Understanding evolution of vortex rings in viscous fluids

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1. Supplementary Material

Vortex rings are classic fluid phenomena. It is one of the simplest form of 3D coherent flows which are central to large number of natural occurrences like tornadoes, hurricanes etc. (Smith *et al.* 2005; Dengler & Reeder 1997; Emanuel 2005). These motions are also observed in wakes of aircrafts, submarines and ocean liners which affect the trajectory of succeeding vessels (Hill 1975; Spalart 1998). Vortices are formed due to various mechanisms like baroclinicity (Dahm *et al.* 1989), fluid instabilities like Richtmyer-Meshkov, gravity driven (Rayleigh-Taylor) and shear driven (Kelvin-Helmholtz) (Manu *et al.* 2015). Studies are also conducted to understand vortex formation during drop splashing (Thomson & Newall 1885; Lee *et al.* 2015). At a more fundamental level, vortex rings are also produced by charged particles propagating in superfluids (Rayfield & Reif 1964). In addition to above, study of vortex rings have attracted many researchers due to its notable existence, at different spatio-temporal scales, in fluid flow (She *et al.* 1990; Pullin & Saffman 1998).

Vortex interactions have also been found to play a major role in various fluid dynamic and heat transfer processes. Once formed, vortex rings are self-sustained and self-propelled. There have been attempts to utilise this feature for designing better exhaust mechanisms (Turner 1960; Fohl 1967). In addition, vortex interactions with stratified mediums exist in atmosphere and oceans resulting in alteration of the rate of mixing (Linden 1973; Orlandi *et al.* 1998). Vortex formation, growth and eventual shedding into the wake is also of major interest because of the propulsive nature of resulting coherent structures (Spedding *et al.* 2003; Dabiri & Gharib 2005). Thus vortex shedding has found its significance in varied engineering applications (e.g. transportation, flow control, etc.) as well as in biological systems (e.g. fish swimming and insect flight) (Moore & Saffman 1973; Weihs 1973; Lugt 1983; Dabiri 2009). Number of studies (Gharib *et al.* 2006; Arvidsson *et al.* 2016) have also highlighted the importance of vortices for diagnosis of various ailments.

Laboratory experiments on vortex rings are usually performed to understand vortex formation, evolution and interaction and decay. Vortices are usually formed by forcing bulk fluid through a constricted opening like an orifice or a nozzle with the help of piston-cylinder arrangement. Recent studies by Gharib *et al.* (1998) showed that there is a maximum cap on the circulation that can be induced by a single vortex ring. According to this study, single vortex is obtained for values of $L/D_n < 4$, where 'L' is the stroke length

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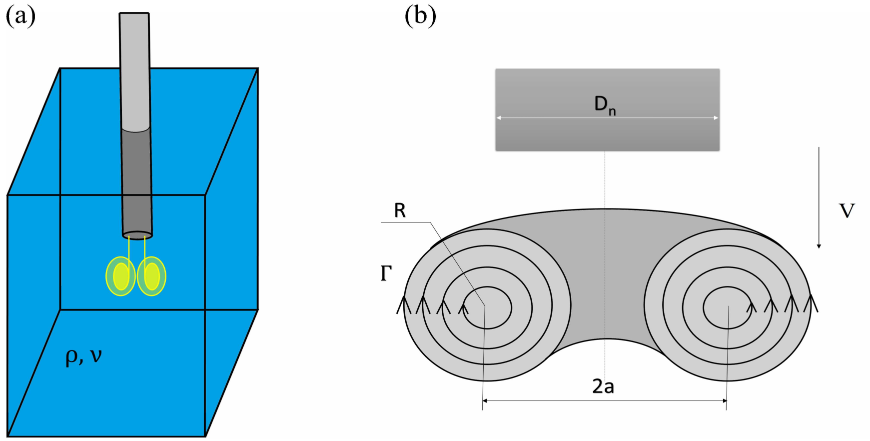

FIGURE 1. Schematic of vortex ring generation and vortex parameters

and 'D_n' is the internal nozzle diameter. For higher values of L/D_n, additional circulation is carried by the wake and a secondary detached vortex is observed. Though stroke length'L' was clearly defined for piston cylinder mechanism, the concept of formation number is equally effective in other cases.

Analytical studies of vortex ring evolution can be traced back to the work of Kelvin (1867) and Helmholtz (1858). Early theoretical studies on formation and evolution of vortex rings were conducted by Lamb (1932) and Sommerfeld (1950) assuming inviscid behaviour. Lamb provided mathematical derivation of Kelvins formula (1.1) for ring velocity

$$V = \frac{\Gamma}{4\pi a}\left[\ln\left(\frac{8a}{r_c}\right) - B + O\left(\frac{r_c}{a}\right)\right] \tag{1.1}$$

Where 'a' is the ring radius and 'r_c' is the size of vortex core. Value of 'B', which takes into account the effect of vorticity distribution, is (1/4) for Rankine vortex, the one where vorticity is evenly distributed in its core. The same constant is (1/2) for hollow cores.

Studies show that a fully developed axisymmetric vortex is governed by two parameters, namely vorticity distribution and toroidal radius as shown in fig. 1. Dyson (1893) and Fraenkel (1972), through rigorous mathematical formulations, provided expression for translational velocity of inviscid vortex rings considering small core parameter 'ϵ' which is a ratio of core radius to ring radi (r_c/a_0)

$$V = \frac{\Gamma}{4\pi a}\left[\ln\left(\frac{8}{\epsilon}\right) - \left(\frac{1}{4}\right) - \frac{3\epsilon^2}{8}\left\{\ln\left(\frac{8}{\epsilon}\right) - \left(\frac{5}{4}\right)\right\} + O(\epsilon^4 \ln \epsilon)\right] \tag{1.2}$$

Detailed study of vortex translation in viscous fluid was conducted by Saffman (1970) and Pullin (1979), where they assumed that the vortex core carried a vorticity distribution as given by Lamb-Oseen (1.3)

$$\omega(r,t) = \frac{\Gamma_0}{4\pi\nu t} e^{-\left(\frac{r^2}{4\nu t}\right)} \tag{1.3}$$

Using this, Saffman also derived the vortex propagation velocity to the 1st order given by,

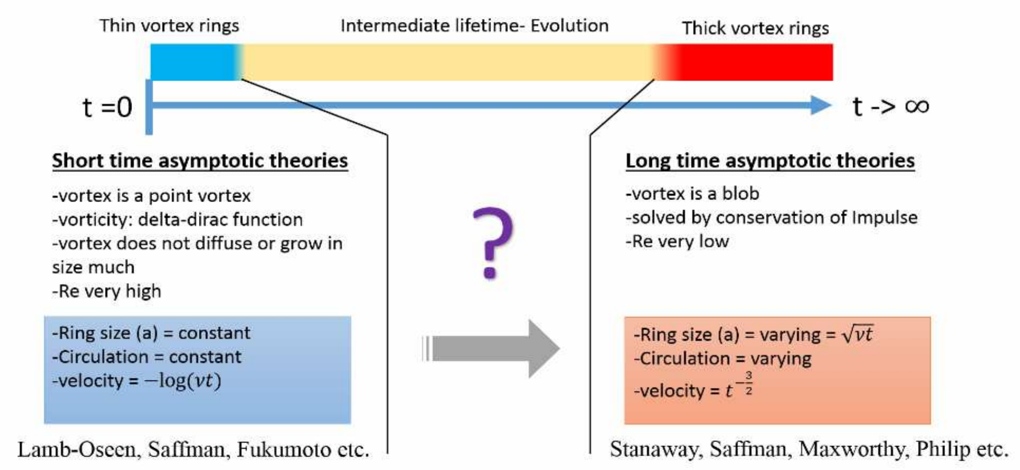

FIGURE 2. Schematic figure representing available literatures with their regime of application

$$V = \frac{\Gamma_0}{4\pi a_0}\left[\ln\left(\frac{8a_0}{\sqrt{4\nu t}}\right) - 0.558 + O\left\{\left(\frac{\nu t}{a_0^2}\right)^{1/2}\ln\left(\frac{\nu t}{a_0^2}\right)\right\}\right] \tag{1.4}$$

Where, expression for Saffman core radius is given by $r_{cs} = \sqrt{4\nu t}$.These equations are valid for finite core sizes in viscous fluids for small time values, typically for t $\ll$ T where T is the piston stroke time. The above results were derived considering (1.3), for which, circulation turns out to be;

$$\Gamma = \int_0^\infty \omega d\psi = constant \tag{1.5}$$

Where, 'ψ' stands for area of selected contour. Thus, constant circulation, when coupled with concept of constant hydrodynamic impulse, mandates vortex ring diameter to remain invariant in time. Thus, thin core model proposed by Saffman could not account for decreasing circulation or ring expansion as observed in the experiments. However, using order of magnitude argument, Saffman proposed relations for translational velocity and radius for thick rings at longer times.

$$a^2 \approx a_0^2 + k'\nu t \tag{1.6}$$

$$V \approx \frac{I}{k}(a_0^2 + k'\nu t)^{-3/2} \tag{1.7}$$

This aspect was further explored by Maxworthy (1972, 1977), where he performed multiple experiments on laminar vortex rings (Re<600). He reported over-prediction in ring expansion given by (1.6). Using his entrainment model, he tried to explain the changes in vortex parameters and provided relations for vortex size, circulation and convective velocity.

$$U \propto t^{-1} \qquad a \propto t^{1/3} \qquad \Gamma \propto t^{-2/3} \tag{1.8}$$

Maxworthy arrived at these empirical relations using scaling arguments and dimensional analysis. Though these equations showed strong resemblance to experimental results, he did not provide any underlying theoretical or mathematical framework. Further studies on vortex translational velocity were performed by Cantwell & Rott (1988) and Stanaway *et al.* (1988) using analytical and numerical methods respectively. Above studies established asymptotic behaviour of vortex velocity, which scales with $t^{-3/2}$ as $t \to \infty$, which is consistent with Saffman's equations (1.6)(1.7). This problem was further studied by Fukumoto & Moffatt (2000) using technique of matched asymptotic expansion for vortex rings in viscous fluids where he accounted for strained elliptic vortex cores. Using this, he extended (1.4) to establish formula for vortex velocity up to third order in 'ϵ'. He also provided relations for vortex ring expansion. However, his higher order model still lacked universal applicability and could not explain dynamics at higher Reynolds numbers. These studies, thus show the lack of consensus and isolation among the proposed theories.

Above mentioned vortex parameters, though intuitively complex, greatly influence vortex interactions with thermal and density stratifications ubiquitous in nature. Numerous studies conducted on different aspects of vortex-interface interactions show that, in addition to parameters like Atwood number and Froude number, microscopic parameters like vorticity distribution and core size also play a key role in determining the nature of interaction (Dahm *et al.* 1989; Orlandi *et al.* 1998). Though significance of these variables have been long known, there have been very little study to incorporate their effects. The major reason for this lack of attention is the complex nature of vortex evolution. Recently, many analytical and numerical studies pertaining to vortex evolution have succeeded in arriving at a near accurate estimation of vortex parameters (Bergdorf *et al.* 2007). However, these equations are either mathematically complex or iterative and hence provide limited insights into understanding derived phenomena like vortex interactions.

In summary, most of the analytical studies on vortex evolution have focused on asymptotic behaviours of its lifetime (figure 2). At the lower end, studies have been conducted for thin vortex rings using classical vorticity equation of Lamb-Oseen (1.3). On the other end, almost all of the studies resort to conservation of hydrodynamic impulse to arrive at long time behaviour. These two extreme approaches, followed throughout the past century, are however fundamentally different from each other. There have been rarely any efforts to link the two. Moreover, any attempt in this direction have been unfruitful and thus the transformation of short time equations into long time regime is a problem, still unresolved. In addition, many different forms of viscous timescales have been used throughout the literature. Parameters like circulation (Γ), viscosity (ν), ring radius (a) and nozzle diameter (D_n) have been used casually and interchangeably, primarily aimed at normalization, with minimum insights and attentions directed to precise understanding of the underlying physical mechanism. Thus the available timescales are far from universal and do not successfully explain the vortex evolution.

REFERENCES

Arvidsson, Per M, Kovács, Sándor J, Töger, Johannes, Borgquist, Rasmus, Heiberg, Einar, Carlsson, Marcus & Arheden, Håkan 2016 Vortex ring behavior provides the epigenetic blueprint for the human heart. *Scientific reports* **6**.

Bergdorf, Michael, Koumoutsakos, Petros & Leonard, Anthony 2007 Direct numerical simulations of vortex rings at re [gamma]= 7500. *Journal of Fluid Mechanics* **581**, 495.

Cantwell, Brian & Rott, Nicholas 1988 The decay of a viscous vortex pair. *The Physics of fluids* **31** (11), 3213–3224.

Dabiri, John O 2009 Optimal vortex formation as a unifying principle in biological propulsion. *Annual Review of Fluid Mechanics* **41**, 17–33.

Dabiri, John O & Gharib, Morteza 2005 The role of optimal vortex formation in biological fluid transport. *Proceedings of the Royal Society of London B: Biological Sciences* **272** (1572), 1557–1560.

Dahm, Werner J.A., Scheil, Christine M. & Tryggvason, Gretar 1989 Dynamics of vortex interaction with a density interface. *J. Fluid Mech.* **205**, 1–43.

Dengler, Klaus & Reeder, Michael J 1997 The effects of convection and baroclinicity on the motion of tropical-cyclone-like vortices. *Quarterly Journal of the Royal Meteorological Society* **123** (539), 699–725.

Dyson, FW 1893 The potential of an anchor ring. part ii. *Philosophical Transactions of the Royal Society of London. A* **184**, 1041–1106.

Emanuel, Kerry 2005 Increasing destructiveness of tropical cyclones over the past 30 years. *Nature* **436** (7051), 686–688.

Fohl, Timothy 1967 Optimization of flow for forcing stack wastes to high altitudes. *Journal of the Air Pollution Control Association* **17** (11), 730–733.

Fraenkel, Lo E 1972 Examples of steady vortex rings of small cross-section in an ideal fluid. *Journal of Fluid Mechanics* **51** (01), 119–135.

Fukumoto, Yasuhide & Moffatt, HK 2000 Motion and expansion of a viscous vortex ring. part 1. a higher-order asymptotic formula for the velocity. *Journal of Fluid Mechanics* **417**, 1–45.

Gharib, Morteza, Rambod, Edmond, Kheradvar, Arash, Sahn, David J & Dabiri, John O 2006 Optimal vortex formation as an index of cardiac health. *Proceedings of the National Academy of Sciences* **103** (16), 6305–6308.

Gharib, Morteza, Rambod, Edmond & Shariff, Karim 1998 A universal time scale for vortex ring formation. *Journal of Fluid Mechanics* **360**, 121–140.

Helmholtz, H 1858 About integrals of hydrodynamic equations related with vortical motions. *J. für die reine Angewandte Mathematik* **55**, 25.

Hill, F.M. 1975 A numerical study of the descent of a vortex pair in a stably stratified atmosphere. *J. Fluid Mech.* **71**, 1–13.

Kelvin, Lord 1867 On vortex atoms. *Proceedings of the Royal Society of Edinburgh* **6**, 94–105.

Lamb, Horace 1932 *Hydrodynamics*. Cambridge university press.

Lee, Ji San, Park, Su Ji, Lee, Jun Ho, Weon, Byung Mook, Fezzaa, Kamel & Je., Jung Ho 2015 Origin and dynamics of vortex rings in drop splashing. *Nature communications* **6**.

Linden, P. F. 1973 The interaction of a vortex ring with a sharp density interface: a model for turbulent entrainment. *J. Fluid Mech.* **60**, 467–480.

Lugt, Hans J 1983 Vortex flow in nature and technology. *New York, Wiley-Interscience, 1983, 305 p. Translation.* **1**.

Manu, KV, Anand, Prateek, Chetia, Utpal Kumar & Basu, Saptarshi 2015 Effects of instabilities and coherent structures on the performance of a thermocline based thermal energy storage. *Applied Thermal Engineering* **87**, 768–778.

Maxworthy, T 1972 The structure and stability of vortex rings. *Journal of Fluid Mechanics* **51** (01), 15–32.

Maxworthy, T 1977 Some experimental studies of vortex rings. *Journal of Fluid Mechanics* **81** (03), 465–495.

Moore, DW & Saffman, PG 1973 Axial flow in laminar trailing vortices. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, , vol. 333, pp. 491–508. The Royal Society.

Orlandi, Paolo, Egermann, P & Hopfinger, EJ 1998 Vortex rings descending in a stratified fluid. *Physics of Fluids* **10** (11), 2819–2827.

Pullin, DI & Saffman, PG 1998 Vortex dynamics in turbulence. *Annual review of fluid mechanics* **30** (1), 31–51.

Pullin, D. I. 1979 Vortex ring formation at tube and orifice openings. *Physics of Fluids* **22** (3), 401–403.

Rayfield, G. W. & Reif, F. 1964 Quantized vortex rings in superfluid helium. *Physical Review* **136**, A1194–A1208.

Saffman, P. G. 1970 The velocity of viscous vortex rings. *Studies in Applied Mathematics* **49** (4), 371–380.

She, Zhen-Su, Jackson, Eric & Orszag, Steven A 1990 Intermittant vortex structures in homogeneous isotropic turbulence. *Nature* **344** (6263), 226.

Smith, Roger K, Montgomery, Michael T & Zhu, Hongyan 2005 Buoyancy in tropical cyclones and other rapidly rotating atmospheric vortices. *Dynamics of atmospheres and oceans* **40** (3), 189–208.

Sommerfeld, Arnold 1950 *Lectures on Theoretical Physics, Vol. Ii: Mechanics of Deformable Bodies*. Academic Press.

Spalart, Philippe R 1998 Airplane trailing vortices. *Annual Review of Fluid Mechanics* **30**, 107–138.

Spedding, GR, Hedenström, Anders & Rosén, Mikael 2003 Quantitative studies of the wakes of freely flying birds in a low-turbulence wind tunnel. *Experiments in Fluids* **34** (2), 291–303.

Stanaway, Sharon K, Cantwell, BJ & Spalart, Philippe R 1988 A numerical study of viscous vortex rings using a spectral method .

Thomson, J. J. & Newall, H. F. 1885 On the formation of vortex rings by drops falling into liquids, and some allied phenomena. *Proceedings of the royal society of London* **39**, 417–436.

Turner, JS 1960 A comparison between buoyant vortex rings and vortex pairs. *Journal of Fluid Mechanics* **7** (03), 419–432.

Weihs, D 1973 Hydromechanics of fish schooling. *Nature* **241** (5387), 290–291.