm{i}\pi/3}$ with k going from 0 to ∞. Thus

$$g(X) = -\frac{3e^{-\mathrm{i}\pi/3}}{2\pi\mathrm{i}}\int_{\infty}^{0}\frac{\mathrm{Ai}'(ke^{-\mathrm{i}\pi/3})e^{-k^3X}}{\mathrm{Ai}(ke^{-\mathrm{i}\pi/3})}\,\mathrm{d}k - \frac{3e^{\mathrm{i}\pi/3}}{2\pi\mathrm{i}}\int_{0}^{\infty}\frac{\mathrm{Ai}'(ke^{\mathrm{i}\pi/3})e^{-k^3X}}{\mathrm{Ai}(ke^{\mathrm{i}\pi/3})}\,\mathrm{d}k, \quad (3.49)$$

or

$$g(X) = \frac{3e^{-5\mathrm{i}\pi/6}}{2\pi}\int_{0}^{\infty}\frac{\mathrm{Ai}'(ke^{-\mathrm{i}\pi/3})e^{-k^3X}}{\mathrm{Ai}(ke^{-\mathrm{i}\pi/3})}\,\mathrm{d}k + \frac{3e^{5\mathrm{i}\pi/6}}{2\pi}\int_{0}^{\infty}\frac{\mathrm{Ai}'(ke^{\mathrm{i}\pi/3})e^{-k^3X}}{\mathrm{Ai}(ke^{\mathrm{i}\pi/3})}\,\mathrm{d}k. \quad (3.50)$$

The integrand decays rapidly and can be evaluated numerically. In FIG. 3.1 the function $g(X)$ is plotted as function of X. As demonstrated in the graph, the limiting behaviour of $g(X)$ is as expected: when $X \to 0$, $g(X) \propto X^{-1/2}$, and when $X \to \infty$, $g(X) \propto X^{-1/3}$. β, as obtained by computing the gradient of $\log_{10}(Nu_x)$ versus $\log_{10}(\tilde{x}/Gz)$, is plotted in FIG. 3.2. $\beta \approx 5/12$ when $X = 1$. Thus, the dimensionless group X can be considered as a kind of criterion for the behaviour of β: $\beta \to 1/2$ when $X \ll 1$, whereas $\beta \to 1/3$ when $X \gg 1$.

The expression

$$G(X) = \frac{0.546\left(X^{1/2.420}+1.058\right)^{2.420/6}}{X^{1/2}} \quad (3.51)$$

is a good approximation to $g(X)$, which is accurate to about 0.5% for $10^{-4} < X < 10^4$.

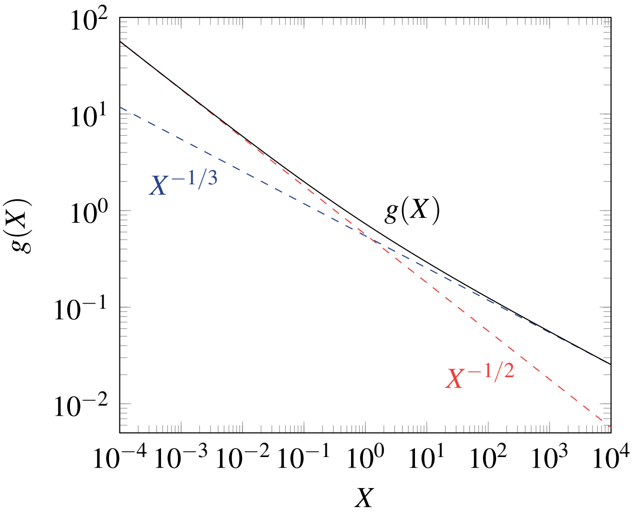

Figure 3.1: $g(X)$ as function of X.

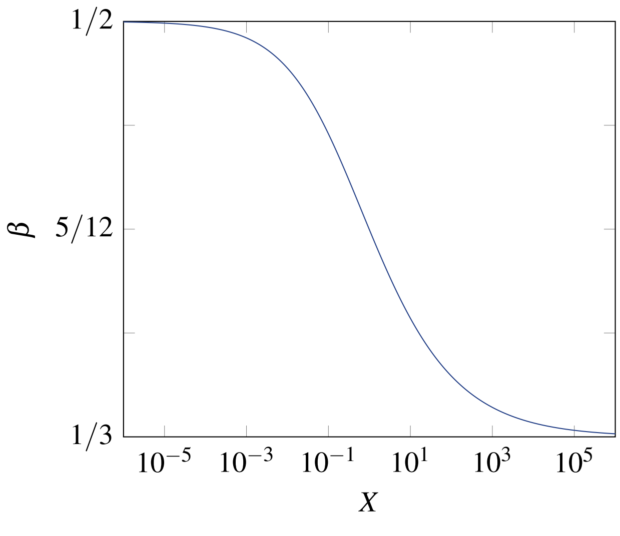

Figure 3.2: β as function of X as obtained from $g(X)$.

4 NUMERICAL PROCEDURE

Numerically, the Graetz-Nusselt problem for finite slip was solved using the pdepe-solver in MATLAB. It is required that the equation to be solved has the following form:

$$c\left(\tilde{r},\tilde{x}/Gz,\Theta,\frac{\partial\Theta}{\partial\tilde{r}}\right)\frac{\partial\Theta}{\partial(\tilde{x}/Gz)}=\tilde{r}^{-m}\frac{\partial\Theta}{\partial\tilde{r}}\left[\tilde{r}^{m}f\left(\tilde{r},\tilde{x}/Gz,\Theta,\frac{\partial}{\partial\tilde{r}}\right)\right]+s\left(\tilde{r},\tilde{x}/Gz,\Theta,\frac{\partial\Theta}{\partial\tilde{r}}\right). \quad (4.1)$$

In this study, $c=\tilde{u}/4$, $f=\partial_{\tilde{r}}\Theta$, and $s=0$. The value of m depends on the coordinate system chosen to solve the equation. For cylindrical coordinates $m=1$. The initial and boundary conditions were defined according to

$$\Theta\left(\tilde{r},(\tilde{x}/Gz)_0\right)=\Theta_0\left(\tilde{r}\right)=1, \quad (4.2)$$

and

$$p\left(\tilde{r},\tilde{x}/Gz,\Theta\right)+q\left(\tilde{r},\tilde{x}/Gz,\right)f\left(\tilde{r},\tilde{x}/Gz,\Theta,\frac{\partial\Theta}{\partial\tilde{r}}\right)=0. \quad (4.3)$$

For the boundary condition $\Theta(1,\tilde{x}/Gz)=0$, $p=\Theta(1,\tilde{x}/Gz)$ and $q=0$. Furthermore, for $\partial_{\tilde{r}}\Theta(0,\tilde{x}/Gz)=0$, $p=0$ and $q=1$. However, boundedness of the solution near $\tilde{r}=0$ requires that the flux f vanishes at $\tilde{r}=0$. For $m>0$, pdepe imposes this boundary condition automatically and it ignores values specified for p and q. The relative and absolute tolerance for the pdepe-solver were set at respectively 10^{-6} and 10^{-12}. For calculating $\partial_{\tilde{r}}\Theta$ at $\tilde{r}=1$ the MATLAB-function pdeval was utilised.

Because heat transport mainly occurs near the inlet and near the wall, the $\tilde{x}\times\tilde{r}=82\times101$-mesh was refined near these boundaries. For $\tilde{x}=0$, the MATLAB logspace-script was used to create an logarithmically equally spaced grid in the x-direction. Refinement in the $\tilde{r}$-direction was obtained by using a linearly spaced vector (linspace) for $\tilde{r}_0$ and the following expression:

$$\tilde{r}=1-\left[0.99(1-\tilde{r}_0^2)+0.01(1-\tilde{r}_0)\right]. \quad (4.4)$$

The values for β_f in the thermally developing regime were obtained by fitting a straight line through $\log_{10}(Nu_x)$ for $-7\leq\log_{10}(\tilde{x}/Gz)\leq-4$ using the polyfit-algorithm.[1] β_l is the gradient of $\log_{10}(Nu_x)$ or $\log_{10}(g(X))$ versus $\log_{10}(\tilde{x}/Gz)$, computed using the 2nd order accurate gradient-algorithm. Nu_∞ is taken as the average of Nu_x for $\tilde{x}/Gz\geq0.1$. To compute the flow-averaged or mixing-cup temperature $\langle\Theta\rangle$ the trapz-script was utilised. Gradients of β and Nu_∞ versus $\tilde{b}$ were calculated using the gradient-algorithm.

The thermal boundary thickness $\tilde{\lambda}_T$ was calculated numerically from the temperature profile, which was approximated by linear intrapolation of the temperature to find $\tilde{r}(\Theta=0.99)$ using two the temperature points closest to $\Theta=0.99$.

[1] Note that changing the range of $\tilde{x}/Gz$-values used to compute β_f results in a different transition point for β_f. Using larger values shifts the transition point upwards. Nonetheless, the distance between the transition points for β_f and Nu_∞ remains at least one order of magnitude large. When using $-7\leq\log_{10}(\tilde{x}/Gz)\leq-4$, the transition point for β_f is located at $\tilde{b}=1.5\times10^{-2}$. This point is located at $\tilde{b}=0.7\times10^{-2}$ when using $-7\leq\log_{10}(\tilde{x}/Gz)\leq-6$. For $-5\leq\log_{10}(\tilde{x}/Gz)\leq-4$, it is located at $\tilde{b}=3\times10^{-2}$.