1 Weakly compressible hollow vortex row

In these supplementary materials to the paper, "The effect of core size on compressible hollow vortex streets" by D. G. Crowdy & V. S. Krishnamurthy, the solution for a weakly compressible single row of hollow vortices as computed by Ardalan *et al* [1] is rederived using the new mathematical approach – based on the Imai-Lamla method and conformal mapping – used in the main body of the paper. This derivation differs from the original approach of Ardalan *et al* [1] and is of interest in its own right.

Let the centroids of the hollow vortices, in a complex $z = x + \mathrm{i}y$ plane, be at $x = nL$ where n is any integer and L is the period of the arrangement. By the periodicity of the row, it is enough to consider a single period window. The notation ∞^+ is used to denote the region of the period window as $y \to +\infty$, while ∞^- denotes the region as $y \to -\infty$. Figure 1 shows a schematic.

Unlike the original derivation Baker *et al* [2] using hodograph variables, Crowdy and Green [6] retrieved the incompressible solution using conformal mapping ideas. Let the unit ζ-disc be transplanted, by the conformal map $z = z_0(\zeta)$, to a single period window of the vortex row containing a single hollow vortex. Since the boundary of the hollow vortex is unknown *a priori*, the challenge is to find the functional form of this conformal mapping; mathematically, this is a free boundary problem. The point $\zeta = \mathrm{i}a$ maps to ∞^+ and $-\mathrm{i}a$ maps to ∞^-. Crowdy & Green [6] show that the incompressible flow solution can be given in the parametric form

$$f = f_0(\zeta) = \frac{\mathrm{i}LU}{2\pi} \log\left(\frac{\zeta^2 + a^2}{\zeta^2 + 1/a^2}\right), \tag{1}$$

$$z = z_0(\zeta) = \frac{L}{\pi}\left[\tan^{-1}(\zeta/a) - a^2 \tan^{-1}(a\zeta)\right]. \tag{2}$$

For given L, the map (2) depends on the single parameter a which reflects the size of each hollow vortex in the row. It is shown in [6] that the solution (1) and (2) is equivalent to that found by Baker, Saffman & Sheffield [2].

1.1 Rayleigh-Jansen expansion

We now construct the first order term in a Rayleigh-Jansen expansion about this incompressible leading order solution. Following [1] we seek weakly compressible solutions for which the circulation of the vortices, the period of the arrangement and constant fluid speed on the boundary take the same values as in the incompressible case.

The new ingredient in our approach is, in addition to expanding the complex potential as in

$$\begin{aligned} f(z,\overline{z}) &= f_0(z) + M^2 f_1(z,\overline{z}) + \ldots, \\ \xi(z,\overline{z}) &= \xi_0(z) + M^2 \xi_1(z,\overline{z}) + \ldots. \end{aligned} \tag{3}$$

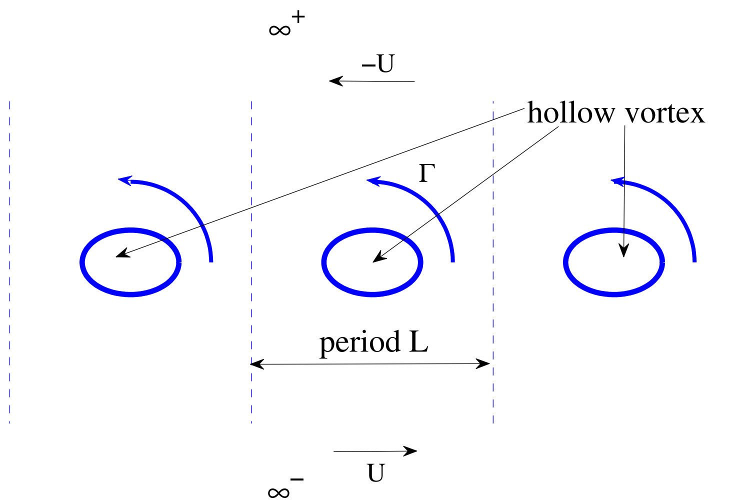

Figure 1: Three periods of a period-L row of hollow vortices.

we will also seek an expansion of a modified conformal mapping function

$$z = z_0(\zeta) + M^2 z_1(\zeta) + \dots, \tag{4}$$

where the first order modification of the mapping, $z_1(\zeta)$, must be found.

The first order correction of the complex potential in (3) is

$$f_1(z, \overline{z}) = \frac{1}{4V_0^2}\xi_0(z)\overline{I(z)} + g(z), \tag{5}$$

where

$$I(z) = \int^z \left(\frac{df_0}{d\tilde{z}}\right)^2 d\tilde{z}, \qquad \xi_0(z) = \frac{df_0}{dz}. \tag{6}$$

This correction can be written in terms of the new parameter ζ variable instead of z. On substitution of (2) into (5) it can be readily shown that

$$f_1 = \frac{\mathrm{i}UL\zeta}{4\pi a}\left[\tan^{-1}(\overline{\zeta}/a) - \frac{1}{a^2}\tan^{-1}(a\overline{\zeta})\right] + G(\zeta), \tag{7}$$

where $G(\zeta)$ is to be found. It is not analytic in the unit disc, and it need not be single-valued there, but it must nevertheless be chosen in order to make the *velocity field* single-valued because encircling either of the two points $\pm\mathrm{i}a$ corresponds to moving to a neighbouring period window (where the velocity field must be identical owing to the required periodicity). Since

$$\tan^{-1}(\zeta) = \frac{\mathrm{i}}{2}\log\left[\frac{1-\mathrm{i}\zeta}{1+\mathrm{i}\zeta}\right], \qquad \overline{\tan^{-1}(\zeta)} = -\frac{\mathrm{i}}{2}\log\left[\frac{1+\mathrm{i}\overline{\zeta}}{1-\mathrm{i}\overline{\zeta}}\right] \tag{8}$$

then we must pick

$$G(\zeta) = -\frac{\mathrm{i}UL\zeta}{4\pi a}\left[\tan^{-1}(\zeta/a)\right] + \tilde{G}(\zeta), \tag{9}$$

where $\tilde{G}(\zeta)$ is single-valued and analytic in the fluid region. Hence

$$f_1 = \frac{\mathrm{i}UL\zeta}{4\pi a}\left[\tan^{-1}(\overline{\zeta}/a) - \tan^{-1}(\zeta/a) - \frac{1}{a^2}\tan^{-1}(a\overline{\zeta})\right] + \tilde{G}(\zeta). \tag{10}$$

The mathematical problem has now been reduced to finding two unknown analytic functions of ζ, namely $\tilde{G}(\zeta)$ and $z_1(\zeta)$. We will show that the function $\tilde{G}(\zeta)$ is determined by imposing the streamline condition on the vortex boundary; once $\tilde{G}(\zeta)$ has been found, $z_1(\zeta)$ is determined from the Bernoulli condition on the vortex boundary.

This general feature remains true in the later Rayleigh-Jansen analysis of the vortex streets. Moreover, it turns out that $\tilde{G}(\zeta)$ and $z_1(\zeta)$ both satisfy a standard boundary value problem known as a Schwarz problem [4, 5]: that is, the problem of finding an analytic function in the unit ζ-disc given either its real, or imaginary, part on the boundary. The general solution of such a problem is given, in principle, by the Poisson integral formula for the unit disc [5]. However, in the analysis to follow, it turns out that we are able to avoid the need for the Poisson integral formula since the required solutions are available by inspection after use of some trigonometric identities.

1.2 Streamline condition

The condition that the hollow vortex boundary is a streamline, meaning that the streamfunction is constant on the hollow vortex boundary, implies the following conditions on the first order correction:

$$\mathrm{Im}[f_1] = 0, \qquad \text{or} \quad \mathrm{Re}[\mathrm{i}f_1] = 0. \tag{11}$$

Hence, on $|\zeta| = 1$ which corresponds to the vortex boundary, and on use of (10), we require that

$$\begin{aligned}
\mathrm{Re}[\mathrm{i}\tilde{G}(\zeta)] &= \mathrm{Re}\left\{\frac{UL\zeta}{4\pi a}\left[\tan^{-1}(\overline{\zeta}/a) - \tan^{-1}(\zeta/a) - \frac{1}{a^2}\tan^{-1}(a\overline{\zeta})\right]\right\} \\
&= \frac{UL}{4\pi a}\left\{\mathrm{Re}\left[\zeta\tan^{-1}(\overline{\zeta}/a)\right] - \mathrm{Re}\left[\zeta\tan^{-1}(\zeta/a)\right] - \frac{1}{a^2}\mathrm{Re}\left[\zeta\tan^{-1}(a\overline{\zeta})\right]\right\} \\
&= \frac{UL}{4\pi a}\left\{\mathrm{Re}\left[\zeta\tan^{-1}(1/\zeta a)\right] - \mathrm{Re}\left[\overline{\zeta}\tan^{-1}(\overline{\zeta}/a)\right] - \frac{1}{a^2}\mathrm{Re}\left[\overline{\zeta}\tan^{-1}(a\zeta)\right]\right\} \\
&= \frac{UL}{4\pi a}\left\{\mathrm{Re}\left[\zeta\cot^{-1}(\zeta a)\right] - \mathrm{Re}\left[\frac{1}{\zeta}\cot^{-1}(\zeta a)\right] - \frac{1}{a^2}\mathrm{Re}\left[\frac{1}{\zeta}\tan^{-1}(a\zeta)\right]\right\},
\end{aligned} \tag{12}$$

where, in the various steps above, we have used the fact that the real part of a complex number is the real part of its conjugate, that $\overline{\zeta} = 1/\zeta$ on $|\zeta| = 1$, as well as the identity

$$\tan^{-1}(1/x) = \cot^{-1}(x). \tag{13}$$

Now
$$\cot^{-1}(x) = \frac{\pi}{2} - \tan^{-1}(x) \tag{14}$$
hence
$$\begin{aligned}\mathrm{Re}[\mathrm{i}\tilde{G}(\zeta)] &= \frac{UL}{4\pi a}\left\{\mathrm{Re}\left[\left(\zeta - \frac{1}{\zeta}\right)\left[\frac{\pi}{2} - \tan^{-1}(\zeta a)\right]\right] - \frac{1}{a^2}\mathrm{Re}\left[\frac{1}{\zeta}\tan^{-1}(a\zeta)\right]\right\} \\ &= \frac{UL}{4\pi a}\left\{\mathrm{Re}\left[\left(\frac{1}{\zeta} - \zeta - \frac{1}{a^2\zeta}\right)\tan^{-1}(a\zeta)\right]\right\}.\end{aligned} \tag{15}$$
The crucial observation is that the function in the square brackets on the right hand side is analytic inside the unit disc and this is precisely the requirement imposed on $\tilde{G}(\zeta)$. Hence we conclude that, up to an unimportant constant,
$$\tilde{G}(\zeta) = \frac{\mathrm{i}UL}{4\pi a}\left[\zeta + \frac{1}{a^2\zeta} - \frac{1}{\zeta}\right]\tan^{-1}(\zeta a). \tag{16}$$
It follows from (10) that
$$\begin{aligned}f_1 = \frac{\mathrm{i}UL\zeta}{4\pi a}\Bigg[&\tan^{-1}(\overline{\zeta}/a) - \tan^{-1}(\zeta/a) - \frac{1}{a^2}\tan^{-1}(a\overline{\zeta}) \\ &+ \left[1 + \frac{1}{a^2\zeta^2} - \frac{1}{\zeta^2}\right]\tan^{-1}(\zeta a)\Bigg].\end{aligned} \tag{17}$$
It is clear that, by the various manipulations above, we have avoided any need for the Poisson integral formula. We also have
$$\xi = \frac{\partial}{\partial z}(f + \overline{f}) = \xi_0(\zeta) + M^2\left[\frac{1}{z_0'(\zeta)}\frac{\partial}{\partial\zeta}(f_1 + \overline{f_1}) - \xi_0(\zeta)\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] + \dots \tag{18}$$
and, on comparison with (3), we identify
$$\xi_1 = \frac{1}{z_0'(\zeta)}\frac{\partial}{\partial\zeta}(f_1 + \overline{f_1}) - \xi_0(\zeta)\frac{z_1'(\zeta)}{z_0'(\zeta)} = \xi_0(\zeta)\left[\frac{1}{f_0'(\zeta)}\frac{\partial}{\partial\zeta}(f_1 + \overline{f_1}) - \frac{z_1'(\zeta)}{z_0'(\zeta)}\right]. \tag{19}$$

1.3 Bernoulli condition

We have yet to determine the perturbed vortex shape, or indeed the fluid speed, and both follow by imposing the Bernoulli condition which takes the form
$$|\xi|^2 = q_0^2, \tag{20}$$
where, following Ardalan *et al* [1], it is assumed that the speed of the fluid on the vortex boundary is unchanged at leading order. Hence
$$(\xi_0 + M^2\xi_1 + \dots)(\overline{\xi_0} + M^2\overline{\xi_1} + \dots) = q_0^2 \tag{21}$$

implying
$$\mathrm{Re}[\xi_1\overline{\xi_0}] = 0. \tag{22}$$
But, from (19),
$$\xi_1\overline{\xi_0} = |\xi_0|^2\left[\frac{1}{f_0'(\zeta)}\frac{\partial}{\partial\zeta}(f_1+\overline{f_1}) - \frac{z_1'(\zeta)}{z_0'(\zeta)}\right]. \tag{23}$$
Now, after some algebra using (17) we find
$$\begin{aligned}\frac{\partial}{\partial\zeta}(f_1+\overline{f_1}) = \frac{\mathrm{i}UL}{4\pi a}&\Bigg[\tan^{-1}(\overline{\zeta}/a) - \tan^{-1}(\zeta/a) - \frac{1}{a^2}\tan^{-1}(a\overline{\zeta})\\ &+\left[1+\frac{1}{a^2\zeta^2}-\frac{1}{\zeta^2}\right]\tan^{-1}(\zeta a)\Bigg]\\ &+\frac{\mathrm{i}UL}{4\pi a}\left[-\frac{a\zeta}{\zeta^2+a^2}+\left(a\zeta+\frac{1}{a\zeta}-\frac{a}{\zeta}\right)\frac{1}{1+a^2\zeta^2}\right]\\ &+\frac{\mathrm{i}UL(1-a^4)\zeta^2\overline{\zeta}}{4\pi a^2(\zeta^2+a^2)(1+\zeta^2a^2)}.\end{aligned} \tag{24}$$
Since $|\xi_0|^2 = q_0^2$ is constant on the boundary the Bernoulli condition is equivalent to
$$\mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] = \mathrm{Re}\left[\frac{1}{f_0'(\zeta)}\frac{\partial}{\partial\zeta}(f_1+\overline{f_1})\right]. \tag{25}$$
On use of (24) this can be written as
$$\begin{aligned}\mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] =&\mathrm{Re}\Bigg\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4a(1-a^4)\zeta}\Bigg[\tan^{-1}(\overline{\zeta}/a)\\ &-\tan^{-1}(\zeta/a)-\frac{1}{a^2}\tan^{-1}(a\overline{\zeta})+\left(1+\frac{1}{a^2\zeta^2}-\frac{1}{\zeta^2}\right)\tan^{-1}(\zeta a)\Bigg]\\ &+\frac{\zeta\overline{\zeta}}{4a^2}-\frac{(1+a^2\zeta^2)}{4(1-a^4)}+\frac{(\zeta^2+a^2)}{4(1-a^4)}\left(1+\frac{1}{a^2\zeta^2}-\frac{1}{\zeta^2}\right)\Bigg\}.\end{aligned} \tag{26}$$
Since $z_0'(\zeta)$ cannot vanish in $|\zeta|<1$ and since $z_1'(\zeta)$ is analytic there, (26) is a second instance of a Schwarz problem in the unit disc for the analytic function in square brackets on the left hand side. While the solution can be written down using the Poisson integral formula, once again we are able to avoid use of this and to find the

solution by inspection. Using similar manipulations as in §1.2, we can rewrite (26) as

$$
\begin{aligned}
\mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] &= \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4a(1-a^4)\zeta}\cot^{-1}(\zeta a)\right\} \\
&- \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4a(1-a^4)\zeta^3}\cot^{-1}(\zeta a)\right\} \\
&- \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4(1-a^4)a^3\zeta^3}\tan^{-1}(\zeta a)\right\} \\
&+ \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4a(1-a^4)\zeta}\left(1+\frac{1}{\zeta^2}-\frac{1}{\zeta^2a^2}\right)\tan^{-1}(\zeta a)\right\} \\
&+ \frac{1}{4(1-a^4)}\mathrm{Re}\left[(\zeta^2+a^2)\left[1+\frac{1}{a^2\zeta^2}-\frac{1}{\zeta^2}\right]\right] - \frac{1}{4(1-a^4)}\mathrm{Re}(1+a^2\zeta^2),
\end{aligned} \tag{27}
$$

where we have neglected a constant on the right hand side since retaining it is equivalent to adding a multiple of z_0 to z_1, which is undesirable since we assume *a priori* that the leading order incompressible solution is fixed. On cancelling a term, reordering, and on use of (14), we find

$$
\begin{aligned}
\mathrm{Re}\left[\frac{z_1'(\zeta)}{z_0'(\zeta)}\right] &= \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4a(1-a^4)\zeta}\left(\cot^{-1}(\zeta a)-\tan^{-1}(\zeta a)\right)\right\} \\
&- \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4a(1-a^4)\zeta^3}\left(\frac{\pi}{2}-\tan^{-1}(\zeta a)\right)\right\} \\
&+ \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{4(1-a^4)a^3\zeta^3}\tan^{-1}(\zeta a)\right\} \\
&- \mathrm{Re}\left\{\frac{(\zeta^2+a^2)(\zeta^2a^2+1)}{2a^3(1-a^4)\zeta^3}\tan^{-1}(\zeta a)\right\} \\
&+ \frac{1}{4(1-a^4)}\mathrm{Re}\left[(\zeta^2+a^2)\left[1+\frac{1}{a^2\zeta^2}-\frac{1}{\zeta^2}\right]\right] - \frac{1}{4(1-a^4)}\mathrm{Re}(1+a^2\zeta^2)
\end{aligned} \tag{28}
$$

which leads to further cancellations and the conclusion that

$$
\frac{z_1'(\zeta)}{z_0'(\zeta)} = \frac{1}{2(1+a^2)\zeta^2} - \frac{(\zeta^2+a^2)(1+\zeta^2a^2)}{2(1+a^2)a^3\zeta^3}\tan^{-1}(\zeta a), \tag{29}
$$

where we notice that the singularity at $\zeta = 0$ is removable. A further integration leads to

$$
z_1(\zeta) = \frac{L(1-a^2)}{4\pi}\left[\left(1-a^2\zeta^2\right)\frac{\tan^{-1}(a\zeta)}{a^2\zeta^2} - \frac{1}{a\zeta}\right]. \tag{30}
$$

On back substitution of (29) into (19) and use of (24), we find

$$
\begin{aligned}
\xi_1 = \xi_0 \Bigg[\frac{(\zeta^2+a^2)(1+\zeta^2 a^2)}{4a(1-a^4)\zeta} \Bigg\{ &\tan^{-1}(\overline{\zeta}/a) - \tan^{-1}(\zeta/a) - \frac{1}{a^2}\tan^{-1}(\overline{\zeta}a) \\
&+ \left(1 - \frac{1}{\zeta^2} + \frac{1}{\zeta^2 a^2}\right)\tan^{-1}(\zeta a) \Bigg\} \Bigg] \\
+ \xi_0 \left[-\frac{1}{2(1+a^2)\zeta^2} + \frac{\zeta\overline{\zeta}}{4a^2} - \frac{(1+a^2\zeta^2)}{4(1-a^4)} + \frac{(\zeta^2+a^2)}{4(1-a^4)}\left[1 + \frac{1}{a^2\zeta^2} - \frac{1}{\zeta^2}\right]\right]&.
\end{aligned} \tag{31}
$$

Crucially, the combination of terms involving $\tan^{-1}$ are exactly that appearing in (17) so that we conclude

$$
\xi_1(\zeta,\overline{\zeta}) = t_0(\zeta) f_1(\zeta,\overline{\zeta}) + t_1(\zeta,\overline{\zeta}) \tag{32}
$$

where

$$
\begin{aligned}
t_0(\zeta) &= \frac{\pi}{aL\zeta(1-a^4)}(\zeta^2+a^2)(1+\zeta^2 a^2), \\
t_1(\zeta,\overline{\zeta}) &= \frac{\mathrm{i}U}{4a^2}\left[\frac{\zeta^2\overline{\zeta}}{a} - \frac{a\zeta(1+a^2\zeta^2)}{1-a^4}\right] - \frac{\mathrm{i}U(\zeta^2+a^2)}{2a^3\zeta(1+a^2)} \\
&\quad + \frac{\mathrm{i}U(\zeta^2+a^2)(1+a^2\zeta^2-a^2)}{4a^3\zeta(1-a^4)}.
\end{aligned} \tag{33}
$$

This completes our analysis.

Using quite different techniques Ardalan *et al* [1] find

$$
\tilde{f} = \tilde{f}_0(\xi) + M^2 \tilde{f}_1(\xi,\overline{\xi}), + \dots, \tag{34}
$$

where we use tildes to denote their solution and

$$
\psi_0 \equiv \mathrm{Im}[\tilde{f}_0(\xi)], \qquad \psi_1 \equiv \mathrm{Im}[\tilde{f}_1(\xi,\overline{\xi})] \tag{35}
$$

are given explicitly in their paper. In the present notation,

$$
\xi = \xi_0 + M^2 \xi_1 + \dots \tag{36}
$$

so that

$$
\begin{aligned}
\tilde{f} &= \tilde{f}_0(\xi_0 + M^2\xi_1 + \dots) + M^2 \tilde{f}_1(\xi_0 + M^2\xi_1 + \dots) + \dots \\
&= \tilde{f}_0(\xi_0) + M^2\left[\xi_1 \tilde{f}_0'(\xi_0) + \tilde{f}_1(\xi_0,\overline{\xi_0})\right] + \dots
\end{aligned} \tag{37}
$$

If the solutions are to coincide, we must have

$$
f_1 = \xi_1 \tilde{f}_0'(\xi_0) + \tilde{f}_1(\xi_0,\overline{\xi_0}). \tag{38}
$$

But from (32) it follows that

$$f_1(\zeta,\overline{\zeta}) = \frac{\xi_1(\zeta,\overline{\zeta})}{t_0(\zeta)} - \frac{t_1(\zeta,\overline{\zeta})}{t_0(\zeta)}. \tag{39}$$

With the normalizations

$$U = a, \qquad L = \frac{4\pi}{a}, \tag{40}$$

which ensures that the leading order solutions are identical, it can be verified that

$$\frac{1}{t_0} = \tilde{f}_0'(\xi_0), \qquad \psi_1 = \mathrm{Im}[\tilde{f}_1(\xi_0,\overline{\xi_0})] = \mathrm{Im}\left[-\frac{t_1}{t_0}\right] \tag{41}$$

thereby confirming that the first-order corrections found by the two independent methods are also identical.

We have demonstrated that a conformal mapping approach, coupled with the Imai-Lamla formulation, faithfully retrieves the results of [1].

It should be noted that our approach offers the advantage that an explicit form (30) for the perturbed shape of the vortices is available; in [1] this was given up to a quadrature.

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