SUPPLEMENTARY MATERIAL:

Sensitivity of the Rayleigh criterion in thermoacoustics

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The equations and sections of the main paper are referenced by square brackets.

1. A remark on the value of the complex acoustic energy

We discuss two physically important cases: the acoustic case and the thermoacoustic case. The acoustic case is defined by $\hat{\dot{Q}} = 0$. From [3.3], it follows that

$$\sigma \int_0^L \left(\bar{\rho}\hat{u}^2 + \frac{1}{\gamma\bar{p}}\hat{p}^2 \right) \, dx = 0,$$

which is fulfilled in the following scenarios:

(i) σ=0, which, due to the momentum and energy equations, implies that $\frac{\partial \hat{p}}{\partial x} = 0$ and $\frac{\partial \hat{u}}{\partial x} = 0$. Physically, this means that the eigenfunctions are trivial in space because of the ideal boundary conditions. Therefore, $\int_0^L (\bar{\rho}\hat{u}^2 + \frac{1}{\gamma\bar{p}}\hat{p}^2) \, dx$=0, i..e, the acoustics are not allowed to propagate;

(ii) $\sigma \neq 0$ and $\int_0^L (\bar{\rho}\hat{u}^2 + \frac{1}{\gamma\bar{p}}\hat{p}^2) \, dx$=0 implies that there are acoustic waves propagating. Furthermore, $\sigma_r = 0$ and $\sigma_i \neq 0$ because the system is conservative, or, in other words, it is self-adjoint (Sturm-Liouville theorem). The limit $\dot{Q} \to 0$ of [3.3] generates an indeterminate form 0/0, which, when solved by Taylor expansion, provides the acoustic natural angular frequency $\sigma = \mathrm{i}\sigma_i$.

The thermoacoustic case is defined by $\int_0^L \hat{p}\hat{\dot{Q}} dx \neq 0$. The problem is non-self-adjoint, hence, $\sigma_r \neq 0$ and $\sigma_i \neq 0$ and [3.3] implies that the complex acoustic energy is not in equilibrium, i.e., $\int_0^L (\bar{\rho}\hat{u}^2 + \frac{1}{\gamma\bar{p}}\hat{p}^2) \, dx \neq 0$. The special thermoacoustic case in which $\int_0^L \hat{p}\hat{\dot{Q}} dx = 0$ with $\hat{\dot{Q}} \neq 0$ implies that the complex acoustic energy is in equilibrium, i.e., $\int_0^L (\bar{\rho}\hat{u}^2 + \frac{1}{\gamma\bar{p}}\hat{p}^2) \, dx$=0. The system is non-self-adjoint but is marginally stable because energy gain/loss mechanisms are in balance.

The same line of reasoning can be used to show that the energy F and Lagrangian G are not zero in a thermoacoustic system [§3.1.1].

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2. An illustrative example for the eigenvalue integral formulae

We consider a simple example, which is amenable to analytical treatment. The variables are non-dimensionalised such that the non-dimensional governing equations read

$$\sigma\hat{u} + \frac{\partial\hat{p}}{\partial x} = 0, \tag{2.1}$$

$$\sigma\hat{p} + \frac{\partial\hat{u}}{\partial x} = \hat{\dot{Q}}. \tag{2.2}$$

We consider a one-mode decomposition of the acoustic variables

$$\hat{p} = \alpha\sin(\pi x), \tag{2.3}$$

$$\hat{u} = \eta\cos(\pi x), \tag{2.4}$$

where π is the non-dimensional acoustic angular frequency of the first acoustic mode. We assume the heat release to be of the form of

$$\hat{\dot{Q}} = (a\hat{u} + b\hat{p} + c\sigma\hat{u} + d\sigma\hat{p})\,\delta(x - x_f), \qquad a, b, c, d \in \mathbb{R}^+. \tag{2.5}$$

By substituting the one-mode decomposition in the governing equations, we obtain

$$\sigma\eta + \alpha\pi = 0, \tag{2.6}$$

$$\sigma\alpha\sin(\pi x) - \eta\pi\sin(\pi x) = (a\eta\cos(\pi x) + b\alpha\sin(\pi x) + c\sigma\eta\cos(\pi x) + d\alpha\sigma\sin(\pi x))\,\delta(x - x_f). \tag{2.7}$$

We integrate the second equation multiplied by $\sin(\pi x)$, which yields

$$\sigma\alpha - \eta\pi = a\eta\sin(2\pi x_f) + 2b\alpha\sin^2(\pi x_f) + c\sigma\eta\sin(2\pi x_f) + 2d\alpha\sigma\sin^2(\pi x). \tag{2.8}$$

The eigenproblem reads

$$\begin{bmatrix} \sigma & \pi \\ -\pi - \underbrace{a\sin(2\pi x_f)}_{\equiv A} - \sigma\underbrace{c\sin(2\pi x_f)}_{\equiv C} & \sigma - \underbrace{2b\sin^2(\pi x_f)}_{\equiv B} - \sigma\underbrace{2d\sin^2(\pi x_f)}_{\equiv D} \end{bmatrix} \begin{bmatrix} \eta \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.9}$$

Zeroing the determinant of the above matrix provides the dispersion relation, which reads

$$\sigma\,(\sigma - B - \sigma D) + \pi\,(\pi + A + \sigma C) = 0, \tag{2.10}$$

hence

$$\sigma^2(1 - D) + \sigma(-B + \pi C) + \pi^2\left(1 + \frac{A}{\pi}\right) = 0. \tag{2.11}$$

The two eigenvalues are given by

$$\sigma_\pm = \frac{B - \pi C \pm \sqrt{(\pi C - B)^2 - 4\pi^2(1 - D)\left(1 + \frac{A}{\pi}\right)}}{2(1 - D)}. \tag{2.12}$$

The eigenvectors are the solutions of (2.9) when $\sigma = \sigma_\pm$. Because the eigenfunctions are defined up to a complex factor, we set $\alpha = 1$. The eigenvector is

$$\begin{bmatrix} \sigma_\pm & \pi \end{bmatrix} \begin{bmatrix} \eta \\ 1 \end{bmatrix} = 0 \qquad \Longrightarrow \begin{bmatrix} \eta \\ \alpha \end{bmatrix}_\pm = \begin{bmatrix} -\frac{\pi}{\sigma_\pm} \\ 1 \end{bmatrix} \tag{2.13}$$

Note that $\sigma_\pm \neq 0$ even if $A, B, C, D = 0$, which corresponds to the natural acoustics.

2.1. *Verification of the eigenvalue integral formulae with one-mode approximation*

By substituting the one-mode decomposition and eigenvector in the non-dimensionalized integral complex formula [3.3], we find

$$
\begin{aligned}
\sigma &= \frac{\int_0^L \hat{p}\dot{\hat{Q}}(\sigma,\hat{p},\hat{u})\ dx}{\int_0^L \left(\hat{u}^2+\hat{p}^2\right)\ dx} \\
&= \frac{\int_0^L \alpha\sin(\pi x)(a\eta\cos(\pi x)+b\alpha\sin(\pi x)+c\sigma\eta\cos(\pi x)+d\sigma\alpha\sin(\pi x))\,\delta(x-x_f)\ dx}{\int_0^L \left(\eta^2\cos^2(\pi x)+\alpha^2\sin^2\pi x\right)\ dx} \\
&= \frac{\alpha\sin(\pi x_f)(a\eta\cos(\pi x_f)+b\alpha\sin(\pi x_f)+c\sigma\eta\cos(\pi x_f)+d\sigma\alpha\sin(\pi x_f))}{\frac{1}{2}\left(\eta^2+\alpha^2\right)} \\
&= \frac{\alpha\left(a\eta\sin(2\pi x_f)+2b\alpha\sin^2(\pi x_f)+c\sigma\eta\sin(2\pi x_f)+2d\sigma\alpha\sin^2(\pi x_f)\right)}{\left(\eta^2+\alpha^2\right)} \\
&= \frac{\left(-\frac{\pi}{\sigma_\pm}\underbrace{a\sin(2\pi x_f)}_{=A}+\underbrace{2b\sin^2(\pi x_f)}_{=B}-\frac{\pi}{\sigma_\pm}\sigma_\pm\underbrace{c\sin(2\pi x_f)}_{=C}+\sigma_\pm\underbrace{2d\sin^2(\pi x_f)}_{=D}\right)}{\left(\frac{\pi^2}{\sigma_\pm^2}+1^2\right)} \\
&= \frac{\left(-\frac{\pi}{\sigma_\pm}A+B-\pi C+\sigma_\pm D\right)}{\left(\frac{\pi^2}{\sigma_\pm^2}+1^2\right)} \\
&= \frac{\left(-\pi\sigma_\pm A+\sigma_\pm^2 B-\pi\sigma_\pm^2 C+\sigma_\pm^3 D\right)}{\left(\pi^2+\sigma_\pm^2\right)}
\end{aligned}
\tag{2.14}
$$

Therefore

$$
\begin{aligned}
0 &= \pi^2+\sigma_\pm^2-\left(-\pi A+\sigma_\pm B-\pi\sigma_\pm C+\sigma_\pm^2 D\right) \\
&= +\sigma_\pm^2(1-D)+\sigma_\pm(-B+\pi C)+\pi^2\left(1+\frac{A}{\pi}\right).
\end{aligned}
\tag{2.15}
$$

This is always fulfilled because of the dispersion relation (2.11).

By substituting the one-mode decomposition and eigenvector in the non-dimensionalized formula for the growth rate [3.4], we find

$$
\begin{aligned}
\sigma_r &= \frac{Re\left(\int_0^L \hat{p}^*\dot{\hat{Q}}(\sigma,\hat{p},\hat{u})\ dx\right)}{\int_0^L \left(|u|^2+|p|^2\right)\ dx} \\
&= \frac{Re\left(\int_0^L \alpha^*\sin(\pi x)(a\eta\cos(\pi x)+b\alpha\sin(\pi x)+c\sigma\eta\cos(\pi x)+d\sigma\alpha\sin(\pi x))\,\delta(x-x_f)\ dx\right)}{\int_0^L \left(|\eta|^2\cos^2(\pi x)+|\alpha|^2\sin^2(\pi x)\right)\ dx} \\
&= \frac{Re\left(\alpha^*\left(a\eta\sin(2\pi x_f)+2b\alpha\sin^2(\pi x_f)+c\sigma\eta\sin(2\pi x_f)+2d\sigma\alpha\sin^2(\pi x_f)\right)\right)}{|\eta|^2+|\alpha|^2} \\
&= \frac{Re\left(\alpha^*\eta A+|\alpha|^2 B+\sigma\alpha^*\eta C+|\alpha|^2\sigma D\right)}{|\eta|^2+|\alpha|^2} \\
&= \frac{Re\left(-\frac{\pi}{\sigma_\pm}A+B-\pi C+\sigma_\pm D\right)}{\left|\frac{-\pi}{\sigma_\pm}\right|^2+1}
\end{aligned}
\tag{2.16}
$$

Therefore

$$
\begin{aligned}
0 &= Re(\sigma_\pm)\left(\left|\frac{-\pi}{\sigma_\pm}\right|^2+1\right) - Re\left(-\frac{\pi}{\sigma_\pm}A + B - \pi C + \sigma_\pm D\right)\\
&= Re\left[\frac{\pi^2}{\sigma_\pm^*} + \sigma_\pm + \frac{\pi}{\sigma_\pm}A - B + \pi C - \sigma_\pm D\right]\\
&= Re\left[\sigma_\pm\frac{\pi^2}{|\sigma_\pm|^2} + \sigma_\pm(1-D) + \sigma^*\frac{\pi}{|\sigma_\pm|^2}A + (-B+\pi C)\right]\\
&= Re\left[\sigma_\pm\pi^2 + \sigma_\pm|\sigma_\pm|^2(1-D) + \sigma^*\pi A + (-B+\pi C)|\sigma_\pm|^2\right]\\
&= Re(\sigma_\pm)\pi^2 + Re(\sigma_\pm)|\sigma_\pm|^2(1-D) + Re(\sigma^*)\pi A + (-B+\pi C)|\sigma_\pm|^2\\
&= Re(\sigma_\pm)|\sigma_\pm|^2(1-D) + Re(\sigma_\pm)\pi^2\left(1+\frac{A}{\pi}\right) + (-B+\pi C)|\sigma_\pm|^2. \qquad (2.17)
\end{aligned}
$$

This equality is always satisfied because $Re(\sigma^*(2.11)) = 0$.

By substituting the one-mode decomposition and eigenvector in the non-dimensionalized formula for the angular frequency [3.5], we find

$$
\begin{aligned}
\sigma_i &= -\frac{Im\left(\int_0^L \hat{p}^*\hat{\dot{Q}}(\sigma,\hat{p},\hat{u})\,dx\right)}{\int_0^L\left(|\hat{u}|^2-|\hat{p}|^2\right)\,dx}\\
&= \frac{Im\left(-\frac{\pi}{\sigma_\pm}A + B - \pi C + \sigma_\pm D\right)}{1-\left|\frac{-\pi}{\sigma_\pm}\right|^2} \qquad (2.18)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
0 &= Im(\sigma)\left(1-\frac{\pi^2}{|\sigma_\pm|^2}\right) - Im\left(-\frac{\pi}{\sigma_\pm}A + B - \pi C + \sigma_\pm D\right)\\
&= Im\left[\sigma_\pm\left(1-\frac{\pi^2}{|\sigma_\pm|^2}\right) + \frac{\pi}{\sigma_\pm}A - B + \pi C - \sigma_\pm D\right]\\
&= Im\left[\sigma_\pm - \sigma_\pm\frac{\pi^2}{|\sigma_\pm|^2} + \sigma^*\frac{\pi}{|\sigma_\pm|^2}A - B + \pi C - \sigma_\pm D\right]\\
&= Im\left[\sigma_\pm|\sigma_\pm|^2(1-D) - \sigma_\pm\pi^2 + \sigma^*\pi A + (-B+\pi C)|\sigma_\pm|^2\right]\\
&= Im(\sigma_\pm)|\sigma_\pm|^2(1-D) - Im(\sigma_\pm)\pi^2\left(1+\frac{A}{\pi}\right) \qquad (2.19)
\end{aligned}
$$

This equality is always satisfied because $Im(\sigma^*(2.11)) = 0$.

3. First variation with localized sources

For localized sources, we equate the first variation of the acoustic energy with the Lagrange multiplier

$$
\delta E = \left(\frac{\gamma-1}{\gamma\bar{p}}\right)\left\langle p^+, \epsilon Q_p\delta(x-x_p)\right\rangle_{V,T} = \epsilon\left(\frac{\gamma-1}{\gamma\bar{p}}\right)\int_0^T p_p^+ Q_p\,dt \qquad (3.1)
$$

to the first variation of the acoustic energy without the Lagrange multiplier

$$\begin{aligned}\delta E = \left(\frac{\gamma-1}{\gamma\bar{p}}\right)\left\langle p+\frac{dp}{d\dot{Q}}\dot{Q},\epsilon\dot{Q}_p\right\rangle_{V,T} &= \\ &= \left(\frac{\gamma-1}{\gamma\bar{p}}\right)\int_0^T\int_V\left(\epsilon p\dot{Q}_p+dp\dot{Q}\right)\,dxdt \\ &= \left(\frac{\gamma-1}{\gamma\bar{p}}\right)\int_0^T\int_V\left(\epsilon pQ_p\delta(x-x_p)+dpQ_f\delta(x-x_f)\right)\,dxdt \\ &= \left(\frac{\gamma-1}{\gamma\bar{p}}\right)\int_0^T\left(\epsilon p_pQ_p+dp_fQ_f\right)\,dt \\ &= \epsilon\left(\frac{\gamma-1}{\gamma\bar{p}}\right)\int_0^T\left(p_p+\frac{dp_f}{\epsilon Q_p}Q_f\right)Q_p\,dt. \end{aligned} \tag{3.2}$$

Because of the Riesz representation theorem, it follows that

$$p_p^+ = p_p + \frac{dp_f}{\epsilon Q_p}Q_f. \tag{3.3}$$

Physically, dp_f is the first variation of the acoustic pressure at the flame location x_f caused by a perturbation of the heat release rate at x_p. This is a nonlocal effect that would not occur if the system were self-adjoint ($Q_f = 0$).