Other Supplementary Material
Observation of resonant interactions among surface gravity waves

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1 Introduction

In this supplementary material, we derive the equations that are used in the main article on degenerated resonance. Hamiltonian theory is presented following Zakharov (1968). Such a model yields slow-time evolution of complex wave amplitudes and receives continuous interest, especially in four-wave interactions (see e.g. Stiassnie & Shemer (2005); Janssen (2009); Leblanc (2009)).

We use this Hamiltonian approach to derive a solution at short-time in degenerated four-wave interactions in order to explain the sinc detuning (or phase mismatch) factor first introduced in Longuet-Higgins & Smith (1966) when the waves are off-resonance. Longuet-Higgins & Smith (1966) based an explanation for this sinc term on the superposition of the near-resonant daughter wave and a free wave generated by the wavemaker. Both waves have the same amplitude to ensure a zero-flux boundary condition. Saying this, they neglected the evanescent waves which are known to make an important contribution to free wave generation (Hudspeth & Sulisz, 1991).

Assuming constant mother-wave amplitudes (see Boyd (2008) in optics), we recover all the previous results in Longuet-Higgins (1962), including the phase-locking observed also in our experiments. We show that for resonance or near-resonance, the total phase is found to be initially locked to $\pi/2$ (valid only for short times) and then to slowly evolve away from this initial value. Concerning the phase mismatch factor, note that Tomita (1989) already used the same Hamiltonian derivation and found the intermediate sine solution for the daughter-wave amplitude yet without linking it to the detuning sinc behavior first described by Longuet-Higgins (1962). In order to improve the accessibility of this type of results, we present the relevant theory in readable form with full details in this supplementary material.

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2 Approximate Hamiltonian Theory

2.1 General case

The dynamical Hamiltonian theory is presented here to account for phase evolution and off-resonance solution; we follow the formalism from Janssen (2009) and Zakharov *et al.* (1992). The potential flow unknowns are the free surface elevation $\eta(\mathbf{x},t)$ and the free surface potential $\psi(\mathbf{x},t)$. The latter is the value of the potential of the flow ϕ taken at the free surface, that is $\psi(\mathbf{x},t)=\phi(\mathbf{x},z=\eta(\mathbf{x},t),t)$. The corresponding space Fourier transforms $\hat{\eta}$ and $\hat{\psi}$ are defined by

$$\hat{\eta}(\mathbf{k},t) = \frac{1}{2\pi}\int d\mathbf{x}\,\eta(\mathbf{x},t)\,\exp\left(-\mathrm{i}\mathbf{k}.\mathbf{x}\right)\ , \qquad (1)$$

$$\hat{\psi}(\mathbf{k},t) = \frac{1}{2\pi}\int d\mathbf{x}\,\psi(\mathbf{x},t)\,\exp\left(-\mathrm{i}\mathbf{k}.\mathbf{x}\right)\ . \qquad (2)$$

For each wavevector $\mathbf{k}$, the corresponding frequency ω is given by the considered dispersion relation $\omega(\mathbf{k})$. These transforms are multiplied by $\sqrt{\omega/k}$ and $\sqrt{k/\omega}$ respectively where $k=|\mathbf{k}|$ so that the resulting amplitudes have the same dimension. After this first canonical transformation[1], the complex action variable $A(\mathbf{k})$ is defined as follows by a second canonical transformation

$$A(\mathbf{k},t) = \frac{1}{2^{1/2}}\left[\left(\frac{\omega}{k}\right)^{1/2}\hat{\eta}(\mathbf{k},t)+\mathrm{i}\left(\frac{k}{\omega}\right)^{1/2}\hat{\psi}(\mathbf{k},t)\right]\ . \qquad (3)$$

We may use later on the following relations

$$\hat{\eta}(\mathbf{k},t) = \left(\frac{k}{2\omega}\right)^{1/2}\left[A(\mathbf{k},t)+A^*(-\mathbf{k},t)\right]\ , \qquad (4)$$

$$\hat{\psi}(\mathbf{k},t) = -\mathrm{i}\left(\frac{\omega}{2k}\right)^{1/2}\left[A(\mathbf{k},t)-A^*(-\mathbf{k},t)\right]\ . \qquad (5)$$

A third canonical transformation from $A(\mathbf{k},t)$ to a new variable $\hat{a}(\mathbf{k},t)$ eliminates the non-resonant interactions describing bound waves in the three-wave and four-wave processes. The reader will refer to Janssen (2009) and Zakharov *et al.* (1992) for more details. By acknowledging the linear evolution of the action variable, a new unknown variable is introduced $B(\mathbf{k},t)=\hat{a}(\mathbf{k},t)\exp(\mathrm{i}\omega(k)t)$ called action amplitude or generalized amplitude spectrum. Further equations for B assess only the nonlinear part. For small wave steepness, the nonlinear evolution of waves with non-decay relation dispersion[2] ($\omega\propto k^{\nu}$ with $\nu<1$) is described by Zakharov's equation (Zakharov, 1968)

$$i\partial_t B_1 = \iiint T_{1234}B_2^*B_3B_4\delta_{1+2-3-4}\,\exp\left(\mathrm{i}\Delta_{1234}t\right)\,d\mathbf{k}_2d\mathbf{k}_3d\mathbf{k}_4\ , \qquad (6)$$

where the frequencies are $\omega_i=\omega(\mathbf{k}_i)$, the frequency detuning or mismatch is $\Delta_{1234}=\omega_1+\omega_2-\omega_3-\omega_4$, $\delta_{1+2-3-4}=\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)$ and $B_i=B(\mathbf{k}_i,t)$ the notation for the action. The interaction coefficients $T_{1234}=T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$ are the kernels given in Krasitskii (1994) or Janssen (2009).

[1] By definition, a canonical transformation preserves the form of the Hamilton's equations describing the system.

[2] All the three-wave interactions are therefore non-resonant.

The free-surface elevation is related to the generalized amplitude spectrum $B(\mathbf{k},t)$ by the above canonical transformations. The nonlinear elevation consists of a linear superposition of free waves and a ensemble of corresponding bound waves. The linear part of the elevation is build as the superposition of free waves

$$\eta_{lin}(\mathbf{x},t) = \int d\mathbf{k} \left(\frac{k}{2\omega}\right)^{1/2} \left[B(\mathbf{k},t)\exp(-\mathrm{i}\omega(k)t) + B^*(-\mathbf{k},t)\exp(\mathrm{i}\omega(k)t)\right] \exp\left(\mathrm{i}\mathbf{k}.\mathbf{x}\right) \ . \tag{7}$$

The bound waves can also be computed by means of the canonical transformations (see Janssen (2009) for the corresponding kernels).

2.2 Degenerated resonance

The degenerated case we study in the paper consists of only three waves, two mother waves 1 and 3 and a daughter wave 4. The wave action amplitude is noted $B(\mathbf{k},t) = B_1(t)\delta(\mathbf{k}-\mathbf{k}_1) + B_3(t)\delta(\mathbf{k}-\mathbf{k}_3) + B_4(t)\delta(\mathbf{k}-\mathbf{k}_4)$ with the resonance condition $2\mathbf{k}_1 - \mathbf{k}_3 - \mathbf{k}_4 = \mathbf{0}$. In the case of an homogeneous wave field, the equations for the degenerated case are deduced from equation (6)

$$\mathrm{i}\partial_t B_1 = (\Omega_1-\omega_1)B_1 + 2T_{1134}\exp(\mathrm{i}\Delta\omega t)B_1^* B_3 B_4 \ , \tag{8a}$$

$$\mathrm{i}\partial_t B_3 = (\Omega_3-\omega_3)B_3 + \ T_{1134}\exp(-\mathrm{i}\Delta\omega t)B_1^2 B_4^* \ , \tag{8b}$$

$$\mathrm{i}\partial_t B_4 = (\Omega_4-\omega_4)B_4 + \ T_{1134}\exp(-\mathrm{i}\Delta\omega t)B_1^2 B_3^* \ , \tag{8c}$$

where $\Delta\omega = 2\omega_1 - \omega_3 - \omega_4$ is the linear frequency detuning. Nonlinear frequencies Ω_i satisfy the following nonlinear dispersion relations

$$\left.\begin{array}{lcl} \Omega_1 &=& \omega_1 + \ T_{1111}|B_1|^2 + 2T_{1313}|B_3|^2 + 2T_{1414}|B_4|^2 \ , \\ \Omega_3 &=& \omega_3 + 2T_{1313}|B_1|^2 + \ T_{3333}|B_3|^2 + 2T_{3434}|B_4|^2 \ , \\ \Omega_4 &=& \omega_4 + 2T_{1414}|B_1|^2 + 2T_{3434}|B_3|^2 + \ T_{4444}|B_4|^2 \ . \end{array}\right\} \tag{9}$$

Here we consider degenerated wave fields where only the two mother waves 1 and 3 are initially present, *i.e.* $B_4(t=0)=0$. In this case, the resonant daughter wave exhibits linear growth in the early stage of the four-wave interaction and an energy exchange happens from mother wave 1 towards mother wave 3 and daughter wave 4. Equations (8) admit self-similar solutions of the form $B_i = B_{i0}\, f_i(\alpha^2 t,\beta)$ for $i=1,3$ with B_{i0} the initial amplitude and f_i functions of unit magnitude; index 0 denotes the solutions at $t=0$. The daughter-wave amplitude may also be written as $B_4 = \alpha\, f_4(\alpha^2 t,\beta)$ where α is the scale $|B_{10}^2 B_{30}|^{1/3}$ and $\beta = |B_{30}/B_{10}|$. Such solutions may be obtained analytically (see Stiassnie & Shemer (2005)) or numerically.

In the following, we define the total detuning $\Delta\Omega = 2\Omega_1 - \Omega_3 - \Omega_4$ and the detuning due to nonlinear effects $\Delta\omega_{nl} = \Delta\Omega - \Delta\omega$.

2.3 Solution at small daughter-wave amplitude

In the early stage of the resonant interaction or for a non-resonant interaction, the daughter-wave amplitude is assumed to be negligible with respect to the mother-wave amplitudes. In equations (8a) and (8b), the first term of the right-hand side is dominant and it follows that the mother waves evolve solely due to the nonlinear dispersion; in other words, they keep a constant amplitude B_{i0} for $i=1$ and 3. In equations (9), the third term of the right-hand side disappear and the mother-wave nonlinear frequency Ω_i as well as the daughter-wave one Ω_4 are also constant.

Reporting this in equations (8a) and (8b), we obtain the mother-wave amplitudes

$$B_i(t) = B_{i0}\,\exp(-\mathrm{i}(\Omega_i - \omega_i)t)\,,\ i = 1\,\mathrm{and}\,3\ . \tag{10}$$

Introducing C_4 such as $B_4(t) = C_4(t)\,\exp(-\mathrm{i}(\Omega_4 - \omega_4)t)$, we obtain from equation (8c) a new equation for C_4

$$\partial_t C_4 = -\mathrm{i}\,T_{1134}\,\exp(-\mathrm{i}\Delta\omega t)B_{10}^2 B_{30}^*\,\exp\left(-\mathrm{i}\Delta\omega_{nl}t\right)\ . \tag{11}$$

We can see that amplitude C_4 accounts for energy transfer as well as for all nonlinear frequency evolution due to the interaction other than the nonlinear dispersion. We have now after straightforward integration of equation (11)

$$B_4 = -\mathrm{i}\,T_{1134}B_{10}^2 B_{30}^*\,\frac{\sin\left(\Delta\Omega t/2\right)}{\Delta\Omega/2}\,\exp(-\mathrm{i}(\Omega_4 - \omega_4 + \Delta\Omega/2)t)\ . \tag{12}$$

This solution (12) is valid as long as $|B_4| \ll |B_{10}|$ and $|B_{30}|$. This expression of the complex amplitude provides the daughter-wave real amplitude and phase. First, since $T_{1134} > 0$ (see Janssen (2009)), the daughter-wave amplitude is

$$|B_4| = T_{1134}\,|B_{10}^2 B_{30}|\,\left|\frac{\sin\left(\Delta\Omega t/2\right)}{\Delta\Omega/2}\right|\ . \tag{13}$$

The linear free surface elevation consists in the surperposition of three waves

$$\eta_{lin} = \frac{1}{2}\left(\sum_i a_i\,\exp(\mathrm{i}(\mathbf{k}_i.\mathbf{x} - \omega_i t + \varphi_i(t))) + c.c.\right)\ ,$$

where i=1, 3 and 4, a_i is the wave amplitude and $\varphi_i(t)$ is the phase due to nonlinear effects, which evolves slowly in time. Using equation (7) and the previous solution for $B_i(t)$, we obtain the mother-wave phases $\varphi_i(t) = -(\Omega_i - \omega_i)\,t + \varphi_{i0}$ for $i = 1$ and 3 where index 0 again denotes initial values. They evolve slowly due to the nonlinear correction in the dispersion relations given in equations (9), whereas the daughter-wave phase taken from equation (12) evolves due to two terms

$$\varphi_4(t) = -(\Omega_4 - \omega_4)t - \Delta\Omega t/2 + 2\varphi_{10} - \varphi_{30} - \frac{\pi}{2}\ . \tag{14}$$

The first term is the slow evolution due to nonlinear dispersion correction and the second term comes from the total frequency detuning $\Delta\Omega = \Delta\omega + \Delta\omega_{nl}$ which contains a fast and a slow terms. The last terms show the phase-locking of the daughter wave. Note that the phase $\varphi_4(t)$ could also be defined by $\varphi_4(t) = -(\Omega_4 - \omega_4)t - \Delta\Omega t/2 + \varphi_{40}$. We now introduce the total phase φ as follows

$$\varphi(t) \equiv 2\varphi_1 - \varphi_3 - \varphi_4 = \frac{\pi}{2} - \frac{\Delta\omega + \Delta\omega_{nl}}{2}t\ . \tag{15}$$

Both the linear and the nonlinear part of the frequency detuning play a role in the evolution of the total phase which then vary with a fast term $\Delta\omega t/2$ and a slow term $\Delta\omega_{nl}t/2$ on a $\alpha^2 t$ time scale.

The following relation holds between the generalized amplitude spectrum B_i for wave i and the free surface amplitude η_i

$$a_i = \sqrt{\frac{2k_i}{\omega_i}}\,B_i\ . \tag{16}$$

It provides us the daughter-wave amplitude a_4 as follows

$$|a_4| = T_{1134} \frac{\omega_1}{2k_1^3} \sqrt{\frac{\omega_3\, k_4}{\omega_4\, k_3^3}} \varepsilon_1^2 \varepsilon_3 \left| \frac{\sin(\Delta\Omega t/2)}{\Delta\Omega/2} \right| , \tag{17a}$$

$$\arg a_4 = -\frac{\pi}{2} + 2 \arg a_{10} - \arg a_{30} - (\Omega_4 - \omega_4 + \Delta\Omega/2) t , \tag{17b}$$

where the steepness is given by $\varepsilon_i = k_i |a_{i0}|$.

2.3.1 Exact linear resonance

At resonance ($\Delta\omega = 0$) and for the initial stage of the interaction ($\Delta\omega_{nl} t \ll 1$), equation (15) gives $\varphi = \pi/2$ and equation (13) predicts a linear growth of the daughter wave, with a maximum growth rate

$$|B_4| = T_{1134} \, |B_{10}^2 B_{30}| \, t \, . \tag{18}$$

From equation (14), phase-locking is expected for the daughter wave whose initial phase is naturally set to $\varphi_{40} = -\pi/2 + 2\varphi_{10} - \varphi_{30}$. These are the results obtained by means of perturbation theory in Longuet-Higgins (1962). Although the phase locking was not explicitly mentionned, it was implicitly accounted for. In Longuet-Higgins (1962), the mother-wave profile is described by $\eta_i = a_i \cos\psi_i$ with $\psi_i = \mathbf{k}_i.\mathbf{x} - \omega_i t$ for $i = 1$ and 3. The evolution of the resonant daughter wave is given by $\eta_4 = a_4 \sin(2\psi_1 - \psi_3)$. It follows using the resonance conditions that $\sin(2\psi_1 - \psi_3) = \cos(\psi_4 - \pi/2)$ and hence the phase of the daughter wave is locked to $-\pi/2$ by comparison to the mother waves.

The bound waves associated with the quartet are given at frequency $2\omega_1 - \omega_3$ by $B_{1134}^{(2)} B_{10}^2 B_{30}^*$ (see e.g. Janssen (2009) for an expression of kernel $B_{1234}^{(2)}$). We have checked numerically (not shown here) that they have negligible amplitudes compared to the resonant daughter wave at the same frequency.

Concerning the phase evolution, the above solutions for φ_i show that all waves phases will evolve with slow nonlinear time (as long as the daughter-wave amplitude is small). From equation (15), we see that the total phase $\varphi = \pi/2 + \Delta\omega_{nl} t/2$ will evolve from its initial $\pi/2$ value on the long time scale $\alpha^2 t$. In other words, the concept of linear resonance ($\Delta\omega = 0$) is valid in the early stage but it does not make sense at longer time since the total phase follow a slow nonlinear evolution.

2.3.2 Off-resonance

We consider now an off-resonance degenerated quartet with a linear frequency detuning $\Delta\omega \neq 0$. Equation (13) gives the expression of the off-resonance daughter-wave amplitude, valid when $|B_4| \ll |B_{10}|$ and $|B_{30}|$. For interpretation and following Longuet-Higgins (1962), we rewrite equation (13) as

$$\frac{|B_4|}{\alpha} = T_{1134} \alpha^2 t \left| \frac{\sin \frac{1}{2}\Delta\Omega t}{\frac{1}{2}\Delta\Omega t} \right| = T_{1134} \alpha^2 t \, \left| \mathrm{sinc} \left(\frac{1}{2} \Delta\Omega t \right) \right| \, . \tag{19}$$

In this equation (19) we have emphasized on

- the scaling $\alpha = \left| B_{10}^2 B_{30} \right|^{1/3}$ of the daughter-wave amplitude,
- the resonant growth $T_{1134}\alpha^2 t$, linear in the slow time scale $\alpha^2 t$,
- the off-resonance correction factor $\mathrm{sinc}\frac{1}{2}\Delta\Omega t$ (known as phase mismatch factor in optics).

This last amplitude modulation should not shadow the real evolution observed in equation (13) which is a sine function.

3 Discussion

Here the Zakharov equation is applied in the context of degenerated resonant interaction of four waves. The strong approximation we made is to restrict the wave spectrum to three interacting waves only, namely two mother waves present at the start and a daughter wave growing in time. The theoretical developments are not limited to surface gravity waves, and should apply to any nonlinear wave system having forbidden three-wave interactions.

The solution is found for both resonant and non-resonant cases when the daughter-wave amplitude is small and the solution agrees well with the ones in Longuet-Higgins (1962) in terms of linear growth rate and in Longuet-Higgins & Smith (1966) in terms of sinc behavior. Note that Longuet-Higgins & Smith (1966) used a different explanation for the sinc term, based on wavemaker free wave emission, which now seems uncorrect in the light of the dynamical theory used here. In their use of zero-flux boundary condition on the wavemaker they neglected evanescent waves which are known to contribute to an important part of the free wave emission (Hudspeth & Sulisz, 1991). Tomita (1989) found also the sine solution for $|B_4|$ contained in equation (13) without linking it however to the detuning sinc behavior first observed experimentally by Longuet-Higgins & Smith (1966) and McGoldrick *et al.* (1966).

3.1 Waves in basins

In the case of mechanically generated waves, the experiments show that the growing daughter wave has a frequency in exact resonance condition $\omega_4 = 2\omega_1 - \omega_3$. The direction θ_4 of the daughter-wave wavenumber $\mathbf{k}_4$ is still unknown ; the condition for wavenumbers may be not fullfilled and a mismatch or detuning can exist $\mathbf{\Delta k} = 2\mathbf{k}_1 - \mathbf{k}_3 - \mathbf{k}_4$. Although the direction of the daughter wave is not specified, we assume that the fastest growing daughter wave is the one with minimal detuning. In other words, the daughter wave propagates along the direction of $2\mathbf{k}_1 - \mathbf{k}_3$ and the corresponding detuning is now $\Delta k = |2\mathbf{k}_1 - \mathbf{k}_3| - k(2\omega_1 - \omega_3)$.

Using equation (16) to convert from generalized to wave amplitudes, we obtain the relation between the interaction kernel T_{1134} and the space amplification factor G in Longuet-Higgins (1962)

$$G = \frac{T_{1134}}{k_1^3} \left(\frac{k_4}{k_1}\right)^{3/4} \left(\frac{k_3}{k_1}\right)^{-5/4} .$$

3.2 Large amplitude

At or near resonance and after a long enough time the daughter wave may reach large amplitude and our assumption $|B_4| \ll |B_1|$ and $|B_3|$ becomes invalid. In that case the exact analytical solution is expressed by means of Jacobian elliptic functions (see Stiassnie & Shemer (2005) for instance). The idea or concept of exact resonance must be limited to only the initial stage. Furthermore, no exact nonlinear resonance conditions can exist since the total phase evolves nonlinearly in time; the detuning $\Delta\Omega$ cannot stay null if a resonant transfer occurs as the amplitudes will evolve and then modify the detuning. In other words, all the four-wave interactions are off-resonance ones. The concept of exact resonance is meaningfull only at the initial stage. It corresponds to a linear growth of the daughter wave with maximum growth rate.

References

Boyd, R. W. 2008 *Nonlinear Optics, Third Edition*, 3rd edn. Academic Press.

Hudspeth, R. T. & Sulisz, W. 1991 Stokes drift in two-dimensional wave flumes. *J. Fluid Mech.* **230**, 209–229.

Janssen, P. A. E. M. 2009 On some consequences of the canonical transformation in the hamiltonian theory of water waves. *J. Fluid Mech.* **637**, 1–44.

Krasitskii, V. P. 1994 On reduced equations in the hamiltonian theory of weakly nonlinear surface waves. *J. Fluid Mech.* **272**, 1 – 20.

Leblanc, S. 2009 Stability of bichromatic gravity waves on deep water. *Eur. J. Mech. / B Fluids* **28** (5), 605–612.

Longuet-Higgins, M. S. 1962 Resonant interactions between two trains of gravity waves. *J. Fluid Mech.* **12**, 321–32,
We have noticed a misprint in equation (6.4) in Longuet-Higgins (1962): the term $-(6+\xi^2)^{1/2}$ should be replaced by $-\mathrm{sgn}(\xi)(6+\xi^2)^{1/2}$ where $\xi = (1-r)/r$.

Longuet-Higgins, M. S. & Smith, N. D. 1966 An experiment on third-order resonant wave interactions. *J. Fluid Mech.* **25**, 417–435.

McGoldrick, L. F., Phillips, O. M., Huang, N. E. & Hodgson, T. H. 1966 Measurements of third-order resonant wave interactions. *J. Fluid Mech.* **25**, 437–456.

Stiassnie, M. & Shemer, L. 2005 On the interaction of four water-waves. *Wave Motion* **41** (4), 307 – 328.

Tomita, H. 1989 Theoretical and experimental investigations of interaction among deep-water gravity waves. *Report of Ship Res. Inst.* **26** (5), 251–350.

Zakharov, V. 1968 Stability of periodic waves of finite amplitude on a surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **2**, 190–198.

Zakharov, V.E., Lvov, V.S. & Falkovich, G. 1992 *Kolmogorov spectra of turbulence*. *Springer series in nonlinear dynamics* vol. 1. Springer-Verlag.