Supplementary Materials: "On damping of two-dimensional piston-mode sloshing in a rectangular moonpool under forced heave motions"

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To provide a self-contained narrative, the Supplementary materials repeat details of the Galerkin method by Faltinsen *et al.* (2007) with small modifications which allow for finding the primary harmonics of the steady-state solution of the quasi-linear boundary value problem. The method focuses on the first harmonics of the quasi-linear problem. All new quantities relative to Faltinsen *et al.* (2007) are framed.

We construct the 2π-periodic solution of the two-dimensional quasi-linear boundary $\boxed{(2.4)\text{-}(2.8)}$ in the mean liquid domain Q_0 shown in figure 1 with a focus on the $\cos t$ and $\sin t$ harmonics. The method employs dividing Q_0 into four subdomains I, II, III and IV by auxiliary interfaces T_1, T_2 and T_3 and setting appropriate transmission conditions on them and accounting for that the motions inside the $(II + III + IV)$-domain are described by

$$\psi_{II-IV}(x,z,t) = \psi|_{(x,z)\in(II+III+IV)} = \varphi^{(1)}(x,z)\cos t + \varphi^{(2)}(x,z)\sin t. \tag{0.1}$$

This makes it possible to define the Neumann traces of $\varphi^{(i)},\ i = 1,2$ on $T_j,\ j = 1,2,3$ as follows

$$T_1: \quad \frac{\partial \varphi^{(i)}}{\partial x}(-b,z) = w_1^{(i)}(z) \quad -h < z < -d, \tag{0.2a}$$

$$T_2: \quad \frac{\partial \varphi^{(i)}}{\partial x}(-\tfrac{1}{2},z) = w_2^{(i)}(z) \quad -h < z < -d, \tag{0.2b}$$

$$T_3: \quad \frac{\partial \varphi^{(i)}}{\partial z}(x,-d) = w_3^{(i)}(x) \quad -\tfrac{1}{2} < x < 0, \tag{0.2c}$$

(the six functions $w_j^{(i)},\ i = 1,2,\ j = 1,2,3$ belong to admissible functional spaces which provide the correctness of the corresponding boundary value problems) and reduce the original wave problem to a system of integral equations relative to $w_j^{(i)},\ i = 1,2,\ j = 1,2,3$.

In contrast to (0.1), $\psi_I(x,z,t) = \psi|_{(x,z)\in I}$ should include an outgoing-wave component. By separating spatial variables in the semi-infinite band and matching with solution (0.1) and the Neumann-traces (0.2a), one obtains the following solution in I as a function of

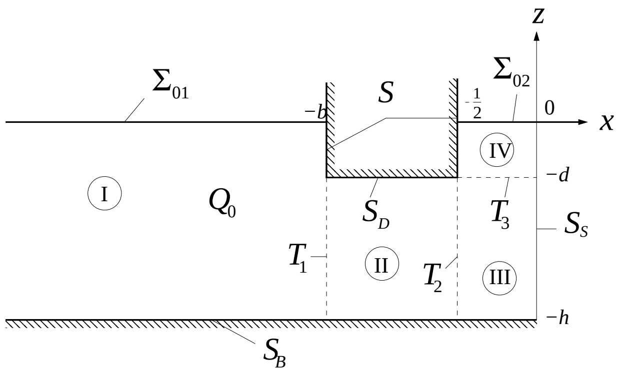

FIGURE 1. Geometric notations to the dimensionless problem (2.6)-(2.10).

$w_1^{(1)}$ and $w_1^{(2)}$

$$\psi_I(x,z,t) = \int_{-h}^{-d} \left(w_1^{(1)}(z_0)\cos t + w_1^{(2)}(z_0)\sin t \right) \; \mathcal{G}_I(x,z;z_0)dz_0 + \\ + \frac{\cosh(\mathcal{K}(z+h))}{\mathcal{K}N_0} \left[\sin(\mathcal{K}(x+b)+t) \int_{-h}^{-d} w_1^{(1)}(z_0)\cosh(\mathcal{K}(z_0+h))dz_0 - \right. \\ \left. - \cos(\mathcal{K}(x+b)+t) \int_{-h}^{-d} w_1^{(2)}(z_0)\cosh(\mathcal{K}(z_0+h))dz_0 \right], \quad (0.3)$$

where

$$\mathcal{G}_I(x,z;z_0) = \sum_{j=1}^{\infty} \frac{\cos(\kappa_j^{(1)}(z_0+h))\cos(\kappa_j^{(1)}(z+h))}{\kappa_j^{(1)}N_j^{(1)}} \exp(\kappa_j^{(1)}(x+b)). \quad (0.4)$$

$\mathcal{K}$ is the root of the transcendental equation

$$\mathcal{K}\tanh(\mathcal{K}h) = \Lambda \quad \text{and} \quad N_0 = \tfrac{1}{2}h(1+\sinh(2\mathcal{K}h)/(2\mathcal{K}h)), \quad (0.5)$$

and $\{\kappa_i^{(1)}\}$ are the positive roots of

$$\kappa_i^{(1)}\tan(\kappa_i^{(1)}h) = -\Lambda \qquad \text{and} \qquad N_i^{(1)} = \tfrac{1}{2}h(1+\sin(2\kappa_i^{(1)}h)/(2\kappa_i^{(1)}h)), \quad i \geqslant 1. \quad (0.6)$$

The pairs $(w_1^{(i)}, w_2^{(i)}),\ i=1,2$, constitute part of the Neumann boundary value problems for the Laplace equation in II. These problems have solutions (generally, to within unknown constants $\mathcal{A}_{-1}^{(i)}$), if and only if, the following solvability conditions is satisfied

$$-\int_{-h}^{-d} w_1^{(i)}(z_0)dz_0 + \int_{-h}^{-d} w_2^{(i)}(z_0)dz_0 + \epsilon\delta_{2i} \underbrace{\int_{-b}^{-\frac{1}{2}} 1\; dx_0}_{(b-\frac{1}{2})} = 0, \quad i=1,2 \quad (0.7)$$

(δ_{ij} is the Kronecker delta). In a physical sense, Eq. (0.7) states instantaneous inflow/outflow balance through I. If (0.7) is true,

$$\varphi_{II}^{(i)}(x,z) = \mathcal{A}_{-1}^{(i)} + \epsilon\delta_{2i}\frac{(z+h)^2-(x+b)^2}{2(h-d)} +$$

$$+\int_{-h}^{-d}\Big[w_1^{(i)}(z_0)\;\mathcal{G}_{II}^{(1)}(x,z;z_0)+w_2^{(i)}(z_0)\;\mathcal{G}_{II}^{(2)}(x,z;z_0)\Big]\,dz_0,\quad i=1,2,\quad (0.8)$$

where

$$\mathcal{G}_{II}^{(1)}(x,z;z_0)=\frac{x}{h-d}-$$
$$-\sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(2)}(z_0+h))\cos(\kappa_j^{(2)}(z+h))}{\kappa_j^{(2)}N_j^{(2)}}\cdot\frac{\cosh(\kappa_j^{(2)}(x+\frac{1}{2}))}{\cosh(\kappa_j^{(2)}(b-\frac{1}{2}))},\quad (0.9a)$$

$$\mathcal{G}_{II}^{(2)}(x,z;z_0)=\sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(2)}(z_0+h))\cos(\kappa_j^{(2)}(z+h))}{\kappa_j^{(2)}N_j^{(2)}}\cdot\frac{\cosh(\kappa_j^{(2)}(x+b))}{\cosh(\kappa_j^{(2)}(b-\frac{1}{2}))}\quad (0.9b)$$

with

$$\kappa_j^{(2)}=\frac{\pi j}{h-d};\qquad N_j^{(2)}=\tfrac{1}{2}(h-d)\tanh(\kappa_j^{(2)}(b-\tfrac{1}{2})),\quad j\geqslant 1.\quad (0.10)$$

Even though the solutions (0.8) are formally determined to within $\mathcal{A}_{-1}^{(i)}$, the actual values of these constants must be computed via the Dirichlet-transmission conditions on T_1, T_2 and T_3.

Analogously, $(w_2^{(i)}, w_3^{(i)}),\ i=1,2$, yield part of the Neumann boundary value problem in III, which needs the solvability condition

$$\int_{-h}^{-d} w_2^{(i)}(z_0)dz_0-\int_{-\frac{1}{2}}^{0} w_3^{(i)}(x_0)dx_0=0,\quad i=1,2.\quad (0.11)$$

Its solution is

$$\varphi_{III}^{(i)}(x,z)=\mathcal{A}_{-2}^{(i)}+\int_{-h}^{-d} w_2^{(i)}(z_0)\;\mathcal{G}_{III}^{(1)}(x,z;z_0)dz_0+$$
$$+\int_{-\frac{1}{2}}^{0} w_3^{(i)}(x_0)\;\mathcal{G}_{III}^{(2)}(x,z;x_0)dx_0,\quad i=1,2,\quad (0.12)$$

where $\mathcal{A}_{-2}^{(i)}$ are also computed from the Dirichlet transmission conditions and

$$\mathcal{G}_{III}^{(1)}(x,z;z_0)=-\frac{x^2-(z+h)^2}{h-d}-$$
$$-\sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(2)}(z_0+h))\cos(\kappa_j^{(2)}(z+h))}{\kappa_j^{(2)}N_j^{(3)}}\cdot\frac{\cosh(\kappa_j^{(2)}x)}{\cosh(\frac{1}{2}\kappa_j^{(2)})},\quad (0.13a)$$

$$\mathcal{G}_{III}^{(2)}(x,z;x_0)=\sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(3)}x_0)\cos(\kappa_j^{(3)}x)}{\kappa_j^{(3)}N_j^{(4)}}\cdot\frac{\cosh(\kappa_j^{(3)}(z+d))}{\cosh(\kappa_j^{(3)}(h-d))}\quad (0.13b)$$

with

$$\kappa_j^{(3)}=2\pi j;\quad N_j^{(3)}=\tfrac{1}{2}(h-d)\tanh(\tfrac{1}{2}\kappa_j^{(2)});\quad N_j^{(4)}=\tfrac{1}{4}\tanh(\kappa_j^{(3)}(h-d)),\quad j\geqslant 1.\ (0.14)$$

Finally, the mixed boundary value problems in IV involving the Neumann boundary condition $(0.2c)$ and extra nonlinear boundary condition on Σ_{02} have to be solved. It has a unique solution if and only if the analogous homogeneous problems have only trivial solutions. This occurs when

$$\Lambda\neq\kappa_j^{(3)}\tanh(\kappa_j^{(3)}d),\quad j\geqslant 1.\quad (0.15)$$

The $\cos t$ and $\sin t$ components of the vertical velocity on T_4 takes the form

$$\left.\frac{\partial \psi_{IV}}{\partial z}\right|_{T_4} = w_3^{(1)}(x)\cos t + w_3^{(2)}(x)\sin t, \tag{0.16}$$

so that the mean relative velocity in the right-hand side of the dynamic boundary condition is

$$\boxed{u_0(t) = \int_{T_4} \left.\frac{\partial \psi_{IV}}{\partial z}\right|_{T_4} \mathrm{d}x = a_1\cos t + (a_2 - \epsilon)\sin t; \quad a_i = 2\int_{-1/2}^{0} w_3^{(i)}(x)\mathrm{d}x.} \tag{0.17}$$

Substituting (0.17) into the dynamic boundary condition, extracting the first harmonics and combining it with the kinematic conditions leads to the Robin boundary condition

$$\Lambda\frac{\partial^2\psi}{\partial t^2} + \frac{\partial \psi}{\partial z} = \boxed{\underbrace{\Lambda_*\frac{2}{3\pi}\Lambda(C_0^{-1} - (2b_*)^{-1})^2}_{\beta}\underbrace{\sqrt{a_1^2 + (a_1^2-\epsilon)^2}}_{\hat{u}_0=\xi(a_1,a_2)}[\underbrace{(a_2-\epsilon)}_{\Xi_1}\cos t + \underbrace{(-a_2)}_{\Xi_2}\sin t]} \tag{0.18}$$

The mixed boundary value problems in IV have the following solution

$$\varphi_{IV}^{(i)}(x,z) = \boxed{-\Xi_i +} \int_{-\frac{1}{2}}^{0} w_3^{(i)}(x_0)\ \mathcal{G}_{IV}(x,z;x_0)dx_0, \quad i=1,2, \tag{0.19}$$

where

$$\begin{aligned}\mathcal{G}_{IV}(x,y;x_0) = {} & 2\left(z+\frac{1}{\Lambda}\right) + \\ & + \sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(3)}x_0)\cos(\kappa_j^{(3)}x)}{\kappa_j^{(3)}N_j^{(5)}}\cdot\frac{\kappa_j^{(3)}\cosh(\kappa_j^{(3)}z) + \Lambda\sinh(\kappa_j^{(3)}z)}{\kappa_j^{(3)}\cosh(\kappa_j^{(3)}d)}\end{aligned} \tag{0.20}$$

and

$$N_j^{(5)} = \frac{1}{4}\left[\frac{\Lambda}{\kappa_j^{(3)}} - \tanh(\kappa_j^{(3)}d)\right]. \tag{0.21}$$

The kernel (0.20) becomes unbounded as Λ tends to the critical values defined by (0.15) because this limit causes $N_j^{(5)} \to 0$ for a certain j.

The problem reduces to integral equations with respect to $w_j^{(i)}$, $j=1,2,3;\ i=1,2,$ by using the Dirichlet transmission conditions on T_j, which imply

$$\psi_I(-b,z,t) = \varphi_{II}^{(1)}(-b,z)\cos t + \varphi_{II}^{(2)}(-b,z)\sin t, \quad -h<z<-d, \quad t \geqslant 0 \tag{0.22}$$

as well as

$$\begin{aligned}&\varphi_{II}^{(i)}(-\tfrac{1}{2},z) = \varphi_{III}^{(i)}(-\tfrac{1}{2},z), \quad -h<z-d,\\ &\varphi_{III}^{(i)}(x,-d) = \varphi_{IV}^{(i)}(x,-d), \quad -\tfrac{1}{2}<x<0, \quad i=1,2.\end{aligned} \tag{0.23}$$

Together with the solvability conditions (0.7) and (0.11), Eqs. (0.22) and (0.23) yield the following system of integral equations

$$\int_{-h}^{-d} w_1^{(1)}(z_0)\mathcal{K}_{1,1}(z,z_0)dz_0 + \int_{-h}^{-d} w_2^{(1)}(z_0)\ \mathcal{K}_{1,2}(z,z_0)dz_0 - \mathcal{A}_{-1}^{(1)} -$$
$$- \frac{\cosh(\mathcal{K}(z+h))}{\mathcal{K}N_0}\int_{-h}^{-d} w_1^{(2)}(z_0)\cosh(\mathcal{K}(z_0+d))dz_0 = 0,$$
$$\int_{-h}^{-d} w_1^{(2)}(z_0)\mathcal{K}_{1,1}(z,z_0)dz_0 + \int_{-h}^{-d} w_2^{(2)}(z_0)\ \mathcal{K}_{1,2}(z,z_0)dz_0 - \mathcal{A}_{-1}^{(2)} + \tag{0.24a}$$
$$+ \frac{\cosh(\mathcal{K}(z+h))}{\mathcal{K}N_0}\int_{-h}^{-d} w_1^{(1)}(z_0)\cosh(\mathcal{K}(z_0+h))dz_0 = \epsilon\frac{(z+h)^2}{2(h-d)},$$
$$-h < z < -d,$$

$$\int_{-h}^{-d} w_1^{(i)}(z_0)\ \mathcal{K}_{2,1}(z,z_0)dz_0 + \int_{-h}^{-d} w_2^{(i)}(z_0)\ \mathcal{K}_{2,2}(z,z_0)dz_0 + \int_{-\frac{1}{2}}^{0} w_3^{(i)}(x_0)\ \mathcal{K}_{2,3}(z,x_0)dx_0 +$$
$$+ \mathcal{A}_{-1}^{(i)} - \mathcal{A}_{-2}^{(i)} = -\epsilon\delta_{2i}\frac{(z+h)^2-(b-\frac{1}{2})^2}{2(h-d)}, \quad -d < z < -h, \tag{0.24b}$$

$$\int_{-h}^{-d} w_2^{(i)}(z_0)\ \mathcal{K}_{3,2}(x,z_0)dz_0 + \int_{-\frac{1}{2}}^{0} w_3^{(i)}(x_0)\ \mathcal{K}_{3,3}(x,x_0)dx_0 + \mathcal{A}_{-2}^{(i)}\boxed{-\Xi_i} = 0, \quad -\tfrac{1}{2} < x < 0, \tag{0.24c}$$

$$\int_{-h}^{-d} w_1^{(i)}(z_0)dz_0 - \int_{-h}^{-d} w_2^{(i)}(z_0)dz_0 = \epsilon\delta_{2i}(b-\tfrac{1}{2}), \tag{0.24d}$$

$$\int_{-h}^{-d} w_2^{(i)}(z_0)\ dz_0 - \int_{-\frac{1}{2}}^{0} w_3^{(i)}(x_0)\ dx_0 = 0, \tag{0.24e}$$

for $i = 1, 2$. The kernels are defined as follows

$$\mathcal{K}_{1,1}(z,z_0) = \mathcal{G}_I(-b,z;z_0) - \mathcal{G}_{II}^{(1)}(-b,z;z_0) = \frac{b}{h-d} +$$
$$+ \sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(1)}(z_0+h))\cos(\kappa_j^{(1)}(z+h))}{\kappa_j^{(1)}N_j^{(1)}} + \sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(2)}(z_0+h))\cos(\kappa_j^{(2)}(z+h))}{\kappa_j^{(2)}N_j^{(2)}}, \tag{0.25a}$$

$$\mathcal{K}_{1,2}(z,z_0) = -\mathcal{G}_{II}^{(2)}(-b,z;z_0) = -\sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(2)}(z_0+h))\cos(\kappa_j^{(2)}(z+h))}{\kappa_j^{(2)}N_j^{(2)}\cosh(\kappa_j^{(2)}(b-\frac{1}{2}))}, \tag{0.25b}$$

$$\mathcal{K}_{2,1}(z,z_0) = \mathcal{G}_{II}^{(1)}(-\tfrac{1}{2},z;z_0) =$$
$$= -\frac{1}{2(h-d)} - \sum_{j=1}^{\infty}\frac{\cos(\kappa_j^{(2)}(z_0+h))\cos(\kappa_j^{(2)}(z+h))}{\kappa_j^{(2)}N_j^{(2)}\cosh(\kappa_j^{(2)}(b-\frac{1}{2}))}, \tag{0.25c}$$

$$\mathcal{K}_{2,2}(z,z_0)=\mathcal{G}^{(2)}_{II}(-\tfrac{1}{2},z;z_0)-\mathcal{G}^{(1)}_{III}(-\tfrac{1}{2},z;z_0)=-\frac{(z+h)^2}{h-d}+\frac{1}{4(h-d)}+$$
$$+\sum_{j=1}^{\infty}\frac{\cos(\kappa^{(2)}_j(z_0+h))\cos(\kappa^{(2)}_j(z+h))}{\kappa^{(2)}_j}\left[\frac{1}{N^{(2)}_j}+\frac{1}{N^{(3)}_j}\right],\quad(0.25d)$$

$$\mathcal{K}_{2,3}(z,x_0)=-\mathcal{G}^{(2)}_{III}(-\tfrac{1}{2},z;x_0)=-\sum_{j=1}^{\infty}(-1)^j\frac{\cos(\kappa^{(3)}_j x_0)}{\kappa^{(3)}_j N^{(4)}_j}\cdot\frac{\cosh(\kappa^{(3)}_j(z+h))}{\cosh(\kappa^{(3)}_j(h-d))},\quad(0.25e)$$

$$\mathcal{K}_{3,2}(x,z_0)=\mathcal{G}^{(1)}_{III}(x,-d;z_0)=$$
$$=h-d-\frac{x^2}{h-d}-\sum_{j=1}^{\infty}(-1)^j\frac{\cos(\kappa^{(2)}_j(z_0+h))}{\kappa^{(2)}_j N^{(3)}_j}\cdot\frac{\cosh(\kappa^{(2)}_j x)}{\cosh(\frac{1}{2}\kappa^{(2)}_j)},\quad(0.25f)$$

$$\mathcal{K}_{3,3}(x,x_0)=\mathcal{G}^{(2)}_{III}(x,-d;x_0)-\mathcal{G}_{IV}(x,-d;x_0)=$$
$$=2\left(d-\frac{1}{\Lambda}\right)+\sum_{j=1}^{\infty}\frac{\cos(\kappa^{(3)}_j x_0)\cos(\kappa^{(3)}_j x)}{\kappa^{(3)}_j N^{(6)}_j},\quad(0.25g)$$

where

$$\frac{1}{N^{(6)}_j}=4\left(\coth(\kappa^{(3)}_j(h-d))-\frac{\kappa^{(3)}_j-\Lambda\ \tanh(\kappa^{(3)}_j d)}{\Lambda-\kappa^{(3)}_j\tanh(\kappa^{(3)}_j d)}\right).$$

The inhomogeneous system of ten integral equations (0.24) couples six unknown functions $w^{(i)}_k,\ k=1,2,3,\ i=1,2$ and four coefficients $\mathcal{A}^{(i)}_{-1},\mathcal{A}^{(i)}_{-2},\ i=1,2$. It can be solved by the Galerkin projective scheme suggesting approximate solutions in the form

$$w^{(i)}_1(z)=\sum_{j=1}^{N_1}\alpha^{(1,i)}_j v^{(1)}_j(z);\quad w^{(i)}_2(z)=\sum_{j=1}^{N_2}\alpha^{(2,i)}_j v^{(1)}_j(z);\quad w^{(i)}_3(x)=\sum_{j=1}^{N_3}\alpha^{(3,i)}_j v^{(2)}_j(x),\quad(0.26)$$

where $\{v^{(1)}_j\}$ and $\{v^{(2)}_j\}$ are two complete systems of functions on $(-h,-d)$ and $(-\frac{1}{2},0)$, respectively. Insertion of (0.26) into (0.24) and use of the projective scheme lead to a system of $2N_1+2N_2+2N_3+4$ linear algebraic system with respect to $2N_1+2N_3+2N_3+4$ variables $\{\alpha^{(1,i)}_j,\ j=1,\dots,N_1\}$, $\{\alpha^{(2,i)}_j,\ j=1,\dots,N_2\}$, $\{\alpha^{(3,i)}_j,\ j=1,\dots,N_3\}$ and $\mathcal{A}^{(i)}_{-j},\ i,j=1,2$.

By introducing the vector

$$\boldsymbol{B}=\Big(\alpha^{(1,1)}_1,\dots,\alpha^{(1,1)}_{N_1};\alpha^{(2,1)}_1,\dots,\alpha^{(2,1)}_{N_2};\alpha^{(3,1)}_1,\dots,\alpha^{(3,1)}_{N_3};\mathcal{A}^{(1)}_{-2},\mathcal{A}^{(1)}_{-1};$$
$$\alpha^{(1,2)}_1,\dots,\alpha^{(1,2)}_{N_1};\alpha^{(2,2)}_1,\dots,\alpha^{(2,2)}_{N_2};\alpha^{(3,2)}_1,\dots,\alpha^{(3,2)}_{N_3};\mathcal{A}^{(2)}_{-2},\mathcal{A}^{(2)}_{-1}\Big)^T,\quad(0.27)$$

the matrix problem following from the Galerkin scheme is as follows

$$\left(\mathcal{P}\boxed{-\xi\mathcal{D}}\right)\boldsymbol{B}=\epsilon\left(\boldsymbol{b}\boxed{-\xi\boldsymbol{d}}\right),\quad(0.28)$$

where $\mathcal{P}$ and $\mathcal{D}$ are the $(2N_1+2N_3+2N_3+4)\times(2N_1+2N_3+2N_3+4)$-matrices,

$$\boxed{\hat{u}_0=\xi=\sqrt{a_1^2+(a_2-\epsilon)^2}}.\quad(0.29)$$

Elements of $\mathcal{P}$ and the right-hand side vector $\boldsymbol{b}$ are integrals over the kernels (0.25) and the functions $\{v_j^{(1)}\}$ and $\{v_j^{(2)}\}$. The matrix $\mathcal{P}$ has the following structure

$$\mathcal{P} = \left\| \begin{array}{c|c} D & \begin{matrix} \text{-p} & \\ & 0 \end{matrix} \\ \hline \begin{matrix} \text{p} & \\ & 0 \end{matrix} & D \end{array} \right\|, \tag{0.30}$$

where the two sub-matrices D and p have dimensions $(N_1+N_2+N_3+2)\times(N_1+N_2+N_3+2)$ and $N_1 \times N_1$, respectively.

Convergence and accuracy of the Galerkin method depend on the functional sets $\{v_j^{(1)}(z)\}$ and $\{v_j^{(2)}(x)\}$. Because the Neumann traces on $T_k,\ k = 1, 2, 3$ are singular at the corner points of the piercing rectangular body, i.e. $w_j^{(i)}(z) \to \infty$ as $z \to -d$ and $w_3^{(i)}(x) \to \infty$ as $x \to -\frac{1}{2}$, the use of a smooth functional basis, for instance, trigonometrical or polynomial, causes weak convergence. On the contrary, accounting for the singular character of the traces should improve the convergence. The local solutions of the complex velocity at the edges $\bar{A}_2$ and $\bar{A}_3$ in the complex plane $Z = \bar{x} + \mathrm{i}\bar{z}$ can be expressed as

$$\frac{dW_0}{dZ}\Big|_{\mathrm{at}\bar{A}_2} = e^{-\mathrm{i}\frac{\pi}{2}}\ \eta_{3a}\sigma\sin(\sigma t) + \sum_{i=1}^{\infty} T_i^{(2)}(t)\left[Z + \bar{B} + \mathrm{i}\bar{d}\right]^{\frac{2i}{3}-1}, \tag{0.31a}$$

$$\frac{dW_0}{dZ}\Big|_{\mathrm{at}\bar{A}_3} = e^{-\mathrm{i}\frac{\pi}{2}}\ \eta_{3a}\sigma\sin(\sigma t) + \sum_{i=1}^{\infty} T_i^{(3)}(t)\left[e^{\mathrm{i}\pi}\left(Z + \frac{L_1}{2} + \mathrm{i}\bar{d}\right)\right]^{\frac{2i}{3}-1}, \tag{0.31b}$$

where W_0 is the complex velocity potential. The first term in (0.31) is caused by the vertical motion of the body.

By conducting direct analytical derivations or noting that summands in (0.31) with i that are divisible by 3 are regular, one can see that terms associated with $T_{3l}^{(2)}(t)$, $T_{3l}^{(3)}(t)$, $l = 1, 2, \ldots$ vanish on the intervals $\bar{T}_k,\ k = 1, 2, 3$. This implies that $\psi(-b, z, t) \sim (z + d)^m,\ \ z \to -d;\quad \psi(x, -d, t) \sim \left(x + \frac{1}{2}\right)^m,\ \ x \to -\frac{1}{2}$, where the numbers m belong to the set

$$\{\pm\tfrac{1}{3} + 2(i-1),\ i \geqslant 1\}. \tag{0.32}$$

The enumeration of (0.32) in ascending order determines a sequence $m_j,\ j \geqslant 1$. The functional basis must satisfy

$$v_j^{(1)} \sim (z + d)^{m_j},\quad z \to -d;\quad v_j^{(2)} \sim \left(x + \tfrac{1}{2}\right)^{m_j},\quad x \to -\tfrac{1}{2}\quad j \geqslant 1. \tag{0.33}$$

Further, accounting for the zero Neumann conditions on S_B and S_S, i.e.

$$(v_j^{(1)})'(-h) = 0 \quad \text{and} \quad (v_j^{(2)})'(0) = 0,\quad j \geqslant 1, \tag{0.34}$$

deduces from (0.33) the following functional sets

$$v_j^{(1)}(z) = \frac{1}{r_j^{(1)}}\left(1 - \left(\frac{z+h}{h-d}\right)^2\right)^{m_j};\quad v_j^{(2)}(x) = \frac{1}{r_j^{(2)}}(1 - (2x)^2)^{m_j},\quad j \geqslant 1, \tag{0.35}$$

where

$$r_j^{(1)} = \sqrt{\frac{(h-d)\sqrt{\pi}\Gamma(2m_j+1)}{2\Gamma(2m_j+\frac{3}{2})}},\quad r_j^{(2)} = \frac{1}{2}\sqrt{\frac{\sqrt{\pi}\Gamma(2m_j+1)}{\Gamma(2m_j+\frac{3}{2})}},\quad j \geqslant 1 \tag{0.36}$$

($\Gamma(\cdot)$ is the gamma-function). The scaling factors $r_j^{(i)},\ i=1,2; j=1,2,3$ appear from the normalization condition $\int_{-h}^{-d}(v_j^{(1)}(z))^2dz = \int_{-\frac{1}{2}}^{0}(v_j^{(2)}(x))^2dx = 1,\ j \geqslant 1$. The non-singular sub-sets $\{v_j^{(1)}\}$ and $\{v_j^{(2)}\}$ with $j \geqslant 2$ constitute complete bases for the functions on the intervals $[-h,-d]$ and $[-\frac{1}{2},0]$, respectively, for functions which satisfy (0.34). The completeness follows from the classical theorem by Müntz theorem.

Elements of $\mathcal{P}$ and the right-hand side vector $\boldsymbol{b}$ are found explicitly via the special functions, but the extra right-hand side vector $\boldsymbol{d}$ and matrix $\mathcal{D}$ have the elements

$$\boxed{\boldsymbol{d} = \{0, \text{except } b_{k+N_1+N_2} = \beta d_k, k=1,\dots,N_3\}},$$

$$\boxed{\mathcal{D} = \{0, \text{except } \mathcal{D}_{k+N_1+N_2,2(N_1+N_2)+N_3+j+2} = -\mathcal{D}_{k+2(N_1+N_2)+N_3+2,N_1+N_2+j} = 2\beta d_k d_j\}},$$

where

$$\boxed{d_k = \frac{\sqrt{\pi}\Gamma(m_k+1)}{4r_k^{(2)}\Gamma(m_k+\frac{3}{2})}}.$$

As explained by Faltinsen *et al.* (2007), the nondimensional resonance condition follows from the equation

$$\det ||D(\Lambda_*)|| = 0, \qquad \Lambda_* \in \left(0, \tfrac{\pi}{2}\tanh\left(\frac{\pi}{2}\right)\right), \tag{0.37}$$

where D is the sub-matrix of dimension $N_1+N_2+N_3+2$ found in (0.30).

The nonlinear equations (0.28) can be solved iteratively starting with $\xi = 0$.