Supplementray Material: Normal stress differences, their origin and constitutive relations for a sheared granular fluid

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Appendix A. Collision integrals of Eq. (3.30-3.33) and their algebraic form

Various collision integrals appearing in (3.30-3.33) can be compactly written as

$$\left.\begin{aligned} \Theta_{x'x'} - \Theta_{y'y'} &= \tfrac{3(1+e)\rho\nu g_0 T}{\pi^{\frac{3}{2}}}\mathcal{J}^{30}_{012}(\eta,\lambda^2,R,\phi), \\ 2\Theta_{x'y'} &= \tfrac{3(1+e)\rho\nu g_0 T}{\pi^{\frac{3}{2}}}\mathcal{J}^{30}_{102}(\eta,\lambda^2,R,\phi), \\ \Theta_{x'x'} + \Theta_{y'y'} &= \tfrac{3(1+e)\rho\nu g_0 T}{\pi^{\frac{3}{2}}}\mathcal{J}^{30}_{002}(\eta,\lambda^2,R,\phi), \\ A_{x'x'} + A_{y'y'} + A_{z'z'} &= -\tfrac{6(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\mathcal{H}^{10}_{003}(\eta,\lambda^2,R,\phi), \end{aligned}\right\}, \tag{A 1}$$

$$\left.\begin{aligned} \widehat{\Gamma}_{z'z'} &= -\tfrac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\Big[\tfrac{1}{3}(1-e)\left(2\mathcal{H}^{12}_{003} - \mathcal{H}^{30}_{003}\right) - 2\eta\left(\mathcal{H}^{31}_{101} - \mathcal{H}^{32}_{011}\right) \\ &\qquad -6\lambda^2\mathcal{H}^{32}_{001} - 4R\mathcal{K}^{31}_{00}\Big], \\ \Gamma_{x'x'} - \Gamma_{y'y'} &= -\tfrac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\Big[(1-e)\mathcal{H}^{30}_{013} + 2\eta\left(2\mathcal{H}^{31}_{111} - \mathcal{H}^{30}_{201} - \mathcal{H}^{32}_{021}\right) \\ &\qquad +6\lambda^2\left(\mathcal{H}^{32}_{011} - \mathcal{H}^{31}_{101}\right) - 4R\left(\mathcal{K}^{30}_{10} - \mathcal{K}^{31}_{01}\right)\Big], \\ \Gamma_{x'y'} &= -\tfrac{6(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{\sigma\pi^{\frac{3}{2}}}\Big[\tfrac{1}{2}(1-e)\mathcal{H}^{30}_{103} + \eta\left(\mathcal{H}^{31}_{201} + \mathcal{H}^{30}_{111} - \mathcal{H}^{32}_{111} - \mathcal{H}^{31}_{021}\right) \\ &\qquad +3\lambda^2\left(\mathcal{H}^{32}_{101} + \mathcal{H}^{31}_{011}\right) + 2R\left(\mathcal{K}^{31}_{10} + \mathcal{K}^{30}_{01}\right)\Big], \end{aligned}\right\}. \tag{A 2}$$

where $\mathcal{H}$, $\mathcal{J}$ and $\mathcal{K}$ are defined in (3.35–3.37).

Substituting the power-series representations (4.1–4.2) into the integrals (3.35-3.37), performing term-by-term integrations, and neglecting the terms beyond fourth order in η, λ, R and $\sin 2\phi$, we have following algebraic expressions for $\mathcal{H}^{\delta p}_{\alpha\beta\gamma}$, $\mathcal{J}^{\delta p}_{\alpha\beta\gamma}$ and $\mathcal{K}^{\delta p}_{\alpha\beta}$:

$$\begin{aligned} \mathcal{H}^{10}_{003} = \frac{\pi}{210}\Big\{&840 + 2688R^2 + 1024R^4 + 768R^2\lambda^2 - 24\eta^2\lambda^2 + 84\Big(\eta^2 + 3\lambda^4\Big) \\ &+3\eta^4 + 672\sqrt{\pi}R\eta\cos 2\phi - 64\eta^2R^2\Big(2 + \cos 4\phi\Big)\Big\}, \end{aligned} \tag{A 3}$$

$$\begin{aligned} \mathcal{H}^{30}_{013} = -\frac{4\pi}{105}\Big[&4\sqrt{\pi}R\Big(21 + 12\lambda^2 + 32R^2\Big)\cos 2\phi + \eta\Big\{42 - \eta^2 + 12\lambda^2 \\ &+32R^2\Big(2 + \cos 4\phi\Big)\Big\}\Big], \end{aligned} \tag{A 4}$$

$$\mathcal{H}^{30}_{103} = \frac{16\pi}{105}R\sin 2\phi\Big\{\sqrt{\pi}\Big(21 + 12\lambda^2 + 32R^2\Big) + 16\eta R\cos 2\phi\Big\}, \tag{A 5}$$

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$$2\mathcal{H}^{12}_{003} - \mathcal{H}^{30}_{003} = -\frac{4\pi}{1155}\Big\{528\sqrt{\pi}R\eta\cos 2\phi + 1386\lambda^2 + 66\Big(\eta^2 - 3\lambda^4\Big) - 33\eta^2\lambda^2 + 1024R^4 + 3\eta^4 + 32R^2\Big(66 - 4\eta^2 + 33\lambda^2 - 2\eta^2\cos 4\phi\Big)\Big\}, \tag{A 6}$$

$$2\eta\Big(\mathcal{H}^{31}_{101} - \mathcal{H}^{32}_{011}\Big) + 6\lambda^2\mathcal{H}^{32}_{001} = \frac{8\pi}{1155}\Big\{22\eta^2 - 64\eta^2R^2 + \eta^4 + 462\lambda^2 + 1056\lambda^2R^2 - 11\eta^2\lambda^2 - 66\lambda^4 + 4\sqrt{\pi}R\Big(33 + 32R^2\Big)\eta\cos 2\phi - 32R^2\eta^2\cos 4\phi\Big\}, \tag{A 7}$$

$$\eta\Big(2\mathcal{H}^{31}_{111} - \mathcal{H}^{30}_{201} - \mathcal{H}^{32}_{021}\Big) + 3\lambda^2\Big(\mathcal{H}^{32}_{011} - \mathcal{H}^{31}_{101}\Big) = -\frac{4\pi}{105}\Big\{36\sqrt{\pi}\lambda^2R\cos 2\phi + \eta\Big(42 - \eta^2 + 12\lambda^2 + 160R^2 - 64R^2\cos 4\phi\Big)\Big\}, \tag{A 8}$$

$$\eta\Big(\mathcal{H}^{31}_{201} + \mathcal{H}^{30}_{111} - \mathcal{H}^{32}_{111} - \mathcal{H}^{31}_{021}\Big) + 3\lambda^2\Big(\mathcal{H}^{32}_{101} + \mathcal{H}^{31}_{011}\Big) = \frac{16\pi}{105}R\sin 2\phi\Big\{9\sqrt{\pi}\lambda^2 - 32\eta R\cos 2\phi\Big\}, \tag{A 9}$$

$$\mathcal{J}^{30}_{012} = -\frac{8\pi}{315}\Big\{21\sqrt{\pi}\eta + 4R\Big(42 - 3\eta^2 + 12\lambda^2 + 32R^2\Big)\cos 2\phi\Big\}, \tag{A 10}$$

$$\mathcal{J}^{30}_{102} = \frac{32\pi}{315}R\sin 2\phi\Big(42 - \eta^2 + 12\lambda^2 + 32R^2\Big), \tag{A 11}$$

$$\mathcal{J}^{30}_{002} = \frac{4\pi}{3465}\Big[33\sqrt{\pi}\Big(35 + 96R^2 + 14\lambda^2\Big) - 8R\eta\Big\{160R^2 - 3\Big(66 + 5\eta^2 - 22\lambda^2\Big)\Big\}\cos 2\phi\Big], \tag{A 12}$$

$$\mathcal{J}^{10}_{002} = \frac{2\pi}{315}\Big[21\sqrt{\pi}\Big(15 + 32R^2\Big) - 8R\eta\Big\{32R^2 - 3\Big(14 + \eta^2 - 4\lambda^2\Big)\Big\}\cos 2\phi\Big], \tag{A 13}$$

$$\mathcal{J}^{12}_{002} = \frac{2\pi}{3465}\Big[33\sqrt{\pi}\Big(35 + 32R^2 - 28\lambda^2\Big) - 8R\eta\Big\{32R^2 - 3\Big(22 + \eta^2\Big)\Big\}\cos 2\phi\Big], \tag{A 14}$$

$$\mathcal{K}^{31}_{00} = \frac{32\pi}{3465}R\Big(66 + 6\eta^2 - 132\lambda^2 + 32R^2 - 24\sqrt{\pi}\eta R\cos 2\phi + 3\eta^2\cos 4\phi\Big), \tag{A 15}$$

$$\mathcal{K}^{30}_{10} - \mathcal{K}^{31}_{01} = \frac{4\pi}{3465}\Big[\sqrt{\pi}\Big\{693 + 32R^2\Big(33 + 10\eta^2 - 18\lambda^2\Big)\Big\}\cos 2\phi - 8R\eta\Big\{209 + 15\eta^2 - 91\lambda^2 - \Big(143 + 15\eta^2 - 37\lambda^2\Big)\cos 4\phi + 40\sqrt{\pi}R\eta\cos 6\phi\Big\}\Big], \tag{A 16}$$

$$\mathcal{K}^{31}_{10} + \mathcal{K}^{30}_{01} = \frac{4\pi}{315}\sin 2\phi\Big\{208\eta R\cos 2\phi + 3\sqrt{\pi}\Big(21 + 32R^2\Big)\Big\}. \tag{A 17}$$

Appendix B. Second moment balance at third and fourth orders and its solution

B.1. *Perturbation solutions at finite density*

We look for perturbation solutions of second moment equations in the form

$$\left.\begin{array}{rcl} \eta &=& \eta^{(2)} + \varepsilon\eta^{(3)} + \varepsilon^2\eta^{(4)} \\ \lambda^2 &=& \lambda^{(2)} + \varepsilon\lambda^{(3)} + \varepsilon^2\lambda^{(4)} \\ R &=& R^{(2)} + \varepsilon R^{(3)} + \varepsilon^2 R^{(4)} \\ \sin 2\phi &=& \sin 2\phi^{(2)} + \varepsilon\sin 2\phi^{(3)} + \varepsilon^2\sin 2\phi^{(4)} \end{array}\right\}. \tag{B 2}$$

Plugging these perturbation series into corresponding third (super-Burnett) and fourth (super-super-Burnett) order equations, we obtain perturbation equations at different orders.

At super-Burnett-order (the third-order in the shear rate), the balance equations for the second moment are

$$\left.\begin{array}{l}
20\sqrt{\pi}\Big\{1+\frac{4}{5}(1+e)\nu g_0\Big\}(\eta^{(3)}R^{(2)}+\eta^{(2)}R^{(3)})\cos 2\phi^{(2)}+256(1+e)\nu g_0 R^{(2)}R^{(3)}\\
\qquad -6(1-e^2)\nu g_0\Big\{\eta^{(2)}\eta^{(3)}+32R^{(2)}R^{(3)}+4\sqrt{\pi}(\eta^{(3)}R^{(2)}+\eta^{(2)}R^{(3)})\cos 2\phi^{(2)}\Big\}=0\\
35\sqrt{\pi}(\eta^{(3)}R^{(2)}+\eta^{(2)}R^{(3)})\cos 2\phi^{(2)}+2(1+e)\nu g_0\Big\{32(1+3e)R^{(2)}R^{(3)}\\
-3(3-e)(\eta^{(2)}\eta^{(3)}+21\lambda^{(2)}\lambda^{(3)})-8\sqrt{\pi}(4-3e)(\eta^{(3)}R^{(2)}+\eta^{(2)}R^{(3)})\cos 2\phi^{(2)}\Big\}=0\\
5\sqrt{\pi}R^{(3)}\cos 2\phi^{(2)}-(1+e)\nu g_0\{3(3-e)\eta^{(3)}+2(1-3e)\sqrt{\pi}R^{(3)}\cos 2\phi^{(2)}\}=0\\
5(\eta^{(3)}-\sin 2\phi^{(3)})+2(1+e)(1-3e)\nu g_0\sin 2\phi^{(3)}=0
\end{array}\right\}. \tag{B 3}$$

The solutions for third-order corrections are zero. At fourth order in the shear rate, the perturbation equations are

$$\left.\begin{array}{l}
1680\sqrt{\pi}\varepsilon^2(\eta^{(4)}R^{(2)}+\eta^{(2)}R^{(4)})\cos 2\phi^{(2)}-3(1-e^2)\nu g_0\Big(168\varepsilon^2\eta^{(2)}\eta^{(4)}+3{\eta^{(2)}}^4\\
+5376\varepsilon^2R^{(2)}R^{(4)}+1024{R^{(2)}}^4-128{R^{(2)}}^2{\eta^{(2)}}^2+768{R^{(2)}}^2{\lambda^{(2)}}^2-24{\eta^{(2)}}^2{\lambda^{(2)}}^2\\
+252{\lambda^{(2)}}^4+672\sqrt{\pi}\varepsilon^2(\eta^{(4)}R^{(2)}+\eta^{(2)}R^{(4)})\cos 2\phi^{(2)}-64{\eta^{(2)}}^2{R^{(2)}}^2\Big)\\
+1344\sqrt{\pi}(1+e)\nu g_0\varepsilon^2(\eta^{(4)}R^{(2)}+\eta^{(2)}R^{(4)})\cos 2\phi^{(2)}\\
+256(1+e)\nu g_0\Big\{{R^{(2)}}^2\Big(-3{\eta^{(2)}}^2+12{\lambda^{(2)}}^2+32{R^{(2)}}^2\Big)+84\varepsilon^2R^{(2)}R^{(4)}\Big\}=0\\
2310\sqrt{\pi}\varepsilon^2(\eta^{(4)}R^{(2)}+\eta^{(2)}R^{(4)})\cos 2\phi^{(2)}+(1+e)\nu g_0\Big[32{R^{(2)}}^2\Big\{8{\eta^{(2)}}^2-165{\lambda^{(2)}}^2\\
-12e{\eta^{(2)}}^2+99e{\lambda^{(2)}}^2\Big\}+4224(1+3e)\varepsilon^2R^{(2)}R^{(4)}-9(3-e)\Big\{{\eta^{(2)}}^4+44\varepsilon^2\eta^{(2)}\eta^{(4)}\\
-11{\eta^{(2)}}^2{\lambda^{(2)}}^2+924\varepsilon^2\lambda^{(2)}\lambda^{(4)}-66{\lambda^{(2)}}^4\Big\}+1024(5+3e){R^{(2)}}^4\\
-528\sqrt{\pi}(4-3e)\varepsilon^2(\eta^{(4)}R^{(2)}+\eta^{(2)}R^{(4)})\cos 2\phi^{(2)}+64(2-3e){\eta^{(2)}}^2{R^{(2)}}^2\Big]=0\\
210\sqrt{\pi}\Big(\varepsilon^2R^{(4)}+{\lambda^{(2)}}^2R^{(2)}\Big)\cos 2\phi^{(2)}-(1+e)\nu g_0\big[12\sqrt{\pi}\{7(1-3e)\varepsilon^2R^{(4)}\\
+4(4-3e){\lambda^{(2)}}^2R^{(2)}-32(1+e){R^{(2)}}^3\}\cos 2\phi^{(2)}+\eta^{(2)}\{126(3-e)\varepsilon^2\eta^{(4)}\\
-3(3-e){\eta^{(2)}}^2+36(3-e){\lambda^{(2)}}^2+96(1-3e){R^{(2)}}^2\}\big]=0\\
105\sqrt{\pi}\{\varepsilon^2\eta^{(4)}-{\lambda^{(2)}}^2\sin 2\phi^{(2)}-\varepsilon^2\sin 2\phi^{(4)}\}-2(1+e)\nu g_0\Big[\big[16(5+3e)\eta^{(2)}R^{(2)}\cos 2\phi^{(2)}\\
-3\sqrt{\pi}\{4(4-3e){\lambda^{(2)}}^2-32(1+e){R^{(2)}}^2\}\big]\sin 2\phi^{(2)}-21\sqrt{\pi}(1-3e)\varepsilon^2\sin 2\phi^{(4)}\Big]=0
\end{array}\right\}. \tag{B 4}$$

The solution of these equations are

$$\begin{array}{l}
\varepsilon^2\eta^{(4)}=\Big[\Big[\sqrt{\pi}\nu g_0\cos^{(2)}2\phi\{5-2(1+e)(1-3e)\nu g_0\}\Big\{1024(1+e)(5+3e){R^{(2)}}^4\\
-192(1+e)(1+3e){R^{(2)}}^2\Big({\eta^{(2)}}^2-4{\lambda^{(2)}}^2\Big)-9(1-e^2)\Big({\eta^{(2)}}^4-8{\eta^{(2)}}^2{\lambda^{(2)}}^2+84{\lambda^{(2)}}^4\Big)\Big\}\Big]\\
-\Big[8\Big\{5\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi+2(1+e)\nu g_0\Big(8(1+3e)R^{(2)}-(1-3e)\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi\Big)\Big\}\\
\times\Big\{210\sqrt{\pi}{\lambda^{(2)}}^2R^{(2)}\cos^{(2)}2\phi-48(1+e)\sqrt{\pi}\nu g_0R^{(2)}\cos^{(2)}2\phi\Big((4-3e){\lambda^{(2)}}^2-8(1+e){R^{(2)}}^2\Big)
\end{array}$$

$$
\frac{-3(1+e)\nu g_0\eta^{(2)}\Big(32(1-3e){R^{(2)}}^2-(3-e)({\eta^{(2)}}^2-{\lambda^{(2)}}^2)\Big)\Big\}\Big]\Big]}{\Big[4\Big[\sqrt{\pi}\cos^{(2)}2\phi\Big\{2\sqrt{\pi}\cos^{(2)}2\phi\{5-2(1+e)(1-3e)\nu g_0\}R^{(2)}-3(1-e^2)\nu g_0\eta^{(2)}\Big\} \times\{5-2(1+e)(1-3e)\nu g_0\}+6(3-e)(1+e)\nu g_0\Big\{5\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi +2(1+e)\nu g_0\Big(8(1+3e)R^{(2)}-(1-3e)\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi\Big)\Big\}\Big]\Big]} \tag{B 5}
$$

$$
\begin{aligned}
\varepsilon^2\lambda^{(4)} = {} & \frac{1}{238848\lambda^{(2)}}\Bigg(\Bigg[\frac{28}{(3-e)}\Big\{1024(5+3e){R^{(2)}}^4 \\
& +96{R^{(2)}}^2\Big(2(2-3e){\eta^{(2)}}^2-11(5-3e){\lambda^{(2)}}^2\Big)-9(3-e)\Big({\eta^{(2)}}^4-11{\eta^{(2)}}^2{\lambda^{(2)}}^2-66{\lambda^{(2)}}^4\Big)\Big\}\Bigg] \\
& -\Bigg[132\Big[35\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi+8(1+e)\nu g_0\Big\{8(1+3e)R^{(2)}-(4-3e)\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi\Big\}\Big] \\
& \times\Big[70\sqrt{\pi}{\lambda^{(2)}}^2R^{(2)}\cos^{(2)}2\phi+(1+e)\nu g_0\Big\{16\sqrt{\pi}R^{(2)}\cos^{(2)}2\phi\Big((1+e){R^{(2)}}^2-(4-3e){\lambda^{(2)}}^2\Big) \\
& \frac{-32(1-3e){\eta^{(2)}}^2R^{(2)}+3(3-e)\eta^{(2)}\Big({\eta^{(2)}}^2-12{\lambda^{(2)}}^2\Big)\Big\}\Big]}{(3-e)(1+e)\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}\nu g_0\cos^{(2)}2\phi}\Bigg] \\
& +\Bigg[1848\varepsilon^2\eta^{(4)}\Bigg[\frac{\sqrt{\pi}\{35-8(1+e)(4-3e)\nu g_0\}R^{(2)}\cos^{(2)}2\phi-6(3-e)(1+e)\nu g_0\eta^{(2)}}{(3-e)(1+e)\nu g_0} \\
& +\frac{3\Big\{35\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi+8(1+e)\nu g_0\Big(8(1+3e)R^{(2)}-(4-3e)\sqrt{\pi}\eta^{(2)}\cos^{(2)}2\phi\Big)\Big\}}{\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}\cos^{(2)}2\phi}\Bigg]\Bigg]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^2R^{(4)} = {} & \frac{1}{42\sqrt{\pi}\{5-2(1+e)(1-3e)\nu g_0\}\cos^{(2)}2\phi}\Big[-210\sqrt{\pi}{\lambda^{(2)}}^2R^{(2)}\cos^{(2)}2\phi \\
& +48(1+e)\sqrt{\pi}\nu g_0R^{(2)}\cos^{(2)}2\phi\Big\{(4-3e){\lambda^{(2)}}^2-8(1+e){R^{(2)}}^2\Big\} \\
& +3(1+e)\nu g_0\eta^{(2)}\Big\{32(1-3e){R^{(2)}}^2-(3-e)\Big({\eta^{(2)}}^2-{\lambda^{(2)}}^2\Big)\Big\} \\
& +\Big\{126(3-e)(1+e)\nu g_0\varepsilon^2\eta^{(4)}\Big\}\Big]
\end{aligned} \tag{B 6}
$$

$$
\varepsilon^2\sin^{(4)}2\phi = \Big[105\sqrt{\pi}\Big({\lambda^{(2)}}^2\sin^{(2)}2\phi-\varepsilon^2\eta^{(4)}\Big)+ \frac{2(1+e)\nu g_0\sin^{(2)}2\phi\Big\{16(5+3e)\eta^{(2)}R^{(2)}\cos^{(2)}2\phi-3\sqrt{\pi}\big(4(4-3e){\lambda^{(2)}}^2-32(1+e){R^{(2)}}^2\big)\Big\}\Big]}{21\sqrt{\pi}\{2(1+e)(1-3e)\nu g_0-5\}} \tag{B 7}
$$

These perturbation solutions are used in §4.2 to construct the complete solution at fourth-order in the shear rate.

Appendix D. Source of second moment tensor

Retaining terms up-to $O(\eta^m\lambda^m R^p\sin^q(2\phi), m+n+p+q \leqslant 4)$, the expressions for the non-zero elements of the source of the second moment tensor (5.16) in USF are given by

$$\begin{aligned}
\aleph_{xx} &= A_{xx} + \widehat{E}_{xx} + \widehat{G}_{xx} + 2\dot{\gamma}\Theta_{xy} \\
&= -\frac{(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{385\sigma\pi^{\frac{1}{2}}}\Big[3080 + 12672R^2 + 5120R^4 + 396\eta^2 - 640\eta^2R^2 + 15\eta^4 \\
&\quad +1848\lambda^2 + 4224\lambda^2R^2 - 132\eta^2\lambda^2 + 660\lambda^4 + 3168\sqrt{\pi}\eta R\cos 2\phi - 320\eta^2R^2\cos 4\phi \\
&\quad +44\eta\sin 2\phi\Big\{42 + 32R^2 - \eta^2 + 12\lambda^2\Big\}\Big] \\
&\quad -\frac{8(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{385\sigma\pi^{\frac{1}{2}}}\Big[22\eta^2 - 64\eta^2R^2 + \eta^4 + 462\lambda^2 + 1056\lambda^2R^2 - 11\eta^2\lambda^2 - 66\lambda^4 \\
&\quad +4\sqrt{\pi}R(33+32R^2)\eta\cos 2\phi - 32R^2\eta^2\cos 4\phi + 11\eta\sin 2\phi\Big\{42 + 224R^2 - \eta^2 + 12\lambda^2\Big\}\Big] \\
&\quad -\frac{8(1+e)\rho\nu g_0 T\dot{\gamma}}{1155\pi^{\frac{1}{2}}}\Big[3\sqrt{\pi}\eta\Big(77 - 32R^2\Big)\cos 2\phi + 32R\Big\{66 + 48R^2 - 2\eta^2 \\
&\quad -\eta^2\cos 4\phi - 44\eta\sin 2\phi\Big\}\Big],
\end{aligned} \tag{D 1}$$

$$\begin{aligned}
\aleph_{yy} &= A_{yy} + \widehat{E}_{yy} + \widehat{G}_{yy} - 2\dot{\gamma}\Theta_{xy} \\
&= -\frac{(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{385\sigma\pi^{\frac{1}{2}}}\Big[3080 + 12672R^2 + 5120R^4 + 396\eta^2 - 640\eta^2R^2 + 15\eta^4 \\
&\quad +1848\lambda^2 + 4224\lambda^2R^2 - 132\eta^2\lambda^2 + 660\lambda^4 + 3168\sqrt{\pi}\eta R\cos 2\phi - 320\eta^2R^2\cos 4\phi \\
&\quad -44\eta\sin 2\phi\Big\{42 + 32R^2 - \eta^2 + 12\lambda^2\Big\}\Big] \\
&\quad -\frac{8(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{385\sigma\pi^{\frac{1}{2}}}\Big[22\eta^2 - 64\eta^2R^2 + \eta^4 + 462\lambda^2 + 1056\lambda^2R^2 - 11\eta^2\lambda^2 - 66\lambda^4 \\
&\quad +4\sqrt{\pi}R(33+32R^2)\eta\cos 2\phi - 32R^2\eta^2\cos 4\phi - 11\eta\sin 2\phi\Big\{42 + 224R^2 - \eta^2 + 12\lambda^2\Big\}\Big] \\
&\quad +\frac{8(1+e)\rho\nu g_0 T\dot{\gamma}}{1155\pi^{\frac{1}{2}}}\Big[3\sqrt{\pi}\eta\Big(77 + 32R^2\Big)\cos 2\phi + 8R\Big\{198 + 160R^2 - 14\eta^2 \\
&\quad +132\lambda^2 - 7\eta^2\cos 4\phi - 176\eta\sin 2\phi\Big],
\end{aligned} \tag{D 2}$$

$$\begin{aligned}
\aleph_{zz} &= A_{zz} + \widehat{E}_{zz} + \widehat{G}_{zz} \\
&= -\frac{(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{385\sigma\pi^{\frac{1}{2}}}\Big[3080 + 4224R^2 + 1024R^4 + 132\eta^2 - 128R^2\eta^2 + 3\eta^4 - 3696\lambda^2 \\
&\quad +1452\lambda^4 + 1056\sqrt{\pi}R\eta\cos 2\phi - 64R^2\eta^2\cos 4\phi\Big] \\
&\quad +\frac{16(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{385\sigma\pi^{\frac{1}{2}}}\Big[22\eta^2 - 64R^2\eta^2 + \eta^4 + 462\lambda^2 + 1056R^2\lambda^2 - 11\eta^2\lambda^2 - 66\lambda^4 \\
&\quad +4\sqrt{\pi}R(33+32R^2)\eta\cos 2\phi - 32R^2\eta^2\cos 4\phi\Big] \\
&\quad +\frac{64(1+e)\rho\nu g_0 T\dot{\gamma}}{1155\pi^{\frac{1}{2}}}R\Big(66 + 32R^2 + 6\eta^2 - 132\lambda^2 - 24\sqrt{\pi}R\eta\cos 2\phi + 3\eta^2\cos 4\phi\Big),
\end{aligned} \tag{D 3}$$

$$
\begin{aligned}
\aleph_{xy} &= A_{xy} + \widehat{E}_{xy} + \widehat{G}_{xy} + \dot{\gamma}\Big(\Theta_{yy} - \Theta_{xx}\Big) \\
&= \frac{4(1-e^2)\rho\nu g_0 T^{\frac{3}{2}}}{35\sigma\pi^{\frac{1}{2}}}\Big[4\sqrt{\pi}R(21+32R^2+12\lambda^2) + \eta(42+96R^2-\eta^2+12\lambda^2)\cos 2\phi\Big] \\
&\quad + \frac{8(1+e)\rho\nu g_0 T^{\frac{3}{2}}}{35\sigma\pi^{\frac{1}{2}}}\Big[36\sqrt{\pi}R\lambda^2 + \eta(42+96R^2-\eta^2+12\lambda^2)\cos 2\phi\Big] \\
&\quad + \frac{4(1+e)\rho\nu g_0 T\dot{\gamma}}{1155\sqrt{\pi}}\Big[\sqrt{\pi}(693+1056R^2-576R^2\lambda^2\cos^2 2\phi - 462\eta\sin 2\phi) \\
&\quad - 4R\eta(132+15\eta^2-145\lambda^2)\cos 2\phi + 60R\eta^3\cos 6\phi \\
&\quad - 148\eta\lambda^2 R\cos 6\phi + 88R\eta^2\sin 4\phi\Big].
\end{aligned} \tag{D 4}
$$

The above expressions are used in §5.4 of this manuscript to obtain an analytical form for the collisional dissipation rate that holds beyond Navier-Stokes order.