

Supplementary material. Degenerate perturbation theory in thermoacoustics: high-order sensitivities and exceptional points

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1. Expansion of eigenvalues at EPs with Puiseux series: theory

Eigenvalues and eigenvectors coalesce at exceptional points. The operators become defective, meaning that the geometric multiplicity of an eigenvalue is smaller than its algebraic multiplicity. At these points, series expansions in integer powers of a parameter are not applicable. They need to be replaced by fractional power series, also known as Puiseux series (Leung 1990; Moiseyev 2011). When dealing with defective eigenvalues in the most general sense, one needs to distinguish two cases. A defective eigenvalue with algebraic multiplicity $a > 1$ is said to be non-derogatory if it has geometric multiplicity $g = 1$, and it is called derogatory if it has geometric multiplicity $a > g > 1$. In the former case, there is only 1 defective eigenvector that spans the entire degenerate subspace. The degenerate eigenvalue can therefore be associated to a single, large Jordan block of dimension a . In the latter case, instead, there is more than one Jordan block associated with the defective eigenvalue. In this supplementary material, we will derive the adjoint-based expansion equations only for the problems relevant to thermoacoustics, for which degenerate eigenvalues have multiplicity $a = 2$. A defective thermoacoustic eigenvalue, therefore, must have geometric multiplicity $g = 1$, and belongs to the non-derogatory case.

Because exceptional points limit the convergence of power series expansions, it may be convenient to expand *at* these points, in order to have the largest possible domain of convergence. Assume we have a defective solution of the nonlinear eigenvalue problem (2.1), with algebraic multiplicity $a = 2$ and geometric multiplicity $g = 1$. The defective eigenvalue and eigenvectors cannot be expressed in terms of power series around the defective point. Instead, they can be written as Puiseux series as

$$s = s_0 + \sum_{j=1}^{\infty} \varepsilon^{\frac{j}{2}} s_j \quad (\text{S1.1a})$$

$$\mathbf{p} = \mathbf{p}_0 + \sum_{j=1}^{\infty} \varepsilon^{\frac{j}{2}} \mathbf{p}_j \quad (\text{S1.1b})$$

The square root that appears in the above equations is representative of the branch-point singularity that occurs at the EP. Any small perturbation away from the EP results in two eigenvalue solutions, the two branches of the square-root function. Substituting these expansions in the nonlinear eigenvalue problem (2.1) and collecting terms of like powers in ε , we obtain the perturbation equations we aim to solve. The equations for the first

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few orders are

$$\varepsilon^0 : \quad \mathbf{L}_{0,0}\mathbf{p}_0 = 0, \quad (\text{S1.2a})$$

$$\varepsilon^{1/2} : \quad \mathbf{L}_{0,0}\mathbf{p}_1 = -s_1\mathbf{L}_{1,0}\mathbf{p}_0, \quad (\text{S1.2b})$$

$$\varepsilon : \quad \mathbf{L}_{0,0}\mathbf{p}_2 = -s_1\mathbf{L}_{1,0}\mathbf{p}_1 - (\mathbf{L}_{0,1} + s_1^2\mathbf{L}_{2,0} + s_2\mathbf{L}_{1,0})\mathbf{p}_0, \quad (\text{S1.2c})$$

$$\varepsilon^{3/2} : \quad \mathbf{L}_{0,0}\mathbf{p}_3 = -s_1\mathbf{L}_{1,0}\mathbf{p}_2 - (\mathbf{L}_{0,1} + s_1^2\mathbf{L}_{2,0} + s_2\mathbf{L}_{1,0})\mathbf{p}_1 \\ - (s_1\mathbf{L}_{1,1} + 2s_1s_2\mathbf{L}_{2,0} + s_1^3\mathbf{L}_{3,0} + s_3\mathbf{L}_{1,0})\mathbf{p}_0. \quad (\text{S1.2d})$$

We describe in the following how these equations can be solved order by order.

1.1. Order 0

At 0th order we have the baseline nonlinear eigenvalue problem. We assume that the operator $\mathbf{L}_{0,0}$ has an eigenvalue s_0 with algebraic multiplicity 2, associated with one eigenvector only, \mathbf{p}_0 . Following López-Gómez & Mora-Corral (2007) and Güttel & Tisseur (2017), we define the generalised eigenvector $\tilde{\mathbf{p}}_1$ such that the Jordan chain $\mathbf{p}(z) = \mathbf{p}_0 + (z - s_0)\tilde{\mathbf{p}}_1$ is a root function of $\mathbf{L}_{0,0}$ at $z = 0$ with multiplicity $a \geq 2$. In particular, for the multiplicity of the root to be at least 2, we must have

$$\left. \frac{\partial \mathbf{L}(z, 0)\mathbf{p}(z)}{\partial z} \right|_{z=s_0} = \mathbf{L}_{1,0}\mathbf{p}_0 + \mathbf{L}_{0,0}\tilde{\mathbf{p}}_1 = 0, \quad (\text{S1.3})$$

from which we can see that the generalised eigenvector of the defective operator is

$$\tilde{\mathbf{p}}_1 = -\mathbf{L}_{0,0}^{-1}\mathbf{L}_{1,0}\mathbf{p}_0, \quad (\text{S1.4})$$

where we denote with $\mathbf{L}_{0,0}^{-1}$ the pseudo-inverse of the matrix $\mathbf{L}_{0,0}$, which is singular. We also calculate the adjoint eigenvector \mathbf{p}_0^\dagger from the adjoint equation $\mathbf{L}_{0,0}^H\mathbf{p}_0^\dagger = \mathbf{0}$, and the generalised adjoint eigenvector from $\mathbf{L}_{0,0}^H\tilde{\mathbf{p}}_1^\dagger = -\mathbf{L}_{1,0}^H\mathbf{p}_0^\dagger$. The self-orthogonality property of defective eigenvalues implies that

$$\langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0}\mathbf{p}_0 \rangle = 0. \quad (\text{S1.5})$$

1.2. Order 1/2

We proceed as usual by imposing a solvability condition that corresponds to the Fredholm alternative, order by order. Note that, because the problem is non-derogatory, the geometric multiplicity is one, and we need to satisfy only one solvability condition, even though the eigenvalue is degenerate. At order $\varepsilon^{1/2}$ we have

$$\langle \mathbf{p}_0^\dagger | \mathbf{L}_{0,0}\mathbf{p}_1 \rangle = -\langle \mathbf{p}_0^\dagger | s_1\mathbf{L}_{1,0}\mathbf{p}_0 \rangle. \quad (\text{S1.6})$$

The l.h.s. is zero, because $\langle \mathbf{L}_{0,0}^H\mathbf{p}_0^\dagger | \mathbf{p}_1 \rangle = 0$ from the definition of adjoint eigenvector. Consequently, for equation (S1.6) to be satisfied, the r.h.s. must vanish too. This is indeed the case because of the self-orthogonality condition (S1.5). Therefore, no solvability condition needs to be imposed at order $\varepsilon^{1/2}$, and we are not able to determine the coefficient s_1 at this order.

Equation (S1.2b) is, hence, always solvable. Using Eq. (S1.4) yields

$$\mathbf{p}_1 = -s_1\mathbf{L}_{0,0}^{-1}\mathbf{L}_{1,0}\mathbf{p}_0 = s_1\tilde{\mathbf{p}}_1 + c_1\mathbf{p}_0, \quad (\text{S1.7})$$

i.e., the first-order correction to the eigenvector is proportional to the generalised eigenvector of the defective problem. The proportionality constant is exactly the first-order eigenvalue correction, which we have not determined yet. We can arbitrary multiples of

\mathbf{p}_0 , belonging to the nullspace of $\mathbf{L}_{0,0}$. The constant c_1 may be determined by imposing a normalisation condition on the perturbed eigenvectors. However, this condition plays no role in the solution of the perturbative equations and is omitted here.

1.3. Order 1

Imposing the solvability condition at this order, we have

$$\begin{aligned} \langle \mathbf{p}_0^\dagger | \mathbf{L}_{0,0} \mathbf{p}_2 \rangle &= \langle \mathbf{p}_0^\dagger | -s_1 \mathbf{L}_{1,0} \mathbf{p}_1 - (\mathbf{L}_{0,1} + s_1^2 \mathbf{L}_{2,0} + s_2 \mathbf{L}_{1,0}) \mathbf{p}_0 \rangle \\ 0 &= \langle \mathbf{p}_0^\dagger | s_1 \mathbf{L}_{1,0} (s_1 \tilde{\mathbf{p}}_1 + c_1 \mathbf{p}_0) + (\mathbf{L}_{0,1} + s_1^2 \mathbf{L}_{2,0} + s_2 \mathbf{L}_{1,0}) \mathbf{p}_0 \rangle \\ 0 &= c_1 s_1 \langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \mathbf{p}_0 \rangle + s_1^2 \langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \tilde{\mathbf{p}}_1 + \mathbf{L}_{2,0} \mathbf{p}_0 \rangle + \langle \mathbf{p}_0^\dagger | \mathbf{L}_{0,1} \mathbf{p}_0 \rangle + s_2 \langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \mathbf{p}_0 \rangle \end{aligned} \quad (\text{S1.8})$$

The first and last term on the r.h.s. (last row) vanish because of self-orthogonality. Therefore c_1 and s_2 play no role in the solvability of the equations, showing respectively that (i) the normalization condition can be omitted and may be imposed at a later stage; (ii) at second order we cannot calculate the eigenvalue correction coefficient s_2 . Instead, the above equation effectively defines s_1 as

$$s_1 = \pm \sqrt{-\frac{\langle \mathbf{p}_0^\dagger | \mathbf{L}_{0,1} \mathbf{p}_0 \rangle}{\langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \tilde{\mathbf{p}}_1 \rangle + \langle \mathbf{p}_0^\dagger | \mathbf{L}_{2,0} \mathbf{p}_0 \rangle}}, \quad (\text{S1.9})$$

in which a square root naturally appears, highlighting that the expansion point is a branch-point.

Having at hand these values, we uniquely determine the first-order eigenvector corrections from Eq. (S1.7)

$$\mathbf{p}_1 = \pm \sqrt{-\frac{\langle \mathbf{p}_0^\dagger | \mathbf{L}_{0,1} \mathbf{p}_0 \rangle}{\langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \tilde{\mathbf{p}}_1 \rangle + \langle \mathbf{p}_0^\dagger | \mathbf{L}_{2,0} \mathbf{p}_0 \rangle}} \tilde{\mathbf{p}}_1 + c_1 \mathbf{p}_0, \quad (\text{S1.10})$$

Substituting these values for s_1 and \mathbf{p}_1 in the equation at order ε , (S1.2c), we guarantee solvability for the eigenvector:

$$\mathbf{p}_2 = -s_1 \mathbf{L}_{0,0}^{-1} [\mathbf{L}_{1,0} \mathbf{p}_1 + (\mathbf{L}_{0,1} + s_1^2 \mathbf{L}_{2,0}) \mathbf{p}_0] - s_2 \mathbf{L}_{0,0}^{-1} \mathbf{L}_{1,0} \mathbf{p}_0 = \mathbf{p}_2^F + s_2 \tilde{\mathbf{p}}_1 + c_2 \mathbf{p}_0, \quad (\text{S1.11})$$

where \mathbf{p}_2^F is the component uniquely determined by the solvability condition, $s_2 \tilde{\mathbf{p}}_1$ is a component proportional to the generalised eigenvector, with proportionality constant equal to the second-order correction to the eigenvalue (to be determined at next order) and $c_2 \mathbf{p}_0$ is an arbitrary component belonging to the nullspace of $\mathbf{L}_{0,0}$, with proportionality constant c_2 determined by a normalization condition.

1.4. Order 3/2

Imposing the solvability condition at this order, we have

$$\begin{aligned} 0 &= \langle \mathbf{p}_0^\dagger | \mathbf{L}_{0,0} \mathbf{p}_3 \rangle = \\ &\langle \mathbf{p}_0^\dagger | -s_1 \mathbf{L}_{1,0} \mathbf{p}_2 - (\mathbf{L}_{0,1} + s_1^2 \mathbf{L}_{2,0} + s_2 \mathbf{L}_{1,0}) \mathbf{p}_1 \\ &\quad - (s_1 \mathbf{L}_{1,1} + 2s_1 s_2 \mathbf{L}_{2,0} + s_1^3 \mathbf{L}_{3,0} + s_3 \mathbf{L}_{1,0}) \mathbf{p}_0 \rangle \end{aligned} \quad (\text{S1.12})$$

The last term (the only one that depends on s_3) vanishes. At this order we need to determine s_2 . Note that s_2 is a second-order quantity (in powers of $\varepsilon^{1/2}$); therefore, it cannot appear quadratically at third order (nor can it at higher powers) as it was the case for s_1 at first order. This is general: at order $j > 1$ we need to determine s_{j-1} ,

which is of order $j - 1$, and therefore can appear only linearly. Indeed, if it appears with integer power $p > 1$, then it is of order at least $p(j - 1)$, which is larger than j . Thus, the equation for s_{j-1} at order j will be linear from this order onwards.

Recall that s_2 is implicitly contained in the expression for \mathbf{p}_2 . Substituting this expression, (S1.11), and the explicit expression for \mathbf{p}_1 (S1.10), we obtain after simplification

$$s_2 = -\frac{1}{2} \frac{\langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \mathbf{p}_2^F + (\mathbf{L}_{0,1} + s_1^2 \mathbf{L}_{2,0}) (\tilde{\mathbf{p}}_1 - \langle \tilde{\mathbf{p}}_1^\dagger | \mathbf{L}_{1,0} \tilde{\mathbf{p}}_1 \rangle \mathbf{p}_0) + (\mathbf{L}_{1,1} + s_1^2 \mathbf{L}_{3,0}) \mathbf{p}_0 \rangle}{\langle \mathbf{p}_0^\dagger | (\langle \mathbf{p}_0^\dagger | \mathbf{L}_{1,0} \tilde{\mathbf{p}}_1 \rangle + \mathbf{L}_{2,0} \mathbf{p}_0) \rangle}, \quad (\text{S1.13})$$

which depends on s_1 only quadratically, and shows that there is a unique solution for s_2 . This value of s_2 uniquely defines \mathbf{p}_2 (up to a normalisation factor), and allows us to solve for \mathbf{p}_3 up to a generalised eigenvector component:

$$\mathbf{p}_3 = \mathbf{p}_3^F + s_3 \tilde{\mathbf{p}}_1 + c_3 \mathbf{p}_0. \quad (\text{S1.14})$$

This procedure can be repeated to arbitrarily high order. Analogous to what is discussed by Mensah *et al.* (2020), recursive expressions for the formulation of the high-order expansion equations can be derived for Puiseux series expansions. These are, however, not needed for the purpose of this manuscript, and a full derivation is not presented.

2. Application of the theory to the 1D Galerkin thermoacoustic model

We report here the equations that have been used for the calculations in §5 of the main body. We recall from Eq. (5.4) that the Galerkin expansion model investigated reads

$$L(s) := s^2 + 2\pi\beta e^{-s\tau} + \pi^2, \quad (\text{S2.1})$$

and that we identify an EP by imposing $L(s) = 0$ and $\partial L / \partial s \equiv L_{1,0} = 0$. Furthermore, since the eigenvalue problem is one dimensional (i.e., it is defined through a scalar equation), the normalised direct and adjoint eigenvectors simply equal unity. From the definition (S1.4), it follows that for this special case, the generalised eigenvector vanishes, $\tilde{\mathbf{p}}_1 = 0$. As a consequence, all the higher-order eigenvector corrections also vanish, $p_j = 0$ for $j = 1, 2, \dots$ – see e.g. (S1.10). From Eq. (2.5), the following expressions for the relevant operator derivatives hold:

$$L_{0,1} = -2s\pi\beta e^{-s\tau} \quad (\text{S2.2a})$$

$$L_{2,0} = \frac{1}{2} (2 + 2\pi\beta\tau^2 e^{-s\tau}) \quad (\text{S2.2b})$$

$$L_{1,1} = 2\pi\beta e^{-s\tau} (s\tau - 1) \quad (\text{S2.2c})$$

$$L_{3,0} = \frac{1}{6} (-2\pi\beta\tau^3 e^{-s\tau}) \quad (\text{S2.2d})$$

Substituting all the quantities calculated above in Eq. (S1.9), the first term of the Puiseux series for the problem at hand is

$$s_1 = \pm \sqrt{-\frac{L_{0,1}}{L_{2,0}}} = \pm \sqrt{\frac{2s\pi\beta e^{-s\tau}}{1 + \pi\beta\tau^2 e^{-s\tau}}}. \quad (\text{S2.3})$$

This needs to be evaluated at the values of s and τ corresponding to the EP, and is the first correction term that enters Eq. (5.7).

Similarly, from Eq. (S1.13), recalling that $p_2^F = \tilde{p}_1 = 0$, the second term in the Puiseux series for the 1D Galerkin thermoacoustic problem reads:

$$s_2 = -\frac{1}{2} \frac{L_{1,1} + s_1^2 L_{3,0}}{L_{2,0}} = -\frac{1}{2} \frac{2\pi\beta e^{-s\tau} (s\tau - 1) + \frac{1}{6} s_1^2 (-2\pi\beta\tau^3 e^{-s\tau})}{1 + \pi\beta\tau^2 e^{-s\tau}}. \quad (\text{S2.4})$$

Combining these results, the second-order Puiseux expansion around the defective point is

$$s \approx s_{\text{def}} + s_1(\Delta\tau)^{1/2} + s_2(\Delta\tau) + \mathcal{O}(\Delta\tau)^{3/2} \quad (\text{S2.5})$$

This is the expression used for the calculations presented in Figure 8 at $N = 2$, which shows that, at this order, the error consistently scales as $(\Delta\tau)^{3/2}$.

3. Derivation of eigenvector coefficients equations for arbitrary splitting order

This proof generalises that shown in the main body in Appendix B for arbitrary splitting order d . To simplify the solvability condition (2.17b) into Eq. (2.18), it is convenient to rewrite the perturbation equations at order n on each branch ζ as

$$\mathbf{L}_{0,0} \hat{\mathbf{p}}_{n,\zeta} = \sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta}, \quad (\text{S3.1})$$

where we have introduced the matrices \mathbf{M} , which, by comparison with Eq. (2.14) and the definition of \mathbf{r}_n , are given by

$$\mathbf{M}_{k,\zeta} \equiv - \left[\mathbf{L}_{0,k} + \sum_{0 < |\mu|_w \leq k} \binom{|\mu|}{\mu} s_\zeta^\mu \mathbf{L}_{|\mu|,k-|\mu|_w} \right]. \quad (\text{S3.2})$$

. The following relation holds:

$$\sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta} = -\mathbf{r}_{n,\zeta} - s_{n,\zeta} \mathbf{L}_{1,0} \hat{\mathbf{p}}_{0,\zeta}. \quad (\text{S3.3})$$

At all orders, the two solvability conditions read

$$\langle \hat{\mathbf{p}}_{0,\zeta}^\dagger | \sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta} \rangle = 0 \quad (\text{S3.4a})$$

$$\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta} \rangle = 0, \quad (\text{S3.4b})$$

which are equivalent to Eqs. (2.17). Provided that these conditions are satisfied, the eigenvector correction at order n is

$$\hat{\mathbf{p}}_{n,\zeta} = \mathbf{L}_{0,0}^g \left[\sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta} \right] + c_{n,\zeta,\zeta} \hat{\mathbf{p}}_{0,\zeta} + c_{n,\zeta,\eta} \hat{\mathbf{p}}_{0,\eta} = \hat{\mathbf{p}}_{n,\zeta}^\perp + c_{n,\zeta,\zeta} \hat{\mathbf{p}}_{0,\zeta} + c_{n,\zeta,\eta} \hat{\mathbf{p}}_{0,\eta}, \quad (\text{S3.5})$$

where $\hat{\mathbf{p}}_{n,\zeta}^\perp$ is the solution of (S3.1) with no components along $\hat{\mathbf{p}}_{0,\zeta}$ nor $\hat{\mathbf{p}}_{0,\eta}$, which belong to the kernel of $\mathbf{L}_{0,0}$. The following relation – which can be proven by induction – holds between the vectors $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}^\perp$:

$$\hat{\mathbf{p}}_{n,\zeta} = \hat{\mathbf{p}}_{n,\zeta}^\perp + \sum_{k=1}^n [c_{k,\zeta,\zeta} \hat{\mathbf{p}}_{n-k,\zeta}^\perp + c_{k,\zeta,\eta} \hat{\mathbf{p}}_{n-k,\eta}^\perp]. \quad (\text{S3.6})$$

We shall now use an induction argument, assuming that we have been able to solve the perturbation equations up to order $n - 1$ included. At order n , we shall then focus on the second of the conditions (S3.4), for which $\eta \neq \zeta$, and we shall use it to derive an equation for the coefficients $c_{n,\zeta,\eta}$, (2.18). We start recalling that, in order to be able to calculate these coefficients, the nominally degenerate eigenvalues must have split at some order $d < n$. This implies that: (i) at orders $k < d$, the two eigenvalue branches are indistinguishable, so that $\mathbf{M}_{k,\zeta} = \mathbf{M}_{k,\eta}$ for $k < d$; (ii) at order $k = d$ the auxiliary eigenvalue problem (2.11) has 2 different simple degenerate eigenvalues, $s_{d,\zeta}$ and $s_{d,\eta}$, and the coefficients α identify the basis into which the degeneracy unfolds. In this basis the auxiliary eigenvalue problem is diagonal, so that

$$\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \mathbf{r}_{d,\zeta} \rangle = -s_{d,\zeta} \delta_{\eta,\zeta}. \quad (\text{S3.7})$$

In terms of the matrices \mathbf{M} , this reads

$$\mathbf{M}_{d,\zeta} = \mathbf{M}_{d,\eta} + s_{d,\eta} \mathbf{L}_{1,0} - s_{d,\zeta} \mathbf{L}_{1,0}. \quad (\text{S3.8})$$

Using (S3.6) the solvability condition (S3.4b) at order n becomes

$$\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^n \mathbf{M}_{k,\zeta} \left[\hat{\mathbf{p}}_{n-k,\zeta}^\perp + \sum_{j=1}^{n-k} [c_{j,\zeta,\zeta} \hat{\mathbf{p}}_{n-k-j,\zeta}^\perp + c_{j,\zeta,\eta} \hat{\mathbf{p}}_{n-k-j,\eta}^\perp] \right] \rangle = 0 \quad (\text{S3.9})$$

By changing the order of the summations and using the linearity of the inner product we then have

$$\begin{aligned} & \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta}^\perp \rangle + \dots \\ & \dots + \sum_{j=1}^{n-1} \left[c_{j,\zeta,\zeta} \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^{n-j} \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k-j,\zeta}^\perp \rangle + c_{j,\zeta,\eta} \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^{n-j} \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k-j,\eta}^\perp \rangle \right] = 0 \end{aligned} \quad (\text{S3.10})$$

Consider now the first inner product on the second line of Eq. (S3.10). By using the change of variable $n - j = m$, it becomes

$$\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^m \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{m-k,\zeta}^\perp \rangle, \quad (\text{S3.11})$$

which, by comparison with Eq. (S3.4b), defines a solvability condition at order m . In the last step, note that, as long as the coefficients c are not determined, the eigenvector corrections \mathbf{p} are equal to \mathbf{p}^\perp . Since we have just defined $m \equiv (n - j) < n$, this condition must already have been solved at previous orders (or we would have not been able to reach order n). Therefore, (S3.11) vanishes, which proves that the coefficients $c_{j,\zeta,\zeta}$ do not influence the solution of the perturbation equations. These coefficients can be uniquely determined if a normalisation condition is imposed on the eigenvector, which is however not discussed here.

The remaining inner product on the second line of Eq. (S3.10), $\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^{n-j} \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k-j,\eta}^\perp \rangle$, on the other hand, is not related to a solvability condition, since the matrix \mathbf{M} is evaluated on branch ζ , whereas the vectors $\hat{\mathbf{p}}^\perp$ are evaluated on branch $\eta \neq \zeta$. However, the following holds:

(i) for $j > n - d$, the index k in the matrices $\mathbf{M}_{k,\alpha}$ can reach the maximum value $n - j = m < d$. From the properties of the matrix \mathbf{M} , for $k < d$ the two branches are

indistinguishable ($\mathbf{M}_{k,\zeta} = \mathbf{M}_{k,\eta}$), and these inner product vanish because of (S3.4a):

$$\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^m \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{m-k,\eta}^\perp \rangle = \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^m \mathbf{M}_{k,\eta} \hat{\mathbf{p}}_{m-k,\eta}^\perp \rangle = 0, \quad \text{for } m = n - j < d. \quad (\text{S3.12})$$

(ii) for $j < n - d$, instead, the inner products $\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^{n-j} \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-j-k,\eta}^\perp \rangle$ are generally non-zero, and provide scaling factors for the coefficients $c_{j,\zeta,\eta}$. At expansion order n the unknown coefficients that we can calculate are however only those for $j = n - d$. The coefficients c for $j < n - d$ are instead known, since they have been determined at previous orders.

By discarding the vanishing terms and by moving on the r.h.s. the known terms in Eq. (S3.10), we have

$$\begin{aligned} c_{n-d,\zeta,\eta} \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^d \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{d-k,\eta}^\perp \rangle &= \\ &= -\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^n \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k,\zeta}^\perp \rangle - \sum_{j=1}^{n-d-1} c_{j,\zeta,\eta} \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^{n-j} \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{n-k-j,\eta}^\perp \rangle = \\ &\equiv \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \mathbf{r}_{n,\zeta}^\perp \rangle, \end{aligned} \quad (\text{S3.13})$$

where we have defined the vector $\mathbf{r}_{n,\zeta}^\perp$, which contains all the information known on branch ζ at order n . We can further simplify the inner product on the l.h.s. by considering the relation between the matrices \mathbf{M} and the vectors \mathbf{r} . From (S3.3), noticing that at orders $k \leq d$ the only coefficient which varies between branches η and ζ is $s_{d,\zeta} \neq s_{d,\eta}$ (because at order d the degeneracy was resolved), we have

$$\sum_{k=1}^d \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{d-k,\eta}^\perp = -\mathbf{r}_{d,\eta} - s_{d,\zeta} \mathbf{L}_{1,0} \hat{\mathbf{p}}_{0,\eta}. \quad (\text{S3.14})$$

Using this relation

$$\begin{aligned} \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \sum_{k=1}^d \mathbf{M}_{k,\zeta} \hat{\mathbf{p}}_{d-k,\eta}^\perp \rangle &= -\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \mathbf{r}_{d,\eta} + s_{d,\zeta} \mathbf{L}_{1,0} \hat{\mathbf{p}}_{0,\eta} \rangle = \\ &= -\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \mathbf{r}_{d,\eta} \rangle - s_{d,\zeta} \langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \mathbf{L}_{1,0} \hat{\mathbf{p}}_{0,\eta} \rangle = s_{d,\eta} - s_{d,\zeta}. \end{aligned} \quad (\text{S3.15})$$

In the last step we have used Eqs. (S3.7) and (2.8). We can finally write an explicit expression for the coefficient $c_{j,\zeta,\eta}$ at order $j = n - d$ by rearranging Eq. (S3.13)

$$c_{n-d,\zeta,\eta} = \frac{\langle \hat{\mathbf{p}}_{0,\eta}^\dagger | \mathbf{r}_{n,\zeta}^\perp \rangle}{s_{d,\eta} - s_{d,\zeta}}, \quad (\text{S3.16})$$

which corresponds to Eq. (2.18) in the main body.