# Supplementary material: Subsonic flow past localised heating elements in boundary layers

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**Supplementary material: Solution of the linearised subsonic flow problem**

The expressions for the perturbed pressure and wall shear are given by

$$\frac{\partial \tilde{P}}{\partial x_b} = \frac{\mathrm{Ai}'(0)}{2\pi \mathrm{Ai}(0)} \int_{-\infty}^{\infty} \frac{(i\omega)^{\frac{5}{3}} \hat{f}(\omega) + (i\omega)^2 \hat{H}(\omega) \Lambda}{i\mathrm{sgn}(\omega)\Theta + (i\omega)^{\frac{4}{3}}} e^{i\omega x_b} d\omega \tag{0.1}$$

and

$$\frac{\partial \hat{U}}{\partial y_b}(x_b, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(i\omega)^{\frac{4}{3}} \hat{f}(\omega) + (i\omega)^{\frac{5}{3}} \hat{H}(\omega) \Lambda}{i\mathrm{sgn}(\omega)\Theta + (i\omega)^{\frac{4}{3}}} e^{i\omega x_b} d\omega \tag{0.2}$$

where $\Theta = -3\mathrm{Ai}'(0)$ and $\Lambda = 3\mathrm{Ai}(0)$

In order to express (0.1) and (0.2) in forms that are easier to deal with numerically, we first write the integrals as

$$\begin{aligned}\frac{\partial \tilde{P}}{\partial x_b} = \frac{\mathrm{Ai}'(0)}{2\pi \mathrm{Ai}(0)} [\int_0^{\infty} & \frac{r^{\frac{5}{3}} e^{-\frac{i5\pi}{6}} \hat{f}(-r) + r^2 e^{-i\pi} \hat{H}(r) \Lambda}{-i\Theta + r^{\frac{4}{3}} e^{-\frac{i2\pi}{3}}} e^{-irx_b} dr \\ & + \int_0^{\infty} \frac{r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{f}(r) + r^2 e^{i\pi} \hat{H}(r) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b} dr]\end{aligned} \tag{0.3}$$

and

$$\begin{aligned}\frac{\partial \hat{U}}{\partial y_b}(x_b, 0) = \frac{1}{2\pi} [\int_0^{\infty} & \frac{r^{\frac{4}{3}} e^{-\frac{i2\pi}{3}} \hat{f}(-r) + r^{\frac{5}{3}} e^{-\frac{i5\pi}{6}} \hat{H}(r) \Lambda}{-i\Theta + r^{\frac{4}{3}} e^{-\frac{i2\pi}{3}}} e^{-irx_b} dr \\ & + \int_0^{\infty} \frac{r^{\frac{4}{3}} e^{\frac{i2\pi}{3}} \hat{f}(r) + r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{H}(r) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b} dr],\end{aligned} \tag{0.4}$$

where we have written $r = |\omega|$. We assume the heating region is of size 1 and located in a domain where $x_b \in [-0.5, 0.5]$ and that the wall temperature

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distribution is given by

$$\tilde{T}_w(x_b) = \begin{cases} 0.2 & |x_b| < 0.5 \\ 0 & |x_b| > 0.5 \end{cases}. \tag{0.5}$$

Applying the Fourier transform to (0.5), yields that

$$\hat{f}(\omega) = \frac{0.4}{\omega}\sin(\frac{\omega}{2}). \tag{0.6}$$

We also assume the hump shape is given by

$$\tilde{H}(x_b) = \tilde{h}\exp(-5x_b^2),$$

and the Fourier transform of $\tilde{H}(x_b)$ is

$$\mathcal{F}\{\tilde{H}(x_b)\} = \hat{H}(\omega) = \tilde{h}\frac{\sqrt{5\pi}}{5}\exp(-\frac{\omega^2}{20}). \tag{0.7}$$

Clearly, the integrand in (0.3) can be expressed as twice the real part of

$$\frac{r^{\frac{5}{3}}e^{\frac{i5\pi}{6}}\hat{f}(r) + r^2 e^{i\pi}\hat{H}(r)\mathit{\Lambda}}{i\mathit{\Theta} + r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}}e^{irx_b}.$$

To find the real part and using (0.6) and (0.7) it can be shown that,

$$\frac{r^{\frac{5}{3}}e^{\frac{i5\pi}{6}}\hat{f}(r) + r^2 e^{i\pi}\hat{H}(r)\mathit{\Lambda}}{i\mathit{\Theta} + r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}}e^{irx_b}$$

$$= \left[\frac{0.4r^{\frac{2}{3}}\sin(\frac{r}{2})}{\sqrt{[\frac{1}{2}\mathit{\Theta} + r^{\frac{4}{3}}\frac{\sqrt{3}}{2}]^2 + [-r^{\frac{4}{3}}\frac{1}{2} - \frac{\sqrt{3}}{2}\mathit{\Theta}]^2}e^{i\boldsymbol{\phi}}} + \frac{r^2(\tilde{h}\frac{\sqrt{5\pi}}{5}e^{-\frac{r^2}{20}})\mathit{\Lambda}}{\sqrt{[r^{\frac{4}{3}}\frac{1}{2}]^2 + [-\mathit{\Theta} - r^{\frac{4}{3}}\frac{\sqrt{3}}{2}]^2}e^{i\alpha}}\right]e^{irx_b}$$

where $\boldsymbol{\phi} = \arctan(\frac{-r^{\frac{4}{3}}\frac{1}{2} - \frac{\sqrt{3}}{2}\mathit{\Theta}}{\frac{1}{2}\mathit{\Theta} + r^{\frac{4}{3}}\frac{\sqrt{3}}{2}})$ and $\alpha = \arctan(\frac{-\mathit{\Theta} - r^{\frac{4}{3}}\frac{\sqrt{3}}{2}}{\frac{1}{2}r^{\frac{4}{3}}})$.

Taking the real part, we obtain

$$\mathfrak{Re}\{\frac{r^{\frac{5}{3}}e^{\frac{i5\pi}{6}}\hat{f}(r) + r^2 e^{i\pi}(\tilde{h}\frac{\sqrt{5\pi}}{5}e^{-\frac{r^2}{20}})\mathit{\Lambda}}{i\mathit{\Theta} + r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}}e^{irx_b}\}$$

$$= \frac{0.4r^{\frac{2}{3}}\sin(\frac{r}{2})\cos(rx_b - \phi)}{\sqrt{[\frac{1}{2}\mathit{\Theta} + r^{\frac{4}{3}}\frac{\sqrt{3}}{2}]^2 + [-r^{\frac{4}{3}}\frac{1}{2} - \frac{\sqrt{3}}{2}\mathit{\Theta}]^2}} + \frac{r^2(\tilde{h}\frac{\sqrt{5\pi}}{5}\exp(-\frac{r^2}{20}))\mathit{\Lambda}\cos(rx_b - \alpha)}{\sqrt{[-r^{\frac{4}{3}}\frac{1}{2}]^2 + [\mathit{\Theta} + r^{\frac{4}{3}}\frac{\sqrt{3}}{2}]^2}}.$$

Therefore, the gradient pressure distribution is given by

$$\frac{\partial\tilde{P}}{\partial x_b} = \frac{\mathrm{Ai}'(0)}{\pi\mathrm{Ai}(0)}\int_0^\infty [\frac{0.4r^{\frac{2}{3}}\sin(\frac{r}{2})\cos(rx_b - \phi)}{[\mathit{\Theta}^2 + \sqrt{3}\mathit{\Theta}r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}} \tag{0.8}$$

$$+\frac{r^2\tilde{h}\frac{\sqrt{5\pi}}{5}\exp(-\frac{r^2}{20})\mathit{\Lambda}\cos(rx_b - \alpha)}{[\mathit{\Theta}^2 + \sqrt{3}\mathit{\Theta}r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}}]dr$$

where $\phi = \arctan(-\frac{r^{\frac{4}{3}}\frac{1}{2}+\frac{\sqrt{3}}{2}\Theta}{\frac{1}{2}\Theta+r^{\frac{4}{3}}\frac{\sqrt{3}}{2}})$ and $\alpha = \arctan(-\frac{\Theta+r^{\frac{4}{3}}\frac{\sqrt{3}}{2}}{\frac{1}{2}r^{\frac{4}{3}}})$.

The perturbed wall shear given by (0.4), can be dealt with in a similar way and we find that

$$\mathfrak{Re}\{\frac{r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}\hat{f}(r)+r^{\frac{5}{3}}e^{\frac{i5\pi}{6}}\hat{H}(r)\Lambda}{i\Theta+r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}}e^{irx_b}\}$$
$$=[\frac{0.4r^{\frac{1}{3}}\sin(\frac{r}{2})\cos(rx_b-\phi_1)}{[\Theta^2+\sqrt{3}\Theta r^{\frac{4}{3}}+r^{\frac{8}{3}}]^{\frac{1}{2}}}+\frac{r^{\frac{5}{3}}(\tilde{h}\frac{\sqrt{5\pi}}{5}e^{-\frac{r^2}{20}})\Lambda\cos(rx_b-\alpha_1)}{[\Theta^2+\sqrt{3}\Theta r^{\frac{4}{3}}+r^{\frac{8}{3}}]^{\frac{1}{2}}}].$$

Therefore, the wall shear distribution is given by

$$\frac{\partial\hat{U}}{\partial y_b}(x_b,0)=\frac{1}{\pi}\int_0^\infty[\frac{0.4r^{\frac{1}{3}}\sin(\frac{r}{2})\cos(rx_b-\phi_1)}{[\Theta^2+\sqrt{3}\Theta r^{\frac{4}{3}}+r^{\frac{8}{3}}]^{\frac{1}{2}}}+$$
$$\frac{r^{\frac{5}{3}}(\tilde{h}\frac{\sqrt{5\pi}}{5}e^{-\frac{r^2}{20}})\Lambda\cos(rx_b-\alpha_1)}{[\Theta^2+\sqrt{3}\Theta r^{\frac{4}{3}}+r^{\frac{8}{3}}]^{\frac{1}{2}}}]dr \tag{0.9}$$

where $\phi_1 = \arctan(\frac{-\frac{\Theta}{2}}{\frac{\sqrt{3}}{2}\Theta+r^{\frac{4}{3}}})$ and $\alpha_1 = \arctan(\frac{-\frac{\sqrt{3}}{2}\Theta-\frac{1}{2}r^{\frac{4}{3}}}{\frac{\sqrt{3}}{2}r^{\frac{4}{3}}+\frac{1}{2}\Theta})$.