Supplementary Material A. Fundamental vertical mode waves

Here we present how to determine the fundamental vertical mode profile of an internal wave of frequency ω for an arbitrary stratification profile $\rho_0(z)$. The mode-1 wave contains most of the internal wave energy and corresponds to the component of largest wavelength. The modal analysis relies on a linearization of (2.1)-(2.4) around a static background state satisfying $\rho = \rho_0(z)$ and $\mathrm{d}p_0/\mathrm{d}z = -\rho_0(z)g$. The small perturbations in ρ and p associated with the internal wave motion are denoted by primed fields, such that $\rho = \rho_0(z) + \rho'(x, z, t)$ and $p = p_0(z) + p'(x, z, t)$.

Under these assumptions, the linearized system becomes

$$\partial_x u + \partial_z w = 0, \tag{A 1}$$

$$\partial_t u = -\frac{1}{\rho_{00}}\partial_x p', \tag{A 2}$$

$$\partial_t w = -\frac{1}{\rho_{00}}\partial_z p' - \frac{\rho'}{\rho_{00}} g, \tag{A 3}$$

$$\partial_t \rho' - \frac{\rho_{00} N^2}{g} w = 0, \tag{A 4}$$

where

$$N(z) = \sqrt{-\frac{g}{\rho_{00}}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}} \tag{A 5}$$

is the buoyancy frequency. Combining (A 1)-(A 4), we end up with a single equation to be solved for the vertical velocity:

$$\left(\frac{\partial^2}{\partial t^2}\boldsymbol{\nabla}^2 + N^2(z)\frac{\partial^2}{\partial x^2}\right) w(x, z, t) = 0. \tag{A 6}$$

Once w is determined, all remaining perturbation profiles are resolved (Gerkema & Zimmerman 2008).

A vertical mode correspond to a solution of (A 6) that has the form of a plane wave propagating in the x direction, with temporal frequency ω and wavenumber k_x:

$$w(x, z, t) = \mathrm{Re}\left\{W(z)\exp(i(k_x x - \omega t))\right\}, \tag{A 7}$$

with i is the imaginary number and $\mathrm{Re}\{\cdot\}$ denotes the real part of the complex argument. Substitution of (A 7) in (A 6) results in the ordinary differential equation for $W(z)$

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + k_x^2\left(\frac{N^2(z)}{\omega^2} - 1\right) W = 0. \tag{A 8}$$

For $z \in [0, H]$, both top and bottom being horizontal surfaces, no-flux boundary conditions lead to $W(0) = W(H) = 0$. For a Sturm-Liouville equation as (A 8), each solution $W^{(n)}(z)$ corresponding to a possible $\left(k_x^{(n)}\right)^2$ is unique and orthogonal to all others. The general solution of (A 6) consists of the superposition of solutions in the form (A 7), with the $W^{(n)}(z)$ and $k_x^{(n)}$ corresponding to each vertical mode. The fundamental mode, which we want to find, is the mode associated to the smallest wavenumber $k_x^{(0)}$.

For a general stratification $\rho_0(z)$, and thus a general $N^2(z)$, (A 8) cannot be solved analytically. We solve (A 8) numerically, by using a finite difference approach. Let $\mathbf{w} = (\mathrm{w}_0, \mathrm{w}_1, \ldots, \mathrm{w}_{M+1})^{\mathsf{T}}$ be the vector with the discretized values of $W(z)$ at the points $z_0, \ldots, z_{M+1}$ uniformly distributed in $[0, H]$. We impose the homogeneous boundary conditions $\mathrm{w}_0 = \mathrm{w}_{M+1} = 0$ and use centered finite differences to approximate the second derivatives in (A 8). The discretized version of (A 8) becomes the linear system

$$\mathbf{A}\mathbf{w} + k_x^2 \mathbf{B}\mathbf{w} = \mathbf{0}, \tag{A 9}$$

where

$$A_{ij} = \begin{cases} -2/\Delta z^2 & \text{for } i = j, \\ 1/\Delta z^2 & \text{for } i = j \pm 1, \\ 0 & \text{otherwise.} \end{cases} \quad B_{ij} = \begin{cases} N^2(z_i)/\omega^2 - 1, & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

with $1 \leqslant i, j \leqslant M$. (A 9) is a generalized eigenvalue problem that can be solved numerically, with generalized eigenvalues $k_x^{(n)}$ and eigenvectors $\mathbf{w}^{(n)}$.

In the event of all $B_{ii} < 0$, which corresponds to a stratification such that $N(z) < \omega$ everywhere, (A 9) has no positive eigenvectors and the solutions will correspond to an evanescent wave (Gerkema & Zimmerman 2008). The mode $\mathbf{w}^{(0)}$ corresponding to the smallest wavenumber $k_x^{(0)}$, numerically computed from (A 9), is a discretization for the continuum fundamental mode vertical velocity $W^{(0)}(z)$.

Finally, from (A 1) the horizontal component vertical mode is also a plane wave

$$u(x, z, t) = \mathrm{Re}\left\{U(z) \exp(i(k_x x - \omega t))\right\}, \tag{A 10}$$

and the modes $U^{(n)}(z)$ can be computed from the $W^{(n)}(z)$. The fundamental mode horizontal velocity component $U^{(0)}(z)$, in particular, relates to the vertical component $W^{(0)}(z)$ as

$$U^{(0)}(z) = \frac{1}{k_x^{(0)}} \frac{\mathrm{d}W^{(0)}}{\mathrm{d}z}, \tag{A 11}$$

and $u = U^{(0)}(z)\sin(\omega t)$ is what is forced at the inlet boundary on the left of the domain as detailed in §2.1. The derivative $\mathrm{d}W^{(0)}/\mathrm{d}z$ is approximated from the discretized $\mathbf{w}^{(0)}$ using centered finite differences. The solution of the resulting linear system provides a discretized $\mathbf{u}^{(0)}$ used to reconstitute $U^{(0)}(z)$. In the main text, for the sake of simplicity, we drop the subscript and note $U(z) \equiv U^{(0)}(z)$.