Weakly nonlinear instability of a Newtonian liquid jet - Supplement

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(Received xx; revised xx; accepted xx)

The present supplementary material to our above paper provides the details of the determination of the solutions for the velocity and pressure fields in the jets due to the unstable deformation of its surface. We provide the solutions for the pressure fields as well since they are essential in the development of the second-order solutions, the method of solution for second order starting from a Poisson equation for this quantity.

Appendix A

A.1. *The pressure field*

The Poisson equation for the modified pressure $\mathcal{P}_{21} = p_{21} + \vec{v}_1^2/2$ reads

$$\varDelta \mathcal{P}_{21} = -\frac{1}{Oh}\mathrm{div}\left[\left(\alpha_1^+ \frac{\psi_2^+}{r^2} + \alpha_1^- \frac{\psi_2^-}{r^2}\right)\vec{\nabla}\psi\right] \tag{A 1}$$

The right-hand side term of this equation can be decomposed in three groups, the first proportional to $e^{-2\alpha_1^+ t}$, the second to $e^{-2\alpha_1^- t}$ and the third to $e^{-(\alpha_1^+ + \alpha_1^-)t}$. We denote by $\mathcal{P}_{21}^+$, $\mathcal{P}_{21}^-$ and $\mathcal{P}_{21}^\pm$ the corresponding modified pressures. Using the representation of the stream function by modified Bessel functions, we re-write equation (A 1) for the three time dependencies and we obtain

$$\begin{aligned}\varDelta \mathcal{P}_{21}^+ = \Bigg(&-\frac{{l^+}^2 - k^2}{2r^2}\Big\{{C_3^+}^2\left[\left({l^+}^2 - 2k^2\right) r^2 I_1^2(l^+ r) + {l^+}^2 r^2 I_0^2(l^+ r)\right] \\ &+ C_1^+ C_3^+\left[krl^+ r I_0(kr) I_0(l^+ r) - k^2 r^2 I_1(kr) I_1(l^+ r)\right]\Big\}\cos 2kz \\ &+ \frac{{l^+}^2 - k^2}{2r^2}\Big\{{C_3^+}^2 {l^+}^2 r^2\left(I_1^2(l^+ r) + I_0^2(l^+ r)\right) \\ &+ C_1^+ C_3^+\left(krl^+ r I_0(kr) I_0(l^+ r) + k^2 r^2 I_1(kr) I_1(l^+ r)\right)\Big\}\Bigg) e^{-2\alpha_1^+ t}\end{aligned} \tag{A 2}$$

$$\begin{aligned}\varDelta \mathcal{P}_{21}^- = \Bigg(&-\frac{{l^-}^2 - k^2}{2r^2}\Big\{{C_3^-}^2\left[\left({l^-}^2 - 2k^2\right) r^2 I_1^2(l^- r) + {l^-}^2 r^2 I_0^2(l^- r)\right] \\ &+ C_1^- C_3^-\left[krl^- r I_0(kr) I_0(l^- r) - k^2 r^2 I_1(kr) I_1(l^- r)\right]\Big\}\cos 2kz \\ &+ \frac{{l^-}^2 - k^2}{2r^2}\Big\{{C_3^-}^2 {l^-}^2 r^2\left(I_1^2(l^- r) + I_0^2(l^- r)\right)\end{aligned} \tag{A 3}$$

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$$
\begin{aligned}
&+\ C_1^- C_3^- \left(krl^- r I_0(kr) I_0(l^- r) + k^2 r^2 I_1(kr) I_1(l^- r)\right)\}\Big) e^{-2\alpha_1^- t} \\
\Delta\mathcal{P}_{21}^{\pm} = &\left(\frac{1}{2Oh}\frac{1}{r^2}\left\{C_3^+ C_3^- \left[\left(\alpha_1^+ (l^{-2} - 2k^2) + \alpha_1^- (l^{+2} - 2k^2)\right) r^2 I_1(l^+ r) I_1(l^- r)\right.\right.\right. \\
&+\ \left.\left(\alpha_1^+ + \alpha_1^-\right) l^+ r I_0(l^+ r) l^- r I_0(l^- r)\right] \qquad \text{(A 4)} \\
&+\ C_1^- C_3^+ \alpha_1^+ \left[l^+ r I_0(l^+ r) kr I_0(kr) - kr I_1(l^+ r) kr I_1(kr)\right] \\
&+\ \left.C_1^+ C_3^- \alpha_1^- \left[l^- r I_0(l^- r) kr I_0(kr) - kr I_1(l^- r) kr I_1(kr)\right]\right\} \cos 2kz \\
&-\ \frac{1}{2Oh}\frac{1}{r^2}\left\{C_3^+ C_3^- \left[\left(\alpha_1^+ l^{-2} + \alpha_1^- l^{+2}\right) r^2 I_1(l^+ r) I_1(l^- r)\right.\right. \\
&+\ \left.\left(\alpha_1^+ + \alpha_1^-\right) l^+ r I_0(l^+ r) l^- r I_0(l^- r)\right] \\
&+\ C_1^- C_3^+ \alpha_1^+ \left[l^+ r I_0(l^+ r) kr I_0(kr) + kr I_1(l^+ r) kr I_1(kr)\right] \\
&+\ \left.\left.C_1^+ C_3^- \alpha_1^- \left[l^- r I_0(l^- r) kr I_0(kr) + kr I_1(l^- r) kr I_1(kr)\right]\right\}\right) e^{-(\alpha_1^+ + \alpha_1^-)t}
\end{aligned}
$$

The solutions are composed from the general solutions of the homogeneous equations and the particular solutions of the inhomogeneous equations. The general solutions of the homogeneous equations corresponding to the three functions of time are

$$\mathcal{P}_{21}^{+\,H}(r,z,t) = C_{21}^+ I_0(2kr) e^{-2\alpha_1^+ t} \cos 2kz \qquad \text{(A 5)}$$

$$\mathcal{P}_{21}^{-\,H}(r,z,t) = C_{21}^- I_0(2kr) e^{-2\alpha_1^- t} \cos 2kz \qquad \text{(A 6)}$$

$$\mathcal{P}_{21}^{\pm\,H}(r,z,t) = C_{21}^{\pm} I_0(2kr) e^{-(\alpha_1^+ + \alpha_1^-)t} \cos 2kz \qquad \text{(A 7)}$$

so that the radial amplitudes of the functions $\mathcal{P}_{21}^H$ are determined by a modified Bessel function of the first kind and zero order of the argument $2kr$. The corresponding modified Bessel function of the second kind was discarded for its divergence on the jet axis. In the above equations, the three coefficients C_{21}^+, C_{21}^-, and C_{21}^- are unknown constants to be determined.

The terms on the right-hand side of equations (A 2)-(A 4) consist of products of modified Bessel functions of the first kind, zero and first order, I_0 and I_1, respectively, and with the arguments kr, $l^+ r$ and $l^- r$. A representation of the particular solutions of the inhomogeneous equations by elementary functions seems to be out of reach. We thus approximate the right-hand sides of these equations using the series expansion of I_0, which reads (Newman 1984)

$$I_0(x) = q_0 + q_2 x^2 + q_4 x^4 + q_6 x^6 + q_8 x^8 + q_{10} x^{10} \cdots =: \sum_{i=1}^{\infty} q_{2(i-1)} x^{2(i-1)} \qquad \text{(A 8)}$$

with the coefficients $q_{2(i-1)}$ from Table 1. The function I_1 is the derivative of I_0 with respect to the argument.

With the approximate right-hand side terms, the inhomogeneous differential equations for the three contributions to $\mathcal{P}_{21}$ read

$$
\begin{aligned}
&r^2 \frac{\partial^2 \mathcal{P}_{21}^+}{\partial r^2} + r \frac{\partial \mathcal{P}_{21}^+}{\partial r} + r^2 \frac{\partial^2 \mathcal{P}_{21}^+}{\partial z^2} = \qquad \text{(A 9)} \\
&= \left(\sum_{i=1}^{N} -\frac{l^{+2} - k^2}{2} C_3^+ l^{+2(i-1)} \sum_{j=1}^{i} q_{2(i-j)} \left\{C_3^+ \left[4\left(l^{+2} - 2k^2\right) j(i-j) q_{2j} + q_{2(j-1)} l^{+2}\right]\right.\right. \\
&+\ \left. C_1^+ k^{2j-1} l^{+1-2j} \left[q_{2(j-1)} l^{+2} - 4j(i-j) q_{2j} k^2\right]\right\} \cos 2kz
\end{aligned}
$$

i	1	2	3	4	5	6	7
$1/q_{2(i-1)}$	1	4	64	2,304	147,456	14,745,600	2,123,366,400
i	8	9					
$1/q_{2(i-1)}$	416,179,814,400	106,542,032,486,400					

TABLE 1. Inverse coefficients of the series expansion of $I_0(x)$ in (A 8). $I_1(x)$ is given as $I_0'(x)$.

$$
\begin{aligned}
&+ \sum_{i=1}^{N} \frac{{l^+}^2 - k^2}{2} C_3^+ {l^+}^{2(i-1)} \sum_{j=1}^{i} q_{2(i-j)} \left\{ C_3^+ {l^+}^2 \left[4j(i-j)q_{2j} + q_{2(j-1)}\right] \right.\\
&+ \left. \left. C_1^+ k^{2j-1} {l^+}^{1-2j} \left[q_{2(j-1)} {l^+}^2 + 4j(i-j)q_{2j}k^2\right]\right\}\right) r^{2i} e^{-2\alpha_1^+ t}
\end{aligned}
$$

$$
r^2 \frac{\partial^2 \mathcal{P}_{21}^-}{\partial r^2} + r \frac{\partial \mathcal{P}_{21}^-}{\partial r} + r^2 \frac{\partial^2 \mathcal{P}_{21}^-}{\partial z^2} = \qquad \text{(A 10)}
$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{N} - \frac{{l^-}^2 - k^2}{2} C_3^- {l^-}^{-2(i-1)} \sum_{j=1}^{i} q_{2(i-j)} \left\{ C_3^- \left[4\left({l^-}^2 - 2k^2\right) j(i-j)q_{2j} + q_{2(j-1)} {l^-}^2\right] \right. \right.\\
&+ \left. C_1^- k^{2j-1} {l^-}^{1-2j} \left[q_{2(j-1)} {l^-}^2 - 4j(i-j)q_{2j}k^2\right]\right\} \cos 2kz\\
&+ \sum_{i=1}^{N} \frac{{l^-}^2 - k^2}{2} C_3^- {l^-}^{2(i-1)} \sum_{j=1}^{i} q_{2(i-j)} \left\{ C_3^- {l^-}^2 \left[4j(i-j)q_{2j} + q_{2(j-1)}\right] \right.\\
&+ \left. \left. C_1^- k^{2j-1} {l^-}^{1-2j} \left[q_{2(j-1)} {l^-}^2 + 4j(i-j)q_{2j}k^2\right]\right\}\right) r^{2i} e^{-2\alpha_1^- t}
\end{aligned}
$$

$$
r^2 \frac{\partial^2 \mathcal{P}_{21}^\pm}{\partial r^2} + r \frac{\partial \mathcal{P}_{21}^\pm}{\partial r} + r^2 \frac{\partial^2 \mathcal{P}_{21}^\pm}{\partial z^2} = \qquad \text{(A 11)}
$$

$$
\begin{aligned}
&= \left(\sum_{i=1}^{N} \frac{1}{2Oh} \sum_{j=1}^{i} q_{2(i-j)} \left\{ C_3^+ C_3^- {l^-}^{2j-1} {l^+}^{2(i-j)-1} \left[\left(\alpha_1^+ \left({l^-}^2 - 2k^2\right) + \alpha_1^- \left({l^+}^2 - 2k^2\right)\right) \right. \right. \right.\\
&\cdot \left. 4j(i-j)q_{2j} + \left(\alpha_1^+ + \alpha_1^-\right) {l^+}^2 q_{2(j-1)}\right] + k^{2(i-j)+1} \left[q_{2(j-1)} - 4j(i-j)q_{2j}\right]\\
&\cdot \left. \left(C_3^+ C_1^- \alpha_1^+ {l^+}^{2j-1} + C_3^- C_1^+ \alpha_1^- {l^-}^{2j-1}\right)\right\} \cos 2kz\\
&- \sum_{i=1}^{N} \frac{1}{2Oh} \sum_{j=1}^{i} q_{2(i-j)} \left\{ C_3^+ C_3^- {l^-}^{2j-1} {l^+}^{2(i-j)-1} \left[\left(\alpha_1^+ {l^-}^2 + \alpha_1^- {l^+}^2\right) 4j(i-j)q_{2j} \right. \right.\\
&+ \left. \left(\alpha_1^+ + \alpha_1^-\right) {l^+}^2 q_{2(j-1)}\right] + k^{2(i-j)+1} \left[q_{2(j-1)} + 4j(i-j)q_{2j}\right]\\
&\cdot \left. \left. \left(C_3^+ C_1^- \alpha_1^+ {l^+}^{2j-1} + C_3^- C_1^+ \alpha_1^- {l^-}^{2j-1}\right)\right\}\right) r^{2i} e^{-(\alpha_1^+ + \alpha_1^-)t}
\end{aligned}
$$

N is the maximum index of the summation and is the parameter of the approximation. We denote the coefficients of r^{2i} in the polynomials with and without dependency on the axial coordinate z as $\gamma_{2i,z}$ and γ_{2i}, respectively, so that (A 9) - (A 11) become

$$r^2\frac{\partial^2\mathcal{P}_{21}^+}{\partial r^2}+r\frac{\partial\mathcal{P}_{21}^+}{\partial r}+r^2\frac{\partial^2\mathcal{P}_{21}^+}{\partial z^2}=\sum_{i=1}^{N}\left(\gamma_{2i,z}^+\cos 2kz+\gamma_{2i}^+\right)r^{2i}e^{-2\alpha_1^+ t} \tag{A 12}$$

$$r^2\frac{\partial^2\mathcal{P}_{21}^-}{\partial r^2}+r\frac{\partial\mathcal{P}_{21}^-}{\partial r}+r^2\frac{\partial^2\mathcal{P}_{21}^-}{\partial z^2}=\sum_{i=1}^{N}\left(\gamma_{2i,z}^-\cos 2kz+\gamma_{2i}^-\right)r^{2i}e^{-2\alpha_1^- t} \tag{A 13}$$

$$r^2\frac{\partial^2\mathcal{P}_{21}^\pm}{\partial r^2}+r\frac{\partial\mathcal{P}_{21}^\pm}{\partial r}+r^2\frac{\partial^2\mathcal{P}_{21}^\pm}{\partial z^2}=\sum_{i=1}^{N}\left(\gamma_{2i,z}^\pm\cos 2kz+\gamma_{2i}^\pm\right)r^{2i}e^{-(\alpha_1^++\alpha_1^-)t} \tag{A 14}$$

where the coefficients $\gamma_{2i,z}$ and γ_{2i} corresponding to each time dependency read

$$\begin{aligned}\gamma_{2i,z}^+ &= -\frac{{l^+}^2-k^2}{2}C_3^+{l^+}^{2(i-1)}\sum_{j=1}^{i}q_{2(i-j)}\left\{C_3^+\left[4\left({l^+}^2-2k^2\right)j(i-j)q_{2j}+q_{2(j-1)}{l^+}^2\right]\right.\\ &\left.+\ C_1^+k^{2j-1}{l^+}^{1-2j}\left[q_{2(j-1)}{l^+}^2-4j(i-j)q_{2j}k^2\right]\right\}\end{aligned} \tag{A 15}$$

$$\begin{aligned}\gamma_{2i,z}^- &= -\frac{{l^-}^2-k^2}{2}C_3^-{l^-}^{2(i-1)}\sum_{j=1}^{i}q_{2(i-j)}\left\{C_3^-\left[4\left({l^-}^2-2k^2\right)j(i-j)q_{2j}+q_{2(j-1)}{l^-}^2\right]\right.\\ &\left.+\ C_1^-k^{2j-1}{l^-}^{-1-2j}\left[q_{2(j-1)}{l^-}^2-4j(i-j)q_{2j}k^2\right]\right\}\end{aligned} \tag{A 16}$$

$$\begin{aligned}\gamma_{2i,z}^\pm &= \frac{1}{2Oh}\sum_{j=1}^{i}q_{2(i-j)}\left\{C_3^+C_3^-{l^-}^{2j-1}{l^+}^{2(i-j)-1}\left[\left(\alpha_1^+\left({l^-}^2-2k^2\right)+\alpha_1^-\left({l^+}^2-2k^2\right)\right)\right.\right.\\ &\cdot\ \left.4j(i-j)q_{2j}+\left(\alpha_1^++\alpha_1^-\right){l^+}^2q_{2(j-1)}\right]+k^{2(i-j)+1}\left[q_{2(j-1)}-4j(i-j)q_{2j}\right]\\ &\cdot\ \left.\left(C_3^+C_1^-\alpha_1^+{l^+}^{2j-1}+C_3^-C_1^+\alpha_1^-{l^-}^{2j-1}\right)\right\}\end{aligned} \tag{A 17}$$

$$\begin{aligned}\gamma_{2i}^+ &= \frac{{l^+}^2-k^2}{2}C_3^+{l^+}^{2(i-1)}\sum_{j=1}^{i}q_{2(i-j)}\left\{C_3^+{l^+}^2\left[4j(i-j)q_{2j}+q_{2(j-1)}\right]\right.\\ &\left.+\ C_1^+k^{2j-1}{l^+}^{1-2j}\left[q_{2(j-1)}{l^+}^2+4j(i-j)q_{2j}k^2\right]\right\}\end{aligned} \tag{A 18}$$

$$\begin{aligned}\gamma_{2i}^- &= \frac{{l^-}^2-k^2}{2}C_3^-{l^-}^{2(i-1)}\sum_{j=1}^{i}q_{2(i-j)}\left\{C_3^-{l^-}^2\left[4j(i-j)q_{2j}+q_{2(j-1)}\right]\right.\\ &\left.+\ C_1^-k^{2j-1}{l^-}^{-1-2j}\left[q_{2(j-1)}{l^-}^2+4j(i-j)q_{2j}k^2\right]\right\}\end{aligned} \tag{A 19}$$

$$\begin{aligned}\gamma_{2i}^\pm &= \frac{-1}{2Oh}\sum_{j=1}^{i}q_{2(i-j)}\left\{C_3^+C_3^-{l^-}^{2j-1}{l^+}^{2(i-j)-1}\left[\left(\alpha_1^+{l^-}^2+\alpha_1^-{l^+}^2\right)4j(i-j)q_{2j}\right.\right.\\ &+\ \left.\left(\alpha_1^++\alpha_1^-\right){l^+}^2q_{2(j-1)}\right]+k^{2(i-j)+1}\left[q_{2(j-1)}+4j(i-j)q_{2j}\right]\\ &\cdot\ \left.\left(C_3^+C_1^-\alpha_1^+{l^+}^{2j-1}+C_3^-C_1^+\alpha_1^-{l^-}^{2j-1}\right)\right\}\end{aligned} \tag{A 20}$$

The particular solutions of the inhomogeneous equations (A 12) - (A 14) are now defined as

$$\mathcal{P}_{21}^{+\,I}(r,z,t) =: \left(\sum_{i=1}^{N}\left(\delta_{2i,z}^{+}\cos 2kz + \delta_{2i}^{+}\right) r^{2i} + P_{21}^{+}\right) e^{-2\alpha_1^{+} t} \tag{A 21}$$

$$\mathcal{P}_{21}^{-\,I}(r,z,t) =: \left(\sum_{i=1}^{N}\left(\delta_{2i,z}^{-}\cos 2kz + \delta_{2i}^{-}\right) r^{2i} + P_{21}^{-}\right) e^{-2\alpha_1^{-} t} \tag{A 22}$$

$$\mathcal{P}_{21}^{\pm\,I}(r,z,t) =: \left(\sum_{i=1}^{N}\left(\delta_{2i,z}^{\pm}\cos 2kz + \delta_{2i}^{\pm}\right) r^{2i} + P_{21}^{\pm}\right) e^{-(\alpha_1^{+}+\alpha_1^{-}) t} \tag{A 23}$$

with the yet unknown coefficients P_{21}. The coefficients $\delta_{2i,z}$ and δ_{2i} are related to $\gamma_{2i,z}$ and γ_{2i} as per

$$\delta_{2,z} = \frac{\gamma_{2,z}}{4} \tag{A 24}$$

$$\delta_{2i,z} = \frac{\left(\gamma_{2i,z} + 4k^2\delta_{2(i-1),z}\right)}{4i^2} \quad \text{for} \quad i \geqslant 2 \tag{A 25}$$

$$\delta_{2i} = \frac{\gamma_{2i}}{4i^2} \quad \text{for all} \quad i \tag{A 26}$$

Using the definition of the modified pressure and its homogeneous and inhomogeneous solutions, for each time dependency, the real second-order pressure contribution p_{21} is obtained as

$$p_{21}(r,z,t) = p_{21}^{+}(r,z,t) + p_{21}^{-}(r,z,t) + p_{21}^{\pm}(r,z,t) \tag{A 27}$$

where

$$\begin{aligned} p_{21}^{+}(r,z,t) = \Bigg[&\left(-\frac{1}{4}\,\hat{\eta}_1^{+\,2}\left(f_r^{+\,2} - f_z^{+\,2}\right) + C_{21}^{+} I_0(2kr) + \sum_{i=1}^{N}\delta_{2i,z}^{+} r^{2i}\right)\cos 2kz \\ &- \frac{1}{4}\,\hat{\eta}_1^{+\,2}\left(f_r^{+\,2} + f_z^{+\,2}\right) + \sum_{i=1}^{N}\delta_{2i}^{+} r^{2i} + P_{21}^{+}\Bigg]\; e^{-2\alpha_1^{+} t} \end{aligned} \tag{A 28}$$

$$\begin{aligned} p_{21}^{-}(r,z,t) = \Bigg[&\left(-\frac{1}{4}\,\hat{\eta}_1^{-\,2}\left(f_r^{-\,2} - f_z^{-\,2}\right) + C_{21}^{-} I_0(2kr) + \sum_{i=1}^{N}\delta_{2i,z}^{-} r^{2i}\right)\cos 2kz \\ &- \frac{1}{4}\,\hat{\eta}_1^{-\,2}\left(f_r^{-\,2} + f_z^{-\,2}\right) + \sum_{i=1}^{N}\delta_{2i}^{-} r^{2i} + P_{21}^{-}\Bigg]\; e^{-2\alpha_1^{-} t} \end{aligned} \tag{A 29}$$

$$\begin{aligned} p_{21}^{\pm}(r,z,t) = \Bigg[&\left(-\frac{1}{2}\hat{\eta}_1^{+}\hat{\eta}_1^{-}\left(f_r^{+} f_r^{-} - f_z^{+} f_z^{-}\right) + C_{21}^{\pm} I_0(2kr) + \sum_{i=1}^{N}\delta_{2i,z}^{\pm} r^{2i}\right)\cos 2kz \\ &- \frac{1}{2}\hat{\eta}_1^{+}\hat{\eta}_1^{-}\left(f_r^{+} f_r^{-} + f_z^{+} f_z^{-}\right) + \sum_{i=1}^{N}\delta_{2i}^{\pm} r^{2i} + P_{21}^{\pm}\Bigg]\; e^{-(\alpha_1^{+}+\alpha_1^{-}) t} \end{aligned} \tag{A 30}$$

A.2. *The velocity components*

The differential equation for u_{r21} reads

$$r^2\frac{\partial^2 u_{r21}}{\partial r^2} + r\frac{\partial u_{r21}}{\partial r} - \left(-\frac{2\alpha_1}{Oh}r^2 + 1\right)u_{r21} + r^2\frac{\partial^2 u_{r21}}{\partial z^2} = \frac{r^2}{Oh}\frac{\partial \mathcal{P}_{21}}{\partial r} - \frac{1}{Oh^2}\frac{\partial \psi_2}{\partial t}\frac{\partial \psi}{\partial r} \tag{A 31}$$

According to the three dependencies on time, we re-write this equation as

$$r^2\frac{\partial^2 u^+_{r21}}{\partial r^2}+r\frac{\partial u^+_{r21}}{\partial r}-\left(-\frac{2\alpha_1^+}{Oh}r^2+1\right)u^+_{r21}+r^2\frac{\partial^2 u^+_{r21}}{\partial z^2}=\frac{r^2}{Oh}\frac{\partial \mathcal{P}^+_{21}}{\partial r}$$
$$+\frac{k^2-l^{+2}}{Oh}C_3^+ rI_1(l^+r)\left[C_1^+ krI_0(kr)+C_3^+l^+rI_0(l^+r)\right]e^{-2\alpha_1^+t}\sin^2 kz \qquad \text{(A 32)}$$

$$r^2\frac{\partial^2 u^-_{r21}}{\partial r^2}+r\frac{\partial u^-_{r21}}{\partial r}-\left(-\frac{2\alpha_1^-}{Oh}r^2+1\right)u^-_{r21}+r^2\frac{\partial^2 u^-_{r21}}{\partial z^2}=\frac{r^2}{Oh}\frac{\partial \mathcal{P}^-_{21}}{\partial r}$$
$$+\frac{k^2-l^{-2}}{Oh}C_3^- rI_1(l^-r)\left[C_1^- krI_0(kr)+C_3^-l^-rI_0(l^-r)\right]e^{-2\alpha_1^-t}\sin^2 kz \qquad \text{(A 33)}$$

$$r^2\frac{\partial^2 u^\pm_{r21}}{\partial r^2}+r\frac{\partial u^\pm_{r21}}{\partial r}-\left(-\frac{\alpha_1^++\alpha_1^-}{Oh}r^2+1\right)u^\pm_{r21}+r^2\frac{\partial^2 u^\pm_{r21}}{\partial z^2}=\frac{r^2}{Oh}\frac{\partial \mathcal{P}^\pm_{21}}{\partial r}$$
$$+\frac{1}{Oh^2}\left[C_3^+\alpha_1^+ rI_1(l^+r)\left(C_1^- krI_0(kr)+C_3^-l^-rI_0(l^-r)\right)\right.$$
$$\left.+C_3^-\alpha_1^- rI_1(l^-r)\left(C_1^+ krI_0(kr)+C_3^+l^+rI_0(l^+r)\right)\right]e^{-(\alpha_1^++\alpha_1^-)t}\sin^2 kz \qquad \text{(A 34)}$$

The terms with products of modified Bessel functions on the right-hand sides are approximated by their series expansions to obtain

$$r^2\frac{\partial^2 u^+_{r21}}{\partial r^2}+r\frac{\partial u^+_{r21}}{\partial r}-\left(-\frac{2\alpha_1^+}{Oh}r^2+1\right)u^+_{r21}+r^2\frac{\partial^2 u^+_{r21}}{\partial z^2}=\frac{r^2}{Oh}\frac{\partial \mathcal{P}^+_{21}}{\partial r} \qquad \text{(A 35)}$$
$$+\sum_{i=1}^{N-1}2C_3^+\frac{\alpha_1^+}{Oh^2}\sum_{j=1}^{i}q_{2(i-j)}q_{2j}j\left(C_1^+k^{2(i-j)+1}l^{+2j-1}\right.$$
$$\left.+C_3^+l^{+2i}\right)r^{2i+1}e^{-2\alpha_1^+t}\sin^2 kz$$

$$r^2\frac{\partial^2 u^-_{r21}}{\partial r^2}+r\frac{\partial u^-_{r21}}{\partial r}-\left(-\frac{2\alpha_1^-}{Oh}r^2+1\right)u^-_{r21}+r^2\frac{\partial^2 u^-_{r21}}{\partial z^2}=\frac{r^2}{Oh}\frac{\partial \mathcal{P}^-_{21}}{\partial r} \qquad \text{(A 36)}$$
$$+\sum_{i=1}^{N-1}2C_3^-\frac{\alpha_1^-}{Oh^2}\sum_{j=1}^{i}q_{2(i-j)}q_{2j}j\left(C_1^-k^{2(i-j)+1}l^{-2j-1}\right.$$
$$\left.+C_3^-l^{-2i}\right)r^{2i+1}e^{-2\alpha_1^-t}\sin^2 kz$$

$$r^2\frac{\partial^2 u^\pm_{r21}}{\partial r^2}+r\frac{\partial u^\pm_{r21}}{\partial r}-\left(-\frac{\alpha_1^++\alpha_1^-}{Oh}r^2+1\right)u^\pm_{r21}+r^2\frac{\partial^2 u^\pm_{r21}}{\partial z^2}=\frac{r^2}{Oh}\frac{\partial \mathcal{P}^\pm_{21}}{\partial r} \qquad \text{(A 37)}$$
$$+\sum_{i=1}^{N-1}\frac{2}{Oh^2}\sum_{j=1}^{i}q_{2(i-j)}q_{2j}j\left(C_3^+\alpha_1^+l^{+2j-1}\left(C_1^-k^{2(i-j)+1}+C_3^-l^{-2(i-j)+1}\right)\right.$$
$$\left.+C_3^-\alpha_1^-l^{-2j-1}\left(C_1^+k^{2(i-j)+1}+C_3^+l^{+2(i-j)+1}\right)\right)r^{2i+1}e^{-(\alpha_1^++\alpha_1^-)t}\sin^2 kz$$

Having substituted the derivatives of the solutions for $\mathcal{P}_{21}$ into the above equations, we see that terms independent of the coordinate z, which emerge from the formulation of $\sin^2 kz$ in terms of $\cos 2kz$, disappear from these differential equations. Denoting the coefficient in the second sum of each equation as ε_{2i+1}, we obtain the differential equations

$$r^2\frac{\partial^2 u^+_{r21}}{\partial r^2}+r\frac{\partial u^+_{r21}}{\partial r}-\left(-\frac{2\alpha_1^+}{Oh}r^2+1\right)u^+_{r21}+r^2\frac{\partial^2 u^+_{r21}}{\partial z^2}= \tag{A 38}$$
$$=\left[\frac{C^+_{21}}{Oh}2kr^2 I_1(2kr)+\sum_{i=1}^{N-1}\left(\frac{2i\delta^+_{2i,z}}{Oh}-\frac{1}{2}\varepsilon^+_{2i+1}\right)r^{2i+1}\right]e^{-2\alpha_1^+ t}\cos 2kz$$

$$r^2\frac{\partial^2 u^-_{r21}}{\partial r^2}+r\frac{\partial u^-_{r21}}{\partial r}-\left(-\frac{2\alpha_1^-}{Oh}r^2+1\right)u^-_{r21}+r^2\frac{\partial^2 u^-_{r21}}{\partial z^2}= \tag{A 39}$$
$$=\left[\frac{C^-_{21}}{Oh}2kr^2 I_1(2kr)+\sum_{i=1}^{N-1}\left(\frac{2i\delta^-_{2i,z}}{Oh}-\frac{1}{2}\varepsilon^-_{2i+1}\right)r^{2i+1}\right]e^{-2\alpha_1^- t}\cos 2kz$$

$$r^2\frac{\partial^2 u^\pm_{r21}}{\partial r^2}+r\frac{\partial u^\pm_{r21}}{\partial r}-\left(-\frac{\alpha_1^++\alpha_1^-}{Oh}r^2+1\right)u^\pm_{r21}+r^2\frac{\partial^2 u^\pm_{r21}}{\partial z^2}= \tag{A 40}$$
$$=\left[\frac{C^\pm_{21}}{Oh}2kr^2 I_1(2kr)+\sum_{i=1}^{N-1}\left(\frac{2i\delta^\pm_{2i,z}}{Oh}-\frac{1}{2}\varepsilon^\pm_{2i+1,z}\right)r^{2i+1}\right]e^{-(\alpha_1^++\alpha_1^-)t}\cos 2kz$$

where the coefficients ε_{2i+1} for each time dependency are defined as

$$\varepsilon^+_{2i+1}=2C_3^+\frac{\alpha_1^+}{Oh^2}\sum_{j=1}^{i}q_{2(i-j)}q_{2j}j\left(C_1^+k^{2(i-j)+1}l^{+2j-1}+C_3^+l^{+2i}\right) \tag{A 41}$$

$$\varepsilon^-_{2i+1}=2C_3^-\frac{\alpha_1^-}{Oh^2}\sum_{j=1}^{i}q_{2(i-j)}q_{2j}j\left(C_1^-k^{2(i-j)+1}l^{-2j-1}+C_3^-l^{-2i}\right) \tag{A 42}$$

$$\varepsilon^\pm_{2i+1}=\frac{2}{Oh^2}\sum_{j=1}^{i}q_{2(i-j)}q_{2j}j\left(C_3^+\alpha_1^+l^{+2j-1}\left(C_1^-k^{2(i-j)+1}+C_3^-l^{-2(i-j)+1}\right)\right.$$
$$\left.+\,C_3^-\alpha_1^-l^{-2j-1}\left(C_1^+k^{2(i-j)+1}+C_3^+l^{+2(i-j)+1}\right)\right) \tag{A 43}$$

The solutions of the homogeneous forms of equations (A 38) - (A 40) read

$$u^+_{r21}{}^H(r,z,t)=D^+_{21}I_1(2m^+r)e^{-2\alpha_1^+t}\cos 2kz \tag{A 44}$$
$$u^-_{r21}{}^H(r,z,t)=D^-_{21}I_1(2m^-r)e^{-2\alpha_1^-t}\cos 2kz \tag{A 45}$$
$$u^\pm_{r21}{}^H(r,z,t)=D^\pm_{21}I_1(2m^\pm r)e^{-(\alpha_1^++\alpha_1^-)t}\cos 2kz \tag{A 46}$$

where we have defined $m^{+2}=k^2-\alpha_1^+/(2Oh)$, $m^{-2}=k^2-\alpha_1^-/(2Oh)$ and $m^{\pm 2}=k^2-(\alpha_1^++\alpha_1^-)/(4Oh)$. Writing the particular solutions of the inhomogeneous equations (A 38) - (A 40) as

$$u^+_{r21}{}^I(r,z,t)=\left[C^+_{21}\frac{k}{\alpha_1^+}I_1(2kr)+\sum_{i=1}^{N-1}\zeta^+_{2i+1}r^{2i+1}\right]e^{-2\alpha_1^+t}\cos 2kz \tag{A 47}$$

$$u^-_{r21}{}^I(r,z,t)=\left[C^-_{21}\frac{k}{\alpha_1^-}I_1(2kr)+\sum_{i=1}^{N-1}\zeta^-_{2i+1}r^{2i+1}\right]e^{-2\alpha_1^-t}\cos 2kz \tag{A 48}$$

$$u^\pm_{r21}{}^I(r,z,t)=\left[C^\pm_{21}\frac{2k}{\alpha_1^++\alpha_1^-}I_1(2kr)+\sum_{i=1}^{N-1}\zeta^\pm_{2i+1}r^{2i+1}\right]e^{-(\alpha_1^++\alpha_1^-)t}\cos 2kz \tag{A 49}$$

with the coefficients ζ_{2i+1} related to coefficients δ_{2i} and ε_{2i+1} by the recursion formulas

$$\zeta_3 = \frac{1}{8}\left(\frac{2\delta_{2,z}}{Oh} - \frac{1}{2}\varepsilon_3\right) \tag{A 50}$$

$$\zeta_{2i+1} = \frac{1}{(2i+1)^2-1}\left(\frac{2i\delta_{2i,z}}{Oh} - \frac{1}{2}\varepsilon_{2i+1} + 4m^2\zeta_{2i-1}\right) \quad \text{for} \quad i \geqslant 2 \tag{A 51}$$

We obtain the second-order radial velocity u_{r21}

$$u_{r21}(r,z,t) = u^+_{r21}(r,z,t) + u^-_{r21}(r,z,t) + u^\pm_{r21}(r,z,t) \tag{A 52}$$

where

$$u^+_{r21} = \left[D^+_{21} I_1(2m^+ r) + C^+_{21}\frac{k}{\alpha^+_1} I_1(2kr) + \sum_{i=1}^{N-1}\zeta^+_{2i+1} r^{2i+1}\right] e^{-2\alpha^+_1 t}\cos 2kz \tag{A 53}$$

$$u^-_{r21} = \left[D^-_{21} I_1(2m^- r) + C^-_{21}\frac{k}{\alpha^-_1} I_1(2kr) + \sum_{i=1}^{N-1}\zeta^-_{2i+1} r^{2i+1}\right] e^{-2\alpha^-_1 t}\cos 2kz \tag{A 54}$$

$$u^\pm_{r21} = \left[D^\pm_{21} I_1(2m^\pm r) + C^\pm_{21}\frac{2k}{\alpha^+_1+\alpha^-_1} I_1(2kr) + \sum_{i=1}^{N-1}\zeta^\pm_{2i+1} r^{2i+1}\right] e^{-(\alpha^+_1+\alpha^-_1)t}\cos 2kz \tag{A 55}$$

From the velocity u_{r21} and the continuity equation (2.25), the second-order axial velocity u_{z21} is deduced as

$$u_{z21}(r,z,t) = u^+_{z21}(r,z,t) + u^-_{z21}(r,z,t) + u^\pm_{z21}(r,z,t) \tag{A 56}$$

where

$$u^+_{z21}(r,z,t) = -\frac{1}{2k}\left[D^+_{21} 2m^+ I_0(2m^+ r) + C^+_{21}\frac{k}{\alpha^+_1} 2kI_0(2kr) + \sum_{i=1}^{N-1}(2i+2)\zeta^+_{2i+1} r^{2i}\right] e^{-2\alpha^+_1 t}\sin 2kz \tag{A 57}$$

$$u^-_{z21}(r,z,t) = -\frac{1}{2k}\left[D^-_{21} 2m^- I_0(2m^- r) + C^-_{21}\frac{k}{\alpha^-_1} 2kI_0(2kr) + \sum_{i=1}^{N-1}(2i+2)\zeta^-_{2i+1} r^{2i}\right] e^{-2\alpha^-_1 t}\sin 2kz \tag{A 58}$$

$$u^\pm_{z21}(r,z,t) = -\frac{1}{2k}\left[D^\pm_{21} 2m^\pm I_0(2m^\pm r) + C^\pm_{21}\frac{2k}{\alpha^+_1+\alpha^-_1} 2kI_0(2kr) + \sum_{i=1}^{N-1}(2i+2)\zeta^\pm_{2i+1} r^{2i}\right] e^{-(\alpha^+_1+\alpha^-_1)t}\sin 2kz \tag{A 59}$$

In these solutions, the coefficients C_{21} and D_{21} are unknown constants to be determined.

A.3. *The coefficients for the second-order solution*

The coefficients C_{21}, D_{21}, F_{21}, G_{21} and P_{21} for each time dependency need to be determined to fully solve the first contribution to the second-order solution.

The coefficients C_{21}, D_{21}, F_{21} are determined by solving the projections of the three boundary conditions on $\cos 2kz$. By doing this, we obtain, for each time dependency

$$C_{21}^{+} = \frac{NC_{21}^{+}}{I_1(2k)DN^{+}}, \quad C_{21}^{-} = \frac{NC_{21}^{-}}{I_1(2k)DN^{-}}, \quad C_{21}^{\pm} = \frac{NC_{21}^{\pm}}{I_1(2k)DN^{\pm}} \tag{A 60}$$

$$D_{21}^{+} = \frac{ND_{21}{}^{+}}{I_1(2m^{+})DN^{+}}, \quad D_{21}^{-} = \frac{ND_{21}{}^{-}}{I_1(2m^{-})DN^{-}}, \quad D_{21}^{\pm} = \frac{ND_{21}^{\pm}}{I_1(2m^{\pm})DN^{\pm}} \tag{A 61}$$

$$F_{21}^{+} = \frac{NF_{21}^{+}}{DN^{+}}, \quad F_{21}^{-} = \frac{NF_{21}{}^{-}}{DN^{-}}, \quad F_{21}^{\pm} = \frac{NF_{21}{}^{\pm}}{DN^{\pm}} \tag{A 62}$$

where the various coefficients NC_{21}, ND_{21}, NF_{21} and DN are defined for each time dependency as

$$NC_{21}^{+} = -\frac{(1-4k^2)(k^2+m^{+\,2})}{4k^3}R_k^{+} + \left(\frac{\alpha_1^{+}Oh(2m^{+}I_c^{+}-1)}{2k^2} + \frac{1-4k^2}{8k^2}\right)R_{zs}^{+} - \frac{\alpha_1^{+}(k^2+m^{+\,2})}{2k^3}R_{zn}^{+} \tag{A 63}$$

$$NC_{21}^{-} = -\frac{(1-4k^2)(k^2+m^{-\,2})}{4k^3}R_k^{-} + \left(\frac{\alpha_1^{-}Oh(2m^{-}I_c^{-}-1)}{2k^2} + \frac{1-4k^2}{8k^2}\right)R_{zs}^{-} - \frac{\alpha_1^{-}(k^2+m^{-\,2})}{2k^3}R_{zn}^{-} \tag{A 64}$$

$$NC_{21}^{\pm} = -\frac{(1-4k^2)(k^2+m^{\pm\,2})}{4k^3}R_k^{\pm} + \left(\frac{(\alpha_1^{+}+\alpha_1^{-})Oh(2m^{\pm}I_c^{\pm}-1)}{4k^2} + \frac{1-4k^2}{8k^2}\right)R_{zs}^{\pm} - \frac{(\alpha_1^{+}+\alpha_1^{-})(k^2+m^{\pm\,2})}{4k^3}R_{zn}^{\pm} \tag{A 65}$$

$$ND_{21}^{+} = \frac{1-4k^2}{2\alpha_1^{+}}R_k^{+} - \frac{\alpha_1^{+}}{4k^2}\left(2Oh\frac{k}{\alpha_1^{+}}(2kI_b-1) - I_b + \frac{k(1-4k^2)}{2\,\alpha_1^{+\,2}}\right)R_{zs}^{+} + R_{zn}^{+} \tag{A 66}$$

$$ND_{21}^{-} = \frac{1-4k^2}{2\alpha_1^{-}}R_k^{-} - \frac{\alpha_1^{-}}{4k^2}\left(2Oh\frac{k}{\alpha_1^{-}}(2kI_b-1) - I_b + \frac{k(1-4k^2)}{2\,\alpha_1^{-\,2}}\right)R_{zs}^{-} + R_{zn}^{-} \tag{A 67}$$

$$ND_{21}^{\pm} = \frac{1-4k^2}{\alpha_1^{+}+\alpha_1^{-}}R_k^{\pm} - \frac{\alpha_1^{+}+\alpha_1^{-}}{8k^2}\left(2Oh\frac{2k}{\alpha_1^{+}+\alpha_1^{-}}(2kI_b-1) - I_b + \frac{2k(1-4k^2)}{(\alpha_1^{+}+\alpha_1^{-})^2}\right)R_{zs}^{\pm} + R_{zn}^{\pm} \tag{A 68}$$

$$NF_{21}^{+} = \left[\frac{2Oh\;m^{+}I_c^{+}}{\alpha_1^{+}} + \frac{k^2+m^{+\,2}}{4k^2}\left(\frac{I_b}{\alpha_1^{+}} + \frac{1}{k}\right)\right]R_k^{+} - \left[\frac{Oh\;m^{+}I_c^{+}}{\alpha_1^{+}2k} + \frac{I_b}{8k}\left(\frac{1}{\alpha_1^{+}} + \frac{1}{k}\right)\right]R_{zs}^{+} + \frac{m^{+\,2}-k^2}{4k\alpha_1^{+}}R_{zn}^{+} \tag{A 69}$$

$$NF_{21}^{-} = \left[\frac{2Oh\; m^{-} I_c^{-}}{\alpha_1^{-}} + \frac{k^2 + {m^{-}}^2}{4k^2}\left(\frac{I_b}{\alpha_1^{-}} + \frac{1}{k}\right)\right] R_k^{-} - \left[\frac{Oh\; m^{-} I_c^{-}}{\alpha_1^{-} 2k} + \right. \tag{A 70}$$
$$\left. + \frac{I_b}{8k}\left(\frac{1}{\alpha_1^{-}} + \frac{1}{k}\right)\right] R_{zs}^{-} + \frac{{m^{-}}^2 - k^2}{4k\alpha_1^{-}} R_{zn}^{-}$$

$$NF_{21}^{\pm} = \left[\frac{4Oh\; m^{\pm} I_c^{\pm}}{(\alpha_1^{-} + \alpha_1^{+})} + \frac{k^2 + {m^{\pm}}^2}{4k^2}\left(\frac{I_b}{\alpha_1^{\pm}} + \frac{1}{k}\right)\right] R_k^{\pm} - \left[\frac{Oh\; m^{\pm} I_c^{\pm}}{(\alpha_1^{-} + \alpha_1^{+})k} + \right. \tag{A 71}$$
$$\left. + \frac{I_b}{8k}\left(\frac{2}{(\alpha_1^{-} + \alpha_1^{+})} + \frac{1}{k}\right)\right] R_{zs}^{\pm} + \frac{{m^{\pm}}^2 - k^2}{2k(\alpha_1^{-} + \alpha_1^{+})} R_{zn}^{\pm}$$

$$DN^{+} = \frac{1 - 4k^2}{8Ohk^2} + 2Oh\left(2m^{+} I_c^{+} - 1\right) - \tag{A 72}$$
$$- \frac{\alpha_1^{+}}{2k^3}({m^{+}}^2 + k^2)\left[2Oh\frac{k}{\alpha_1^{+}}\left(2kI_b - 1\right) - I_b\right]$$

$$DN^{-} = \frac{1 - 4k^2}{8Ohk^2} + 2Oh\left(2m^{-} I_c^{-} - 1\right) - \tag{A 73}$$
$$- \frac{\alpha_1^{-}}{2k^3}({m^{-}}^2 + k^2)\left[2Oh\frac{k}{\alpha_1^{-}}\left(2kI_b - 1\right) - I_b\right]$$

$$DN^{\pm} = \frac{1 - 4k^2}{8Ohk^2} + 2Oh\left(2m^{\pm} I_c^{\pm} - 1\right) - \frac{\alpha_1^{+} + \alpha_1^{-}}{4k^3}({m^{\pm}}^2 + k^2) \tag{A 74}$$
$$\cdot \left[2Oh\frac{2k}{\alpha_1^{+} + \alpha_1^{-}}\left(2kI_b - 1\right) - I_b\right]$$

In these coefficients, we have defined $I_b = I_0(2k)/I_1(2k)$, $I_c^{+} = I_0(2m^{+})/I_1(2m^{+})$, $I_c^{-} = I_0(2m^{-})/I_1(2m^{-})$, and $I_c^{\pm} = I_0(2m^{\pm})/I_1(2m^{\pm})$. The terms R_k, R_{zs} and R_{zn} are the right-hand sides of the kinematic, tangential-stress and normal-stress boundary conditions of second order, (2.28)-(2.30), respectively. These terms for each time dependency read

$$R_k^{+} = \frac{{\hat{\eta}_1^{+}}^2}{2}\left[\left(2k^2 Oh - \alpha_1^{+}\right)\left(1 - 2kI_a\right) - 2k^2 Oh\left(1 - 2l^{+} I_a^{+}\right)\right] - Ap_{ur21}^{+} \tag{A 75}$$

$$R_k^{-} = \frac{{\hat{\eta}_1^{-}}^2}{2}\left[\left(2k^2 Oh - \alpha_1^{-}\right)\left(1 - 2kI_a\right) - 2k^2 Oh\left(1 - 2l^{-} I_a^{-}\right)\right] - Ap_{ur21}^{-} \tag{A 76}$$

$$R_k^{\pm} = \frac{\hat{\eta}_1^{+}\hat{\eta}_1^{-}}{2}\left[\left(4k^2 Oh - \left(\alpha_1^{+} + \alpha_1^{-}\right)\right)\left(1 - 2kI_a\right) - 4k^2 Oh\left(1 - l^{+} I_a^{+} - l^{-} I_a^{-}\right)\right] \tag{A 77}$$
$$- Ap_{ur21}^{\pm}$$

$$R_{zs}^{+} = - {\hat{\eta}_1^{+}}^2 k\left[3kOh\left({l^{+}}^2 + k^2\right) I_a - l^{+} Oh\left({l^{+}}^2 + 5k^2\right) I_a^{+} + \alpha_1^{+}\right]$$
$$- \frac{1}{2k} Ap_{uz21}^{+'} - 2kAp_{ur21}^{+} \tag{A 78}$$

$$R_{zs}^{-} = - {\hat{\eta}_1^{-}}^2 k\left[3kOh\left({l^{-}}^2 + k^2\right) I_a - l^{-} Oh\left({l^{-}}^2 + 5k^2\right) I_a^{-} + \alpha_1^{-}\right]$$
$$- \frac{1}{2k} Ap_{uz21}^{-'} - 2kAp_{ur21}^{-} \tag{A 79}$$

$$R_{zs}^{\pm} = -\hat{\eta}_1^{+}\hat{\eta}_1^{-} k\left[3kOh\left({l^{+}}^2 + {l^{-}}^2 + 2k^2\right) I_a - l^{+} Oh\left({l^{+}}^2 + 5k^2\right) I_a^{+}\right. \tag{A 80}$$
$$\left. - l^{-} Oh\left({l^{-}}^2 + 5k^2\right) I_a^{-} + \left(\alpha_1^{+} + \alpha_1^{-}\right)\right] - \frac{1}{2k} Ap_{uz21}^{\pm'} - 2kAp_{ur21}^{\pm}$$

$$R_{zn}^{+} = -\frac{1}{4}\,\hat{\eta}_1^{+\,2}\left(f_r^{+\,2} - f_z^{+\,2}\right) - \frac{\hat{\eta}_1^{+\,2}}{2}\left[\alpha_1^{+\,2} + 1 + \frac{k^2}{2} - 2k^2\alpha_1^{+}Oh + 2Oh\; f_r^{+\,\prime\prime}\right] + Ap_{p21}^{+} - 2OhAp_{ur21}^{+\,\prime} \qquad \text{(A 81)}$$

$$R_{zn}^{-} = -\frac{1}{4}\,\hat{\eta}_1^{-\,2}\left(f_r^{-\,2} - f_z^{-\,2}\right) - \frac{\hat{\eta}_1^{-\,2}}{2}\left[\alpha_1^{-\,2} + 1 + \frac{k^2}{2} - 2k^2\alpha_1^{-}Oh + 2Oh\; f_r^{-\,\prime\prime}\right] + Ap_{p21}^{-} - 2OhAp_{ur21}^{-\,\prime} \qquad \text{(A 82)}$$

$$R_{zn}^{\pm} = -\frac{1}{4}2\hat{\eta}_1^{+}\hat{\eta}_1^{-}\left(f_r^{+}f_r^{-} - f_z^{+}f_z^{-}\right) - \frac{\hat{\eta}_1^{+}\hat{\eta}_1^{-}}{2}\left[\left(\alpha_1^{+\,2} + \alpha_1^{-\,2}\right) + 2 + k^2 - 2k^2Oh\left(\alpha_1^{+} + \alpha_1^{-}\right) + 2Oh\left(f_r^{+\,\prime\prime} + f_r^{-\,\prime\prime}\right)\right] + Ap_{p21}^{\pm} - 2OhAp_{ur21}^{\pm\,\prime} \qquad \text{(A 83)}$$

where we have defined $I_a = I_0(k)/I_1(k)$, $I_a^{+} = I_0(l^{+})/I_1(l^{+})$ and $I_a^{-} = I_0(l^{-})/I_1(l^{-})$. The terms Ap with the various subscripts, and without or with prime, are the approximated terms expressed as sums in the solutions for u_{r21}, u_{z21} or p_{21}, as denoted in the subscripts, and the prime refers to the derivative with respect to the radial coordinate r, all evaluated at $r = 1$.

The coefficients G_{21} and P_{21} are determined by solving the remaining projections of the boundary conditions. We obtain two relations with the kinematic and normal-stress boundary conditions, which allows for the determination of G_{21} and P_{21}, for each time dependency, as

$$G_{21}^{+} = -\frac{\hat{\eta}_1^{+\,2}}{4} \qquad \text{(A 84)}$$

$$G_{21}^{-} = -\frac{\hat{\eta}_1^{-\,2}}{4} \qquad \text{(A 85)}$$

$$G_{21}^{\pm} = -\frac{\hat{\eta}_1^{+}\hat{\eta}_1^{-}}{2} \qquad \text{(A 86)}$$

$$P_{21}^{+} = \frac{\hat{\eta}_1^{+\,2}}{4}\left[3 - k^2 + f_r^{+\,2} + f_z^{+\,2} + 2\,\alpha_1^{+\,2} - 4k^2\alpha_1^{+}Oh + 4Oh\; f_r^{+\,\prime\prime}\right] - Ap_{p21}^{+} \qquad \text{(A 87)}$$

$$P_{21}^{-} = \frac{\hat{\eta}_1^{-\,2}}{4}\left[3 - k^2 + f_r^{-\,2} + f_z^{-\,2} + 2\,\alpha_1^{-\,2} - 4k^2\alpha_1^{-}Oh + 4Oh\; f_r^{-\,\prime\prime}\right] - Ap_{p21}^{-} \qquad \text{(A 88)}$$

$$P_{21}^{\pm} = \frac{\hat{\eta}_1^{+}\hat{\eta}_1^{-}}{2}\left[3 - k^2 + f_r^{+}f_r^{-} + f_z^{+}f_z^{-} + \left(\alpha_1^{+\,2} + \alpha_1^{-\,2}\right) - 2k^2Oh\left(\alpha_1^{+} + \alpha_1^{-}\right) + 2Oh\left(f_r^{+\,\prime\prime} + f_r^{-\,\prime\prime}\right)\right] - Ap_{p21}^{\pm} \qquad \text{(A 89)}$$

In the expressions of the coefficients P_{21} the primes again denote derivatives with respect to the radial coordinate and all the functions of the radial coordinate are to be taken at $r = 1$.

Finally, the two initial conditions (2.31) reveal the amplitudes $\hat{\eta}_{22}^{p}$ and $\hat{\eta}_{22}^{m}$ for the second contributions to the second-order solutions as

$$\hat{\eta}_{22}^{p} = -\frac{1}{\alpha_2^{p} - \alpha_2^{m}}\left[F_{21}^{+}\left(2\alpha_1^{+} - \alpha_2^{m}\right) + F_{21}^{-}\left(2\alpha_1^{-} - \alpha_2^{m}\right) + F_{21}^{\pm}\left(\alpha_1^{+} + \alpha_1^{-} - \alpha_2^{m}\right)\right] \qquad \text{(A 90)}$$

$$\hat{\eta}_{22}^{m} = \frac{1}{\alpha_2^{p} - \alpha_2^{m}}\left[F_{21}^{+}\left(2\alpha_1^{+} - \alpha_2^{p}\right) + F_{21}^{-}\left(2\alpha_1^{-} - \alpha_2^{p}\right) + F_{21}^{\pm}\left(\alpha_1^{+} + \alpha_1^{-} - \alpha_2^{p}\right)\right] \qquad \text{(A 91)}$$

A.4. *The coefficients for the inviscid solution*

To retrieve the inviscid solution from the viscous solution, one sets $R_{zs} = 0$, $Ap_{ur21} = 0$, $Ap_{uz21} = 0$, $Ap_{21} = 0$ and takes the limit $Oh \to 0$ for which $\alpha_1^+ + \alpha_1^- = 0$ and $\alpha_2^p + \alpha_2^m = 0$. Note that the parameter Oh appears in the modified wavenumbers l and m. This reveals the following coefficients

$$C_{21}^{+} = C_{21}^{-} = C_{21} = -\frac{\alpha_1^2}{2kI_1(2k)}\left[4F_{21} + \frac{1}{4}(1 - 2kI_a)\right] \tag{A 92}$$

$$C_{21}^{\pm} = 0 \tag{A 93}$$

$$F_{21}^{+} = F_{21}^{-} = F_{21} = \frac{2I_b\alpha_1^2(1 - 2kI_a) + k\left[k^2 + 2 + \alpha_1^2\left(3 - I_a^2\right)\right]}{8I_b\left(\alpha_2^2 - 4\alpha_1^2\right)} \tag{A 94}$$

$$F_{21}^{\pm} = \frac{2 + k^2 + \alpha_1^2\left(1 + I_a^2\right)}{8\left(1 - 4k^2\right)} \tag{A 95}$$

$$G_{21}^{+} = G_{21}^{-} = G_{21} = -\frac{1}{16} \tag{A 96}$$

$$G_{21}^{\pm} = -\frac{1}{8} \tag{A 97}$$

$$P_{21}^{+} = P_{21}^{-} = P_{21} = \frac{1}{16}\left[3 - k^2 + \alpha_1^2\left(3 + I_a^2\right)\right] \tag{A 98}$$

$$P_{21}^{\pm} = \frac{1}{8}\left[3 - k^2 + \alpha_1^2\left(1 - I_a^2\right)\right] \tag{A 99}$$

$$\hat{\eta}_{22}^{p} = \hat{\eta}_{22}^{m} = \hat{\eta}_{22} = -\frac{1}{2}\left(2F_{21} + F_{21}^{\pm}\right) \tag{A 100}$$

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